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PARADOXES: Apple Pie Paradox

Experience shows that probability is a topic many people find difficult, in that mistakes are easier to make than in (some) other subject areas. This is one of six Statistical Highlights (#46 to #51) which discuss probabilistic subtleties and mistakes (which sometimes lead to so-called 'paradoxes'), with a view to helping the reader recognize and deal correctly with such matters. [These and related statistical issues are also discussed in Figure 7.12 of the STAT 220 Course Materials.]

1. The Context

At the restaurant where I often eat lunch, the dessert menu includes choice of three pies: **A**pple, **B**lueberry and **C**herry. As a fruit pie aficionado, over many years I have kept a record of my enjoyment of these pies on a scale of 1 (least) to 6 (most). As is common in situations like this, I think of the enjoyment data I have collected as generating probability functions for three probabilistically independent discrete random variables A, B and C (shown at the right). We see that apple pie gives me consistent medium enjoyment, blueberry is either very poor or very good about half the time each, and cherry is still more variable in three categories of poor a little more than half the time, quite good and excellent both around one time in five.

a	1	2	3	4	5	6
$f(a)$	0	0	1	0	0	0

-----(1)

b	1	2	3	4	5	6
$f(b)$	0.51	0	0	0	0.49	0

-----(2)

c	1	2	3	4	5	6
$f(c)$	0	0.56	0	0.22	0	0.22

-----(3)

2. Which choice of the three pies is likely to give greatest enjoyment?

We can answer this question using the *joint* probability function of the three random variables A, B and C. With each random variable having 6 values, we can think of the joint p.f. as a 6×6×6 cubical array of 216 probabilities (all but 6 of which are zero in this instance). If we take A as the *vertical* axis of the cube, the 6×6 'layer' with $A = 3$ (shown at the right) is the only relevant one here as it contains the six non-zero probabilities; under the assumptuion of probabilistic independence, their values are the *product* of the relevant three values from the individual p.f.s (1), (2) and (3) at the right above – for example, 0.2856 is 1×0.51×0.56 and is the probability of the event $A = 3 \cap B = 1 \cap C = 2$. [∩ denotes 'intersection'.]

$c \backslash b$	1	2	3	4	5	6	
1	0	0	0	0	0	0	0
2	0.2856	0	0	0	0.2744	0	0.56
3	0	0	0	0	0.	0	0
4	0.1122	0	0	0	0.1078	0	0.22
5	0	0	0	0	0	0	0
6	0.1122	0	0	0	0.1078	0	0.22
	0.51	0	0	0	0.49	0	

-----(4)

Table (4) shows that the *highest* enjoyment index is likely to arise from choosing **blueberry** pie, the *lowest* from choosing **apple**, based on the following probabilities.

- The two events $A = 3 \cap B = 5 \cap C = 2$ and $A = 3 \cap B = 5 \cap C = 4$, where B has the highest enjoyment index, have probabilities of 0.2744 and 0.1078 which sum to **0.3822**;
- The three events $A = 3 \cap B = 1 \cap C = 4$, $A = 3 \cap B = 1 \cap C = 6$ and $A = 3 \cap B = 5 \cap C = 6$, where C has the highest enjoyment index, have probabilities of 0.1122, 0.1122 and 0.1078 which sum to **0.3322**;
- The event $A = 3 \cap B = 1 \cap C = 2$, where A has the highest enjoyment index, has probability **0.2856**.

3. What is the situation when there are only *two* pies to choose from?

Sometimes, when I am ready to order dessert, the restaurant is out of one pie so I have only *two* choices; there are three cases.

∗ When the choice is between apple and blueberry, looking at Tables (1) and (2) above, I choose apple because I will enjoy it more 51% of the time (when its index of 3 is higher than blueberry's 1).

∗ When the choice is between apple and cherry, looking at Tables (1) and (3) above, I again choose apple because I will enjoy it more 56% of the time (when its index of 3 is higer than cherry's 2).

∗ When the choice is between blueberry and cherry, looking at Tables (2) and (3) above, I choose cherry because blueberry's index of 1 is lower (slightly) more than half the time.

Thus, in the case of only two pies to choose from, the previous favourite of bluebarry is rejected in favour of apple and cherry; the enjoyment ranking is now apple-cherry-blueberry, a *reversal* of blueberry-cherry-apple from before. The interchange of apple and blueberry, depending on whether two or three pies are available, is the 'paradox' of this Highlight #47 title.

4. How do we resolve the 'Apple Pie Paradox'?

We get more insight into the less formal calculations for the pairwise comparisons in Section 3 from the three relevant bivariate probability functions for A and B, A and C, B and C; these p.f.s are given overleaf on page HL47.2 as Tables (5), (6) and (7).

∗ The *bi*variate probability function of A and B in Table (5) shows, as in Section 3 but more formally, that the event $A = 3 \cap B = 1$ (where the enjoyment index for apple is higher) is (slightly) more probable that the event $A = 3 \cap B = 5$ (where

 (*continued overleaf*)

the enjoyment index for apple is lower); this is the (slight) preference for apple over blueberry.

