# Figure 8.1. RESPONSE MODELS: Least Squares Estimating of Parameters

This Figure summarizes the least squares estimating process, using both calculus and linear algebra, for the parameters of the common STAT 231 response models; a more evocative name than *least squares* would be *minimum sum of squared residuals.* Matters omitted from some derivations in this Figure, which the reader should supply, include:

- second derivative demonstrations of *minima* (Step 6);
- for Model 2, extension from the case $\mathrm{i} = 1, 2$ to the case $\mathrm{i} = 1, 2, \ldots, \mathrm{q}$;
- definitions of vector notation (Step 3);
- vector diagrams (Step 4).

A **general** form for a response model – a response variate expressed as a **structural component** plus **noise** – is shown in two versions at the right;

- in the upper version, the structural component $\mu_j$ is a linear combination of q (unknown) parameters $\theta_\mathrm{k}$ and (known) explanatory variate values $\mathrm{x}_{j\mathrm{k}}$;
- in the lower version, the explanatory variate values form the $\mathrm{n} \times \mathrm{q}$ matrix $\mathbf{X}$ and the parameters are the $\mathrm{q} \times 1$ column vector $\underset{\sim}{\Theta}$.

$$Y_j = \mu_j + R_j, \quad j = 1, 2, \ldots, \mathrm{n},$$
$$\underset{\sim}{\boldsymbol{Y}} = \mathbf{X}\underset{\sim}{\Theta} + \underset{\sim}{\boldsymbol{R}} \qquad R_j \sim G(0, \sigma), \text{ independent, EPS;}$$
$$\mu_j = \theta_1 \mathrm{x}_{j1} + \ldots + \theta_\mathrm{k} \mathrm{x}_{j\mathrm{k}} + \ldots + \theta_\mathrm{q} \mathrm{x}_{j\mathrm{q}}$$
[explanatory variates $\mathrm{X}_1, \ldots, \mathrm{X}_\mathrm{k}, \ldots, \mathrm{X}_\mathrm{q}$]

– $\underset{\sim}{\boldsymbol{Y}}$ and $\underset{\sim}{\boldsymbol{R}}$ are $\mathrm{n} \times 1$ column vectors with $\mathrm{n} \gg \mathrm{q}$; *i.e.*, there are *many* more observations than parameters to be estimated.

The response models given below are special cases of this general form; their numbering (1, 1a, 1b, 2, 3, 3a, 4, 4a) is only for convenience in this Figure (and Figure 5.9) and does *not* carry over to the Course Notes; 'EPS' denotes *equiprobable selecting.*

**Model 1:** $Y_j = \mu + R_j, \quad j = 1, 2, \ldots, \mathrm{n}; \quad R_j \sim G(0, \sigma);$ independent, EPS.

**Step 1:** $Y_j = \mu + R_j, \qquad j = 1, 2, \ldots, \mathrm{n}.$

**Step 2:** $y_j = \mu + r_j, \qquad j = 1, 2, \ldots, \mathrm{n}.$

**Step 3:** $r_j = y_j - \mu, \qquad j = 1, 2, \ldots, \mathrm{n}.$

**Step 4:** $g(\mu) = \sum_{j=1}^{\mathrm{n}} r_j^2 = \sum_{j=1}^{\mathrm{n}} [y_j - \mu]^2.$

**Step 5:** $\dfrac{dg}{d\mu} = -2\sum_{j=1}^{\mathrm{n}} [y_j - \mu],$

which is zero when: $\sum_{j=1}^{\mathrm{n}} [y_j - \hat{\mu}] = 0;$

*i.e.*, $\sum_{j=1}^{\mathrm{n}} y_j - \mathrm{n}\hat{\mu} = 0,$

or: $\hat{\mu} = \frac{1}{\mathrm{n}} \sum_{j=1}^{\mathrm{n}} y_j = \bar{y},$ the **estimate** of $\mu$.

**Step 6:** $\dfrac{d^2 g}{d\mu^2} = -2\sum_{j=1}^{\mathrm{n}} [-1] = 2\mathrm{n},$

which is **positive**, as required for a **minimum**.

