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PARADOXES: Monty Hall / Three Prisoners' Dilemma

Experience shows that probability is a topic many people find difficult, in that mistakes are easier to make than in (some) other subject areas. This is one of six Statistical Highlights (#46 to #51) which discuss probabilistic subtleties and mistakes (which sometimes lead to so-called 'paradoxes'), with a view to helping the reader recognize and deal correctly with such matters. [These and related statistical issues are also discussed in Figure 7.12 of the STAT 220 Course Materials.]

1. Monty Hall Context

Monty Hall was a popular American TV game-show host. The game, which started in 1963 and was called *Let's make a deal*, had three closed doors (here designated **A**, **B** and **C**) visible to the contestant; behind one door was a valuable prize like a car, behind each of the others was a goat. The contestant chose a door and would receive whatever was behind it. *Un*like the contestant, Monty Hall knew which door had the valuable prize behind it; before the doors were opened to reveal the contestant's winnings, Monty Hall would open one of the two doors *not* selected and, using his prior knowledge, reveal a goat. He then offered the contestant the opportunity to *change* their choice of door. It might *seem* that Monly's opening a door provided *no* information to the contestant about where the valuable prize *actually* was, because at least one of the two doors *not* selected *must* have a goat behind it. The Question of interest in this Highlight #49 is: *Is this reasoning correct or can information about the probability of winning the valuable prize be gleaned from Monty's door opening?* That is; *Is there any probabilistic advantage to the contestant switching doors?*

2. Probability and conditional probability calculations

Assuming (without loss of generality) that the valuable prize is behind door **A**, we consider three *equally* probable situations.

- the contestant chooses door **A** so Monty can open either door **B** or door **C**;
- the contestant chooses door **B** so Monty has to open door **C**;
- the contestant chooses door **C** so Monty has to open door **B**.

In the first situation, the contestant loses by switching but they win by switching in the other two situations. Thus, switching gives the contestant a probability of ⅔ of winning, which is the long-recognized probabilistic advantage of switching over the (wrong) 'intuitive' probability of ½ and it answers affirmatively the question posed at the end of Section 1 above.

A refinement of the situation where Monty has a *choice* of which of two doors to open involves using conditional probability; this refinement does not *materially* affect the advantage of switching, and its discussion has the *dis*advantage that the more formal reasoning may make it less accessible to readers without the relevant specialized knowledge.

Define: event A – the valuable prize is behind door **A**;
event B – the valuable prize is behind door **B**;
event C – the valuable prize is behind door **C**;
event M_C – Monty opens door **C**.

$$\Pr(A) = \Pr(B) = \Pr(C) = 1/3 \qquad \text{-----(1)}$$

Assume the game show selects the door for the valuable prize equiprobably so, for the contestant, the first three events each have probability ⅓, as in equation (1) at the right above. [This assumption is reasonable because, if selecting is *not* equiprobable, the game show history would allow the more probable door to become public knowledge.] The discussion below in this Section 2 has the contestant (initially) select door **A**, which is why the focus in this Section 2 is on event A.

As shown at the right, we first use Bayes' rule as in equation (2) and then expand the denominator as in equation (3) – the vertical lines (|) denote *conditional on* (or *given*).

$$\Pr(A|M_C) = \frac{\Pr(M_C|A)\times\Pr(A)}{\Pr(M_C)} \qquad \text{-----(2)}$$

$$= \frac{\Pr(M_C|A)\times\Pr(A)}{\Pr(M_C|A)\times\Pr(A) + \Pr(M_C|B)\times\Pr(B) + \Pr(M_C|C)\times\Pr(C)} \qquad \text{-----(3)}$$

Monty can*not* open door **C** if it has the valuable prize so equation (4) follows. $\Pr(M_C|C) = 0$ -----(4)

It is door **C** that Monty *must* open if door **B** has the valuable prize so equation (5) follows. $\Pr(M_C|B) = 1$ -----(5)

Using the probabilities provided by equations (1), (4) and (5) in equation (3) yields equation (6). Our focus on event A in equations (2), (3) and (6) is because the contestant chose door **A**.