* The bivariate p.f. of A and C in Table (6) shows that the event $A=3 \cap C=2$ (where the enjoyment index for apple is higher) is more probable than the two events $A=3 \cap C=4$ and $A=3 \cap C=6$ (where the enjoyment index for apple is lower) – their probabilities sum to 0.44, less than 0.56; this is the preference for apple over cherry.
* The bivariate p.f. of B and C in Table (7) shows that the two events $B=5 \cap C=2$ and $B=5 \cap C=4$ (where the enjoyment index for blueberry is higher) is less probable – their probabilities sum to 0.3822 – than the four events $B=1 \cap C=2$, $B=1 \cap C=4$, $B=1 \cap C=6$ and $B=5 \cap C=6$ (where the enjoyment index for blueberry is lower) – their probabilities sum to 0.6178; this is the preference for cherry over bluebarry.

  We note in passing that the bivariate p.f. in Table (7) is, in this instance, the *same* as the p.f. in Table (4) overleaf on page HL47.1, which is the 'layer' with $A=3$ of the trivariate p.f. The difference is that, in the three-way situation, we take the six probabilities in *three* groups (as at the end of Section 2 overleaf) compared with *two* groups in the pairwise situation in this Section 4.

$b \backslash a$	1	2	3	4	5	6	
1	0	0	0.51	0	0	0	0.51
2	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0
5	0	0	0.49	0	0	0	0.49
6	0	0	0	0	0	0	0
	0	0	1	0	0	0	

-----(5)

$c \backslash a$	1	2	3	4	5	6	
1	0	0	0	0	0	0	0
2	0	0	0.56	0	0	0	0.56
3	0	0	0	0	0	0	0
4	0	0	0.22	0	0	0	0.22
5	0	0	0	0	0	0	0
6	0	0	0.22	0	0	0	0.22
	0	0	1	0	0	0	

-----(6)

$c \backslash b$	1	2	3	4	5	6	
1	0	0	0	0	0	0	0
2	0.2856	0	0	0	0.2744	0	0.56
3	0	0	0	0	0	0	0
4	01122	0	0	0	0.1078	0	0.22
5	0	0	0	0	0	0	0
6	01122	0	0	0	0.1078	0	0.22
	0.51	0	0	0	0.49	0	

-----(7)

The foregoing discussion indicates that we should restrain our surprise at different rankings of pie enjoyment under three-way and pairwise situations for several reasons.

- The probabilities for the two situations come from *different* (although related trivariate and bivariate) probability distributions, implying at least the *possibility* of differing outcomes from different situations.
- The probabilites that yield these different rankings are generated from *multiple* components so it is *un*surprisung if (carefully chosen) sets of these components can be made to yield seemingly anomalous rankings.
  - This reminds us that ranking *uni*variate information is usually straight-forward, ranking *multi*varite information may be (much) less so.
- The probability calculations involve a *modelling* assumption of probabilistic independence, a mathematical idealization whose real-world implications are routinely troublesome.
- The respective *means* of the three univariate p.f.s (1), (2) and (3), at the upper right overleaf on page HL47.1, are 3.00, 2.96 and 3.32, an enjoyment ranking of cherry-apple-blueberry which differs from the two previous rankings – see Table HL47.1 at the right.
  - This is another reminder that, when probability distributions are involved, different approaches to a Question may yield different Answers.

Table HL47.1: Pie Enjoyment

Approach	**Ranking**
Univariate	cherry-apple-blueberry
Bivariate	apple-cherry-blueberry
Trivariate	blueberry-cherry-apple

- Enjoyment ranking based *strictly* on probabilities may obscure practical considerations – for instance, for blueberry pie, a restaurant patron might ignore the *small* probability difference (0.51 *vs* 0.49) in favour of its *large* enjoyment index difference (1 *vs* 5).

SOURCE: Typewritten summary of a colloquium talk by Prof. Blyth at Queen's University, Kingston, Ontario, on November 2, 1972, kindly provided to the writer by Prof. Ross Honsburger, likely in (early) 1983.

This Highlight #47 is an elaboration of the typewritten summary; the pie names have been permuted here for alphabetic convenience in the Highlight titles.

NOTES: 1. Unsurprisingly, this context involves a difference between practice and theory – it would require a sensitive palate to routinely distinguish the enjoyment corresponding to the relatively similar (although theoretically distinct) **bold** probabilities overleaf near the middle of page HL47.1.

2. With outcomes of 1 to 6 for the pie enjoyment index, the probabilistic structure of this 'paradox' would remain the same if a (realistic) context could be found to make plausible comparing two- and three-dice situations, although the restricted outcomes for *these* dice (arising from zero probabilities) make them unusual, a matter that arises again in Statisical Highlight #48.

   - Whether the context is three pies or three dice, there is no *overlap* of outcomes from the three, which means that *ties* are precluded – see also the comments on the lower half of page HL48.2 in Statistical Highlight #48..