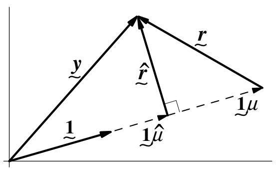
**Also:** $$\hat{\sigma} = \sqrt{\frac{\sum_{j=1}^{\mathrm{n}} \hat{r}_j^2}{\mathrm{n} - \mathrm{q}}} = \sqrt{\frac{\sum_{j=1}^{\mathrm{n}} [y_j - \hat{\mu}]^2}{\mathrm{n} - 1}} = \sqrt{\frac{\sum_{j=1}^{\mathrm{n}} [y_j - \bar{y}]^2}{\mathrm{n} - 1}}.$$

**Step 1:** $Y_j = \mu + R_j, \qquad j = 1, 2, \ldots, \mathrm{n}.$

**Step 2:** $y_j = \mu + r_j, \qquad j = 1, 2, \ldots, \mathrm{n}.$

**Step 3:** $\underset{\sim}{\boldsymbol{y}} = \underset{\sim}{\mathbf{1}}\mu + \underset{\sim}{\boldsymbol{r}},$ where: $\underset{\sim}{\boldsymbol{y}} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_\mathrm{n} \end{pmatrix}, \quad \underset{\sim}{\mathbf{1}} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \quad \underset{\sim}{\boldsymbol{r}} = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_\mathrm{n} \end{pmatrix}.$

**Step 4:** The least squares estimate of $\mu$ is determined by the vector $\underset{\sim}{\hat{\boldsymbol{r}}}$, the vector $\underset{\sim}{\boldsymbol{r}}$ of **minimum** length; this occurs when $\underset{\sim}{\boldsymbol{r}}$ is **perpendicular** to the vector $\underset{\sim}{\mathbf{1}}$, so we set the dot (or inner) product $\underset{\sim}{\mathbf{1}} \cdot \underset{\sim}{\hat{\boldsymbol{r}}}$ to zero.




**Step 5:** $\underset{\sim}{\mathbf{1}}^\mathrm{T} \underset{\sim}{\hat{\boldsymbol{r}}} = 0, \qquad$ *i.e.*, $\underset{\sim}{\mathbf{1}}^\mathrm{T}(\underset{\sim}{\boldsymbol{y}} - \underset{\sim}{\mathbf{1}}\hat{\mu}) = 0,$

*i.e.*, $\underset{\sim}{\mathbf{1}}^\mathrm{T}\underset{\sim}{\boldsymbol{y}} - \underset{\sim}{\mathbf{1}}^\mathrm{T}\underset{\sim}{\mathbf{1}}\hat{\mu} = 0,$

or: $\hat{\mu} = (\underset{\sim}{\mathbf{1}}^\mathrm{T}\underset{\sim}{\mathbf{1}})^{-1}(\underset{\sim}{\mathbf{1}}^\mathrm{T}\underset{\sim}{\boldsymbol{y}}) = \mathrm{n}^{-1}\sum_{j=1}^{\mathrm{n}} y_j = \bar{y},$ the **estimate** of $\mu$.

**Also:** $$\hat{\sigma} = \sqrt{\frac{\underset{\sim}{\hat{\boldsymbol{r}}}^\mathrm{T}\underset{\sim}{\hat{\boldsymbol{r}}}}{\mathrm{n} - \mathrm{q}}} = \sqrt{\frac{(\underset{\sim}{\boldsymbol{y}} - \underset{\sim}{\mathbf{1}}\hat{\mu})^\mathrm{T}(\underset{\sim}{\boldsymbol{y}} - \underset{\sim}{\mathbf{1}}\hat{\mu})}{\mathrm{n} - 1}} = \sqrt{\frac{\sum_{j=1}^{\mathrm{n}} [y_j - \bar{y}]^2}{\mathrm{n} - 1}}.$$

**Model 1a:** ${}_\mathrm{M}Y_j = \tau + \delta + R_j, \quad j = 1, 2, \ldots, \mathrm{m}; \quad R_j \sim G(0, \sigma);$ independent, EPS.

If we take the response variate as $Y_j = {}_\mathrm{M}Y_j - \tau$, the **difference** between the **measured** value and the **true** value:

**Model 1b:** $Y_j = \delta + R_j, \quad j = 1, 2, \ldots, \mathrm{m}; \quad R_j \sim G(0, \sigma);$ independent, EPS;

thus, Model 1a rewritten as Model 1b is equivalent to Model 1, except the structural component is $\delta$ instead of $\mu$.
Least squares estimating of $\delta$ and $\sigma$ in Model 1b is therefore equivalent to that for $\mu$ and $\sigma$ in Model 1 above.