$$\Pr(A|M_C) = \frac{\Pr(M_C|A)\times 1/3}{\Pr(M_C|A)\times 1/3 + 1\times 1/3 + 0\times 1/3} = \frac{\Pr(M_C|A)}{\Pr(M_C|A)+1} \qquad \text{-----(6)}$$

Because Monty has opened door **C**, equation (7) follows. $\Pr(B|M_C) = 1 - \Pr(A|M_C)$ -----(7)

Equation (6) shows (unexpectedly?) that, given the contestant's choice of door **A**, the probanility valuable prize is behind door **A** depends on the process [*un*known to the contestant but constrained by equation (5)] Monty uses to decides it is door **C** he will open.

Table HL49.1

$\Pr(M_C\|A)$	$\Pr(A\|M_C)$	$\Pr(B\|M_C)$
1	½	½
½	⅓	⅔
¼	⅕	⅘
1/10	1/11	10/11

Based on equations (6) and (7), Table HL49.1 at the right shows values of $\Pr(A|M_C)$ and $\Pr(B|M_C)$ for four (decreasing) values of $\Pr(M_C|A)$; five matters are relevant here.

- ∗ When the valuable prize is behind door **A** so Monty can choose to open door **B** or door **C**, the first two values in the first column of Table HL49.1 at the right mean that:

(*continued overleaf*)

- ○ Monty will *always* open door **C** – $\Pr(M_C|A) = 1$;
- ○ Monty selects *equi*probably between door **B** and door **C** – $\Pr(M_C|A) = ½$ – which yields the probabilistic advantage of switching – $\Pr(B|M_C) = ⅔$ – previous;y identified overleaf at the start of Section 2.

∗ As the value of $\Pr(M_C|A)$ in the first column decreases, the second column also decreases but the third *in*creases.

∗ A value of *zero* is not present at the bottom of the first column because door **C** *has* been opened; were 0 to be included, the respective second and third column values would be 0 and 1.

∗ Each pair of probabilities across the second and third columns add to 1 because, with door **C** opened, the valuable prize *must* be behind either door **A** or door **B** – recall equation (7) overleaf on page HL49.1.

∗ Initially, $\Pr(A) = ⅓$ [equation (1) at the upper right overleaf on page HL49.1]; however, when conditioned on event M_C, it retains this value when $\Pr(M_C|A) = ½$ but, for other values, it can *in*crease to as high as ½ or become arbitrarily close to zero.

Thus, Table HL49.1 at the lower right overleaf on page HL49.1 (again) answers the question at the end of Section 1 overleaf: *The contestant can appreciably* in*cease their probability of getting the valuable prize by switching doors (despite not knowing Monty's process for deciding to open door* **C**). [This has been recognized for many years from previous discussions of Monty Hall.]

3. How do we account for the probabilistic advantage of switching doors?

Table HL49.1 and the discussion overleaf on page HL49.1 identify the (easily overlooked) distinction between:

- knowing that at least one of doors **B** and **C** must have a goat behind it; **AND:**
- of the two doors **B** and **C**, knowing one that *actually* has a goat behind it.

The contestant faces the first situation initially, the second situation *after* Monty has opened door **C**.

- About half the time (anticipating door **C** will be opened) when the valuable prize is behind door **A** [the contestant's choice and event *A* of equation (1) at the upper right overleaf], it is (perhaps) surprising that Monty's opening door **C** has such an effect on the probabilities relevant to the decision whether to switch doors – recall Table HL40.1 at the lower right overleaf.
- For the other (roughly) half the time when the valuable prize is behind door **B**, Monty has no choice but to open door **C**; this recurring (about half the time) constraint on Monty's choice of door to open is the reason underlying why there is:
  - ○ a probabilistic advantage for the contestant to switch doors;
  - ○ the distinction (made above) between two situations that might be mistakenly taken as equivalent.

NOTE: 1. Throughout the foregoing discussion, the particular letter of a given door is of no significance; we could have **B** instead of **A**, **C** instead of **B** and **A** instead of **C** (or another such permutation) without affecting the reasoning.

