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PARADOXES: Ali-Baba Paradox

Experience shows that probability is a topic many people find difficult, in that mistakes are easier to make than in (some) other subject areas. This is one of six Statistical Highlights (#46 to #51) which discuss probabilistic subtleties and mistakes (which sometimes lead to so-called 'paradoxes'), with a view to helping the reader recognize and deal correctly with such matters. [These and related statistical issues are also discussed in Figure 7.12 of the STAT 220 Course Materials.]

1. The Ali-Baba Paradox

Two envelopes, identical in appearance, contain sums of money \$x and \$2x respectively; they are given under equiprobable selecting (EPS) to individuals **A** and **B**. With*out* opening the envelopes, **A** and **B** then have the opportunity to *exchange* envelopes; does either expect to gain from doing so?

A reasons as follows: if his envelope contains \$y, then **B**'s contains either \$½y or \$2y with equal probability because the envelopes were given to **A** and **B** under EPS. Thus, the random variable Y representing the dollar amount in **B**'s envelope has probability function [p.f.] (1) shown at the right, and so Y has expected value \$5⁄4y; **A** therefore expects to gain \$¼y by exchanging envelopes. Of course, **B** can reason the same way and so she *also* expects to gain \$¼y by exchanging; a gain under exchanging envelopes by *both* **A** and **B** is the 'paradox' (or, more accurately, the contradiction)

y	½y	2y
$f(y)$	0.5	0.5

-----(1)

To resolve the matter, first consider *three* envelopes – one pink envelope containing \$x which is given to **A**, and two blue envelopes containing \$½x and \$2x, one of which is given to **B** under EPS. **A** now *does* expect to gain by exchanging because:

- formally, the expected value calculation above for Y now applies, OR:
- informally, **A** can, with equal probability, gain \$x but lose only \$½x and so, *under repetition*, will gain \$¼x on average.

However, *un*like the original situation with *two* envelopes, **B**, reasoning like **A**, now expects to *lose* \$¼x, so there is no 'paradox'.

In the original situation of two envelopes, we now see that **A**'s reasoning is correct about the *values* of the random variable Y but wrong about their *probabilities*, despite EPS being used to give out the envelopes. In fact, the two values of Y have probabilities of 0 and 1, but A does not know which value has which probability. This illustrates the subtlety (and, hence, the danger) of the long-standing *wrong* idea that lack of knowledge (or 'ignorance') can be modelled by a *uniform* probability distribution. Making this mistake here leads to **A** and **B** treating as *one* case what is actually two cases which, incidentally, *do* involve uniform distributions as a consequence of the use of EPS.

∗ If the two amounts of money are \$½x and \$x so p.f. (2) at the right applies, **A** is equally likely to gain or lose, and **B** to lose or gain, \$½y by exchanging and there is no 'paradox'.

y	½y	y
$f(y)$	0.5	0.5

-----(2)

∗ If the two amounts of money are \$x and \$2x so p.f. (3) at the right applies, **A** is equally likely to gain or lose, and **B** to lose or gain, \$y by exchanging, again with no 'paradox'.

y	y	2y
$f(y)$	0.5	0.5

-----(3)

Both cases agree with our intuitive idea of no net gain (and no 'paradox') by exchanging envelopes.

NOTES: 1. The Ali-Baba paradox is sometimes called the *exchange paradox*.

2. It is misleading to call the matter first described a 'paradox' – it has more in common with an *argument by contradiction*, where we reason to an impossibility (here, **A** and **B** *both* expect to gain by exchanging envelopes) and so demonstrate there must be a *wrong* statement (here, the probabilities assigned) in the chain of reasoning.

3. The *probabilities* of the two cases modelled by p.f.s (2) and (3) at the right above do *not* appear to be involved in resolving the 'paradox', possibly with the proviso that neither can have *zero* probability; their probabilities depend on how the two sums of money are determined.
   - If the sums are set and then placed in the envelopes under EPS, the two cases are *equally* probable;
   - if a sum is placed in one envelope and a process, probabilistic or otherwise, used to decide on the sum (half or double) for the other envelope, the two cases may have *un*equal or *unknown* probabilities.

4. A more technical discussion of how to resolve the Ali-Baba Paradox is given by Christensen, R. and J. Utts: Bayesian Resolution of the "Exchange Paradox", *The American Statistician* **46**, No. 4, 274-276 (1992).

   Other attempts (of surprising variety) to resolve the 'paradox', together with some background and references, are given in the Statistical Society of Canada *Liaison* **12**, No. 1, 34-39 and No. 2, 20-25 (1998).
   - The concluding paragraph of the second of the latter two articles is:

     > More than probability theory is involved in disentangling these enigmas. The proper domain for these discussions is game theory, and questions of rationality and the representation of knowledge of rationality arise. A troubling question is whether rationality is equivalent to maximizing expected gain, particularly in cases like the Pareto example where the law of large numbers does not apply.

   Readers may find it is *not* necessarily the case that the more technical a discussion of how to resolve the Ali-Baba paradox, the greater its success.