**Model 3:** $Y_{\mathrm{i}j} = \mu_\mathrm{i} + \gamma_j + R_{\mathrm{i}j}, \quad \mathrm{i} = 1, 2, \; j = 1, 2, \ldots, \mathrm{n}; \quad R_{\mathrm{i}j} \sim G(0, \sigma);$ independent, EPS.

Using this model which is appropriate for a **blocked** Plan, we work with the **differences** of the responses of the pairs of units within the n blocks:

$Y_{1j} - Y_{2j} = \mu_1 - \mu_2 + \gamma_j - \gamma_j + R_{1j} - R_{2j},$ which becomes:

**Model 3a:** $Y_j = \mu_\mathrm{d} + R_j, \quad j = 1, 2, \ldots, \mathrm{n}; \quad R_j \sim G(0, \sigma_\mathrm{d});$ independent, EPS;

thus, Model 3a for the **differences** is equivalent to Model 1, but with parameters $\mu_\mathrm{d}$ and $\sigma_\mathrm{d}$ instead of $\mu$ and $\sigma$.
Least squares estimating of $\mu_\mathrm{d}$ and $\sigma_\mathrm{d}$ in Model 3a is therefore equivalent to that for $\mu$ and $\sigma$ in Model 1 above.

**Model 4a:** $Y_j = \beta \mathrm{x}_j + R_j, \quad j = 1, 2, \ldots, \mathrm{n}; \quad R_j \sim G(0, \sigma);$ independent, EPS;

establish the results at the right using **both** calculus and linear algebra.

$$\hat{\beta} = \frac{\sum_{j=1}^{\mathrm{n}} x_j y_j}{\sum_{j=1}^{\mathrm{n}} x_j^2} \qquad \hat{\sigma} = \sqrt{\frac{\sum_{j=1}^{\mathrm{n}} y_j^2 - \hat{\beta}\sum_{j=1}^{\mathrm{n}} x_j y_j}{\mathrm{n} - 1}}$$



(*continued overleaf*)

**Model 2:** $Y_{ij} = \mu_i + R_{ij}$, $i = 1, 2$, $j = 1, 2, ...., n_i$; $R_{ij} \sim G(0, \sigma)$; independent, EPS. [think of $n_1 + n_2$ as n]

**Step 1:** $Y_{ij} = \mu_i + R_{ij}$, $i = 1, 2$, $j = 1, 2, ...., n_i$.

**Step 2:** $y_{ij} = \mu_i + r_{ij}$, $i = 1, 2$, $j = 1, 2, ...., n_i$.

**Step 3:** $r_{ij} = y_{ij} - \mu_i$, $i = 1, 2$, $j = 1, 2, ...., n_i$.

**Step 4:** $g(\mu_1, \mu_2) = \sum_{i=1}^{2}\sum_{j=1}^{n_i} r_{ij}^2 = \sum_{j=1}^{n_1}[y_{1j} - \mu_1]^2 + \sum_{j=1}^{n_2}[y_{2j} - \mu_2]^2$.

**Step 5:** $\dfrac{\partial g}{\partial \mu_1} = -2\sum_{j=1}^{n_1}[y_{1j} - \mu_1] + 0$,

which is zero when: $\sum_{j=1}^{n_1}[y_{1j} - \hat{\mu}_1] = 0$;

*i.e.*, $\sum_{j=1}^{n_1} y_{1j} - n_1\hat{\mu}_1 = 0$,

or: $\hat{\mu}_1 = \frac{1}{n_1}\sum_{j=1}^{n_1} y_{1j} = \bar{y}_1$, the **estimate** of $\mu_1$.

Similarly: $\hat{\mu}_2 = \frac{1}{n_2}\sum_{j=1}^{n_2} y_{2j} = \bar{y}_2$, the **estimate** of $\mu_2$.

**Step 6:** Show the two estimates are **minima** using the four second partial derivatives.

**Also:** $\hat{\sigma} = \sqrt{\dfrac{\sum_{i=1}^{2}\sum_{j=1}^{n_i}\hat{r}_{ij}^2}{n - q}} = \sqrt{\dfrac{\sum_{j=1}^{n_1}[y_{1j} - \hat{\mu}_1]^2 + \sum_{j=1}^{n_2}[y_{2j} - \hat{\mu}_2]^2}{n_1 + n_2 - 2}} = \sqrt{\dfrac{(n_1 - 1)\hat{\sigma}_1^2 + (n_2 - 1)\hat{\sigma}_2^2}{n_1 + n_2 - 2}}$.