4. The Three Prisoners' Dilemma Context

The Monty Hall context is based on a real-world situation with a possible substantial benefit to a contestant. The Three Prisoners' Dilemma is a *dark* (hypothetical) situation with catastrophy (execution) awaiting one of three prisoners. The contestant in Monty Hall would like to *in*crease their probability of getting the valuable prize, any prisoner wants to *de*crease their probability of execution, or (obviously) *not* to *in*crease it. We recognize these differences in the two contexts from the start of discussion, despite a number of similarities; these matters are summarized in Table HL49.2 at the right below.

Three prisoners, here designated **A**, **B** and **C**, are told that their warder has equiprobably selected one of them to be executed next day. Prisoner **A** (naturally enough) would like to know if he was selected but realizes the warder would refuse to answer such a direct question; prisoner **A** therefore decides to ask (confidentially) which of **B** and **C** is to be spared, knowing that one of them must be; the warder replies **C**. Prisoner **A** is now more distressed – before he asked, he had a probability of ⅓ of being executed next day, now it has risen to ½. *Is prisoner* **A***'s reasoning correct?*

Table HL49.2

Monty Hall context	Three Prisoners' Dilemma context
Three doors, **A**, **B**, **C**	Three prisoners, **A**, **B**, **C**
One 'good' outcome	One 'bad' outcome
Two 'bad' outcomes	Two 'good' outcomes
Doors indifferent to outcomes	Prisoners involved in outcomes
Monty Hall with inside knowledge	Warder with inside knowledge
One revealed 'bad' outcome	One revealed 'good' outcome
Contestant – selects door **A**, say	a Prisoner – **A**, say
Question: switch doors?	Question: probability of execution?

5. Probabilities

Similarites between the Monty Hall and the Three Prisoners' Dilemma contexts means that they involve the *same* probability structure; probabilities relevant to **A**'s reasoning after the warder's answer are those in the first two columns of Table HL49.1 at the lower right overleaf, shown again at the right in Table HL49.3, with the events now relevant defined above it.

event *A* – Prisoner **A** is to be executed next day;
event *B* – Prisoner **B** is to be executed next day;
event *C* – Prisoner **C** is to be executed next day;
event W_C – Warder says **C** is to be spared.

The third column of Table HL49.1 is not of immediate concern for Prisoner **A**, but its implications of statistical interest are pursued in Section 6 on the facing page HL49.3.

Table HL49.3 shows that **A**'s reasoning *is* correct in the sense that her/his probability of execution may have been altered, but with the caveat about its dependence on the (unknown to **A**) process the warder uses to identify **C** as being spared in the situation where **A** is the one to be executed the next day.

Table HL49.3

$\Pr(W_c|A)$	$\Pr(A|W_c)$
1	½
½	⅓
¼	⅕
1⁄10	1⁄11

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[The discussion follows exactly that of the Monty Hall context on the lower half of page HL49.1 and on the facing page HL49.2, omitting consideration of **B**'s probabilities.] Distinguishing between:

- knowing that at least one of prisoners **B** and **C** will be spared, **AND:**
- of the two prisoners **B** and **C**, knowing one that will *actually* be spared,

is (of course) also involved, being the situation faced by prisoner **A** before and after questioning the warder.

6. Permuting A, B and C

In the Monty Hall context, which letter is assigned to which door has no effect on the reasoning, provided (of course) that the initial assignments are consistantly maintained throughout the discussion. Also, the contestant is *distinct* from the doors although this person's function probabilistically is only to focus attention on one door (the contestant's probabilistically-*un*important initial choice) and to provide a plausible rationale to involve possibly switching doors.

In the Three Prisoners' Dilemma context, focus on prisoner **A** does not preclude the case of prisoner **B** also (confidentially) questioning the warder and, like prisoner **A**, finding out that prisoner **C** is to be spared. This would correspond to *two* versions of Table HL49.1 at the lower right of page HL49.1 – the version as given for prisoner **A** as questioner, another version for prisoner **B** as questioner with the *headings* of the second and third columns interchanged. This means that each prisoner, based on having the *same* information, has the *same* probabilities for themselves and each sees the *same* set of probabilities for the other prisoner. Only an 'external' observer might view the two table versions as seemingly incompatible.