(*continued overleaf*)

NOTES: (cont.) 5. The useful idea of identifying distinct *actual* cases [like the probability functions (2) and (3) overleaf near the middle of page HL46.1] in a *composite* situation [like that modelled by the probability function (1) overleaf] recurs on page HL49.3 in discussion of what is called the Three Prisoner's Dilemma in Statistical Highlight #49.

2. Postscript – What can we learn from studying paradoxes?

Interest in (so-called) paradoxes is long-standing – Zeno's paradoxes likely date from the fifth century BC. Studying paradoxes tends to be the purview of a subset of the credentialed members of a population; engagement of a wider audience may be limited by three characteristics, one or more of which are often associated with paradoxes:

- *un*realistic aspects of their context;
- subtleties in their reasoning;
- (sometimes unnecessary) technicalities in attempts to resolve them.

However, paradoxes like those discussed in Statistical Highlights #46 to #51 raise the matter of (hopefully sensible) decision-making in the presence of uncertainty; because people deal with uncertainty on a daily basis, good decision-making in such circumstances, within the limits of individual abilities, would be a useful skill for *everyone* to acquire.

In addition, the mental exercise of engaging with (seemingly) paradoxical issues may help foster a mind-set more capable of distinguishing valid intellectual argument from emotional commitment in matters as diverse as vaccine hesitancy and atheism.

More broadly, when resolving 'paradoxes' (or arguments by contradiction), we should seek the *simplest* resolution. In the case of the Ali-Baba paradox, if **A** had reasoned from the beginning that, if I have the envelope with $x, then **B** has $2x so I gain $x by exchanging; conversely, if I have the envelope with $2x, I *lose* $x by exchanging. I therefore have no incentive to exchange (and, of course, neither does **B**) and there is *no* 'paradox' (and little need for further discussion, however erudite).

In the same vein, in Zeno's paradox of Archilles chasing the tortoise and being able to run twice as fast, it is clear that Archilles *will* catch the tortoise, although it will take him longer the further behind the tortoise he starts. The 'paradox' (again, really an argument by contradiction) based on the idea that, for any point reached by Archilles, the tortoise will always have moved on and so will never be overtaken, is *clearly* wrong, so what is the *simplest* demonstration of the mistake in reasoning? It would be nice if the technical 'modern' recognition that an *infinite* series can have a *finite* sum is not needed. [We can also speculate on whether these matters bear on the **pleonastic fallacy** – the belief that an absolute *qualitative* difference can be overcome by a successive accumulation of extremely small and entirely relative *quantitative* steps.]

3. Commonalities and Differences

In addition to the 'paradox' theme, the overarching commonality of Statistical Highlights #46 to #51 is the involvement of probabilities/proportions, which are multiplied, added and (sometimes) divided to yield other proportions. However, we can distinguish four groups of these Highlights on the basis of differences in what is identified as the source of their particular 'paradox'.

- Highlight #46: misapplication of 'equiprobable';
- Highlights #47 and #48: comparisons from multivariate discrete distributions;
- Highlights #49 and #50: comparisons of conditional probabilities;
- Highlight #51: comparing proportions.

It is noteworthy that the multiplication of probabilities in Statistical Highlights #47 and #48 under a modelling assumption of probabilistic independence recurs in the weighted average 'explanation' of Simpson's Paradox in Table HL51.8 at the upper right of page HL51.4 in Statistical Highlight #51, although the eight **bold** proportions in this table are expressed as percentages. Multiplication of probabilities is also involved in Statistical Highlights #49 and #50 in the form of probabilistic *de*pendence of two events that is modelled by conditional probability – see equation (9) on the lower half of page HL50.2

It may be that the ultimate source of the 'paradoxes' of Statistical Highlights #47 to #51 lies is the *bi*variate nature of proportions (their numerator and denominator), together with under-appreciation of what may emerge when proportions are combined – for example, in models involving probabilistic independence or conditional probabilities – and then used to generate 'rankings'. This idea is discussed in Section 4 on pages HL51.3 and HL51.4 in Statistical Highlight #51.

A different context in which proportions again need careful treatment is discussed in Note 19 on page 2.122 in Figure 2.15 in the STAT 332 Course Materials. This Note includes a reminder of:

- the need to distinguish percent from percentage points – for instance, a change from 35 percent to 38 percent in a party's support among voters in a political poll is an increase of 3 *percentage points*, **not** 3 *percent* – see, for example, the middle column of the article EM9317, *Street drug use waning in Metro.*
- a ratio expressed (perhaps for greater-sounding impact) as ... *an increase of 500 per cent* would be clearer if described as ... *a 6-fold increase.*
  - It is also likely that such wording is often *mis*used and/or *mis*understood to mean a *5*-fold increase.

We also remember the *differing* magnitudes of compensatory percentages; for example, a *de*crease of 20% in a quantity (like the value of an investment) requires an *in*crease of 25% to restore its original value, to undo a decrease of one third (33⅓%) requires a subsequent increase of one half (50%) and a 50% decrease needs a 100% increase to make it good.