**Step 1:** $Y_{ij} = \mu_i + R_{ij}$, $i = 1, 2$, $j = 1, 2, ...., n_i$.

**Step 2:** $y_{ij} = \mu_i + r_{ij}$, $i = 1, 2$, $j = 1, 2, ...., n_i$.

**Step 3:** $\underset{\sim}{y} = \underset{\sim}{a}\mu_1 + \underset{\sim}{b}\mu_2 + \underset{\sim}{r}$,
where: $\underset{\sim}{y}$, $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{r}$ are $n_1 + n_2 \times 1$ column vectors, with $\underset{\sim}{a}^T\underset{\sim}{a} = n_1$, $\underset{\sim}{b}^T\underset{\sim}{b} = n_2$, $\underset{\sim}{a}^T\underset{\sim}{b} = \underset{\sim}{b}^T\underset{\sim}{a} = 0$.

**Step 4:** The least squares estimates of $\mu_1$ and $\mu_2$ are determined by the vector $\underset{\sim}{\hat{r}}$, the vector $\underset{\sim}{r}$ of **minimum** length; this occurs when $\underset{\sim}{r}$ is **perpendicular** to the plane which is *span*($\underset{\sim}{a}$, $\underset{\sim}{b}$), so we set the dot (or inner) products $\underset{\sim}{a}\cdot\underset{\sim}{\hat{r}}$ and $\underset{\sim}{b}\cdot\underset{\sim}{\hat{r}}$ to zero.
(Illustrate this matter with a **diagram**.)

**Step 5:** $\underset{\sim}{a}^T\underset{\sim}{\hat{r}} = 0$, *i.e.*, $\underset{\sim}{a}^T(\underset{\sim}{y} - \underset{\sim}{a}\hat{\mu}_1 - \underset{\sim}{b}\hat{\mu}_2) = 0$,

*i.e.*, $\underset{\sim}{a}^T\underset{\sim}{y} - \underset{\sim}{a}^T\underset{\sim}{a}\hat{\mu}_1 = 0$ [because: $\underset{\sim}{a}^T\underset{\sim}{b} = 0$],

or: $\hat{\mu}_1 = (\underset{\sim}{a}^T\underset{\sim}{a})^{-1}(\underset{\sim}{a}^T\underset{\sim}{y}) = n_1^{-1}\sum_{j=1}^{n_1} y_{1j} = \bar{y}_1$, the **estimate** of $\mu_1$.

Similarly: $\hat{\mu}_2 = (\underset{\sim}{b}^T\underset{\sim}{b})^{-1}(\underset{\sim}{b}^T\underset{\sim}{y}) = n_2^{-1}\sum_{j=1}^{n_2} y_{2j} = \bar{y}_2$, the **estimate** of $\mu_2$.

**Also:** $\hat{\sigma} = \sqrt{\dfrac{\underset{\sim}{\hat{r}}^T\underset{\sim}{\hat{r}}}{n - q}} = \sqrt{\dfrac{(\underset{\sim}{y} - \underset{\sim}{a}\hat{\mu}_1 - \underset{\sim}{b}\hat{\mu}_2)^T(\underset{\sim}{y} - \underset{\sim}{a}\hat{\mu}_1 - \underset{\sim}{b}\hat{\mu}_2)}{n_1 + n_2 - 2}} = \text{etc.}$

**Model 4:** $Y_j = \alpha + \beta_1(x_j - \bar{x}) + R_j$, $j = 1, 2, ...., n$; $R_j \sim G(0, \sigma)$; independent, EPS.