It may be helpful for such an observer to distinguish between:

+ the state of affairs in the real world, whatever it may be, which is *actual*; **AND:**
– a prisoner's (*in*complete) state of knowledge, where we describe the uncertainty in *inferences* about (actual) real-world situations in terms of *probabilities*.

There are three real-world situations possible in both contexts of this Highlight #49; using the notation E and S for 'executed' and 'spared' in the Three Prisoners' Dilemma, these situations are listed (with internal order **A**, **B**, **C**) at the right and labeled 1, 2 and 3. Only *one* of these three (known to the warder) is *actual*.

1.	E	S	S
2.	S	E	S
3.	S	S	E

∗ Situation 1 being actual (where prisoners **B** and **C** are both spared) corresponds to our foregoing discussion of the process the warder uses to decide to identify prisoner **C** as being spared and its probabilistic consequences as quantified in Tables HL49.3 and HL49.1; its occurrence in about *half* the situations with prisoner **C** spared is the source of *both* prisoner **A**'s possble increase in probability to as much as ½ of being the one to be executed next day when prisoner **A** questions the warder *and* the (substantially) increased probabilities of being that one after prisoner **B** (confidentially) questions the warder.

  ○ When prisoner **B** questions the warder and situation 1 is actual, it serves for prisoner **B** the same role as situation 2 being actual does for prisoner **A** after **A** questions the warder.

∗ Situation 2 being actual leaves the warder with no choice but to name prisoner **C** as being spared when questioned by prisoner **A**; its occurrence, in about half the situations with **C** spared, is the source of prisoner **B**'s increased probabilities of being the one to be executed next day in the third column of Table HL49.1 on page HL49.1.

  ○ When **A** questions the warder and situation 2 is actual, it serves for **A** the same role as situation 1 being actual does for prisoner **B** after **B** questions the warder.

∗ Situation 3 being actual is possible before prisoner **A** (or **B**) questions the warder but is then ruled out by the warder's answer.

Thus, with the initial three reduced to two possible real-world situations (one of which *is* actual), there is no incompatibility in prisoners **A** and **B** using the *same* knowledge and reasoning to generate the *same* probabilities, with differing assignments based on *which one* questioned the warder.

– *Before* questioning the warder, both prisoners would also reason the same way from the same lesser knowledge to the same probabilities but these probabilities (of course) differ from those the prisoners generate with greater post-warder knowledge.

NOTES: 2. In the case of prisoner **B** *also* (confidentially) questioning the warder, discussed above in Section 6, an answer that prisoner **A** is to be spared is not permitted under the conditions of the discussion in this Highlight #49 but, if it were and prisoner **A** and prisoner **B** were to *combine* their post-warder knowledge, it becomes *known* that prisoner **B** is the one selected equiprobably to be executed the next day.

3. Distinguishing the three *actual* situations in the Three Prisoners' Dilemma (as the the right above) is reminiscent of distinguishing the two actual envelope cases in the Ali-Baba Paradox near the middle of page HL46.1 in Statistical Highlight #46.

4. The shorter title **Prisoner's Dilemma** is (according to a Google search in March, 2025) a paradox in decision analysis in which two individuals acting in their own self-interest do not produce the optimul outcome for the pair considered as a group. Key take-aways from *this* Prisoner's Dilemma are said to be:
   - individuals have an incentive to choose an outcome that is less-than-optimal for their group as a whole;

(*continued overleaf*)

NOTES: (cont.) 4.
- the prisoner's dilemma occurs in many aspects of the economy (and in other contexts);
- in the classic dilemma, individuals receive the greatest payoffs if they betray the group rather than cooperate;
- under repetition, it is possible for each individual to devise a strategy that rewards cooperation;
- there are many methods which achieve better collective outcomes despite apparently unfavourable individual outcomes.

SOURCE: Oldford, R.W.: *Probability, problems, and paradoxes oictured by eikosograms*, April 21, 2003 (35 pages, 17 references);
url http://math.uwaterloo,ca > examples > paper PDF
Using a different approach, Prof. Oldford discusses Monty Hall, the Three Prisoners' Dilemma and other 'paradoxes'.