**Step 1:** $Y_j = \alpha + \beta_1(x_j - \bar{x}) + R_j$, $j = 1, 2, ...., n$.

**Step 2:** $y_j = \alpha + \beta_1(x_j - \bar{x}) + r_j$, $j = 1, 2, ...., n$.

**Step 3:** $r_j = y_j - \alpha - \beta_1(x_j - \bar{x})$, $j = 1, 2, ...., n$.

**Step 4:** $g(\alpha, \beta_1) = \sum_{j=1}^{n} r_j^2 = \sum_{j=1}^{n}[y_j - \alpha - \beta_1(x_j - \bar{x})]^2$.

**Step 5:** $\dfrac{\partial g}{\partial \alpha} = -2\sum_{j=1}^{n}[y_j - \alpha - \beta_1(x_j - \bar{x})]$,

which is zero when:

$\sum_{j=1}^{n}[y_j - \hat{\alpha} - \hat{\beta}_1(x_j - \bar{x})] = 0$;

*i.e.*, $\sum_{j=1}^{n} y_j - n\hat{\alpha} - \hat{\beta}_1\sum_{j=1}^{n}(x_j - \bar{x}) = 0$,

*i.e.*, $\sum_{j=1}^{n} y_j - n\hat{\alpha} = 0$ [because: $\sum_{j=1}^{n}(x_j - \bar{x}) \equiv 0$],

or: $\hat{\alpha} = \frac{1}{n}\sum_{j=1}^{n} y_j = \bar{y}$, the **estimate** of $\alpha$.

Also: $\dfrac{\partial g}{\partial \beta_1} = -2\sum_{j=1}^{n}[y_j - \alpha - \beta_1(x_j - \bar{x})](x_j - \bar{x})$,

which is zero when:

$\sum_{j=1}^{n}[y_j - \hat{\alpha} - \hat{\beta}_1(x_j - \bar{x})](x_j - \bar{x}) = 0$;

*i.e.*, $\sum_{j=1}^{n} y_j(x_j - \bar{x}) - \hat{\alpha}\sum_{j=1}^{n}(x_j - \bar{x}) - \hat{\beta}_1\sum_{j=1}^{n}(x_j - \bar{x})^2 = 0$,

*i.e.*, $\sum_{j=1}^{n} y_j(x_j - \bar{x}) - \hat{\beta}_1\sum_{j=1}^{n}(x_j - \bar{x})^2 = 0$ [because: $\sum_{j=1}^{n}(x_j - \bar{x}) \equiv 0$],

or: $\hat{\beta}_1 = \dfrac{\sum_{j=1}^{n} y_j(x_j - \bar{x})}{\sum_{j=1}^{n}(x_j - \bar{x})^2} \equiv \dfrac{\sum_{j=1}^{n} y_j(x_j - \bar{x})}{SS_x}$, the **estimate** of $\beta_1$.

**Step 6:** Show the two estimates are **minima** using the four second partial derivatives.

**Also:** $\hat{\sigma} = \sqrt{\dfrac{\sum_{j=1}^{n}\hat{r}_j^2}{n - q}} = \sqrt{\dfrac{\sum_{j=1}^{n}[y_j - \bar{y} - \hat{\beta}_1(x_j - \bar{x})]^2}{n - 2}} = \sqrt{\dfrac{SS_y - \hat{\beta}_1 SS_{xy}}{n - 2}}$; $SS_y = \sum_{j=1}^{n}(y_j - \bar{y})^2$, $SS_{xy} = \sum_{j=1}^{n}(x_j - \bar{x})(y_j - \bar{y}) \equiv \sum_{j=1}^{n}(x_j - \bar{x})y_j$.

**Step 1:** $Y_j = \alpha + \beta_1(x_j - \bar{x}) + R_j$, $j = 1, 2, ...., n$.

**Step 2:** $y_j = \alpha + \beta_1(x_j - \bar{x}) + r_j$, $j = 1, 2, ...., n$.

**Step 3:** $\underset{\sim}{y} = \underset{\sim}{\mathbf{1}}\alpha + \underset{\sim}{\mathbf{x}}\beta_1 + \underset{\sim}{r}$, where $\underset{\sim}{\mathbf{x}} = \begin{pmatrix} x_1 - \bar{x} \\ x_2 - \bar{x} \\ \vdots \\ x_n - \bar{x} \end{pmatrix}$

**Step 4:** The least squares estimates of $\alpha$ and $\beta_1$ are determined by the vector $\underset{\sim}{\hat{r}}$, the vector $\underset{\sim}{r}$ of **minimum** length; this occurs when $\underset{\sim}{r}$ is **perpendicular** to the plane which is *span*($\underset{\sim}{\mathbf{1}}$, $\underset{\sim}{\mathbf{x}}$), so we set the dot (or inner) products $\underset{\sim}{\mathbf{1}}\cdot\underset{\sim}{\hat{r}}$ and $\underset{\sim}{\mathbf{x}}\cdot\underset{\sim}{\hat{r}}$ to zero.
(Illustrate this matter with a **diagram**.)

**Step 5:** $\underset{\sim}{\mathbf{1}}^T\underset{\sim}{\hat{r}} = 0$, *i.e.*, $\underset{\sim}{\mathbf{1}}^T(\underset{\sim}{y} - \underset{\sim}{\mathbf{1}}\hat{\alpha} - \underset{\sim}{\mathbf{x}}\hat{\beta}_1) = 0$,

*i.e.*, $\underset{\sim}{\mathbf{1}}^T\underset{\sim}{y} - \underset{\sim}{\mathbf{1}}^T\underset{\sim}{\mathbf{1}}\hat{\alpha} - \underset{\sim}{\mathbf{1}}^T\underset{\sim}{\mathbf{x}}\hat{\beta}_1 = 0$,

*i.e.*, $\underset{\sim}{\mathbf{1}}^T\underset{\sim}{y} - \underset{\sim}{\mathbf{1}}^T\underset{\sim}{\mathbf{1}}\hat{\alpha} = 0$ [because: $\underset{\sim}{\mathbf{1}}^T\underset{\sim}{\mathbf{x}} = \sum_{j=1}^{n}(x_j - \bar{x}) \equiv 0$],

or: $\hat{\alpha} = (\underset{\sim}{\mathbf{1}}^T\underset{\sim}{\mathbf{1}})^{-1}(\underset{\sim}{\mathbf{1}}^T\underset{\sim}{y}) = n^{-1}\sum_{j=1}^{n} y_j = \bar{y}$, the **estimate** of $\alpha$.

Also: $\underset{\sim}{\mathbf{x}}^T\underset{\sim}{\hat{r}} = 0$, *i.e.*, $\underset{\sim}{\mathbf{x}}^T(\underset{\sim}{y} - \underset{\sim}{\mathbf{1}}\hat{\alpha} - \underset{\sim}{\mathbf{x}}\hat{\beta}_1) = 0$,

*i.e.*, $\underset{\sim}{\mathbf{x}}^T\underset{\sim}{y} - \underset{\sim}{\mathbf{x}}^T\underset{\sim}{\mathbf{1}}\hat{\alpha} - \underset{\sim}{\mathbf{x}}^T\underset{\sim}{\mathbf{x}}\hat{\beta}_1 = 0$,

*i.e.*, $\underset{\sim}{\mathbf{x}}^T\underset{\sim}{y} - \underset{\sim}{\mathbf{x}}^T\underset{\sim}{\mathbf{x}}\hat{\beta}_1 = 0$ [because: $\underset{\sim}{\mathbf{x}}^T\underset{\sim}{\mathbf{1}} = \underset{\sim}{\mathbf{1}}^T\underset{\sim}{\mathbf{x}} \equiv 0$],

or: $\hat{\beta}_1 = (\underset{\sim}{\mathbf{x}}^T\underset{\sim}{\mathbf{x}})^{-1}(\underset{\sim}{\mathbf{x}}^T\underset{\sim}{y}) = [\sum_{j=1}^{n}(x_j - \bar{x})^2]^{-1}[\sum_{j=1}^{n}(x_j - \bar{x})y_j]$

$= \dfrac{\sum_{j=1}^{n} y_j(x_j - \bar{x})}{\sum_{j=1}^{n}(x_j - \bar{x})^2} \equiv \dfrac{\sum_{j=1}^{n} y_j(x_j - \bar{x})}{SS_x}$, the **estimate** of $\beta_1$.

**Also:** $\hat{\sigma} = \sqrt{\dfrac{\underset{\sim}{\hat{r}}^T\underset{\sim}{\hat{r}}}{n - q}} = \sqrt{\dfrac{(\underset{\sim}{y} - \underset{\sim}{\mathbf{1}}\hat{\alpha} - \underset{\sim}{\mathbf{x}}\hat{\beta}_1)^T(\underset{\sim}{y} - \underset{\sim}{\mathbf{1}}\hat{\alpha} - \underset{\sim}{\mathbf{x}}\hat{\beta}_1)}{n - 2}} = \text{etc.}$