Characterizing Fractional Degree Stochastic Dominance by Invariance Laws

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Classic notions of stochastic dominance have integer degrees. Recent studies have imposed distinct preference conditions for refinement, resulting in a range of fractional degree stochastic dominance rules. However, preference conditions are generally not mutually exclusive, making it challenging to establish a strict criterion for rule selection. To address this, we establish fractional degree stochastic dominance rules based exclusively on invariance laws under a general condition that is applicable to all intermediate utility sets. This approach enables practitioners to rely solely on the mutual compatibility and exclusivity of invariance properties to compare and select the appropriate rules. We illustrate the usefulness of our approach through an application to the problem of mutual fund selection.

Key words: stochastic dominance, fractional degree, invariance laws, mutual fund selection.

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1. Introduction

Stochastic dominance, as a ranking rule, captures the consensus among clearly defined groups of decision makers when comparing random variables representing monetary outcomes. Due to its nonparametric nature, stochastic dominance has been widely used to assess changes in risk prospects in decision analysis (Bawa 1982, Levy 1992, Müller and Stoyan 2002, Liu and Neilson 2019). Historically, the concept of stochastic dominance was initially defined using integer degrees, primarily because it relies on integer-degree derivatives of utility functions to classify decision makers. Fishburn (1976) was the first to extend this concept to fractional degrees, using power-like test functions. Different from the approach of Fishburn (1976) focusing on mathematical forms of the test functions, Müller et al. (2017) and Huang et al. (2020) identified two economically intuitive methods to interpolate between integer-degree stochastic dominance. Both studies highlighted the importance of refining the degree to enhance the discriminatory power of stochastic dominance.

Clearly, there are multiple reasonable approaches to interpolating integer-degree stochastic dominance, and the specific rule for fractional degree stochastic dominance is not unique. The various fractional degree stochastic dominance rules proposed in the literature are typically based on different conditions imposed on preferences. However, distinct preference conditions are generally not mutually exclusive, as commonly used utility functions often satisfy several of these conditions simultaneously. For instance, Müller et al. (2017) add a constraint to the ratio of marginal utilities, while Huang et al. (2020) add a constraint on the degree of absolute risk aversion to interpolate between first- and second-degree stochastic dominance. However, many popular utility functions, such as smooth parametric functions or piecewise functions with concave kinks due to loss aversion, satisfy both Müller et al. and Huang et al.'s conditions (Huang et al. 2020, Section 2.3). This phenomenon makes it difficult to establish a definitive criterion for users to differentiate and select among these rules by comparing preference conditions alone.

In this paper, we take a different angle, developing fractional degree stochastic dominance rules from the perspective of operational invariance with respect to transformations. We only adopt a weak and general assumption about preferences, without imposing any substantive conditions on utility functions, and construct different fractional degree stochastic dominance rules based exclusively on invariance properties. The resulting rules each satisfy as many of these properties as possible, allowing practitioners to rely solely on the mutual compatibility and exclusivity of these properties to compare and select the appropriate rules. This approach helps overcome the difficulty of choice, especially when preference conditions are hard to compare.

In stochastic environments, transformations of random variables are ubiquitous in a wide variety of decision-making processes. Given the prevalence of these transformations, the applicability of an ordering rule can be greatly enhanced if its associated ranking is invariant with respect to the transformations deemed plausible in applications. This perspective was emphasized by Levy in his survey of early findings on stochastic dominance, where he wrote, “whether the stochastic dominance relationship is invariant to the transformation of the random variables [...] is quite important from a practical point of view” (Levy 1992, p.560). In the literature, invariance laws for transformations have been established for integer-degree stochastic dominance and applied to problems such as efficient portfolio construction (Levy and Sarnat 1971, Levy and Kroll 1978), the study of risky behavior (Eeckhoudt et al. 2009), and optimization with stochastic dominance constraints (Zhang and Homem-de Mello 2017). In this paper, we extend this line of inquiry by constructing fractional degree stochastic dominance rules based exclusively on these laws.

Motivated by economic considerations, we examine four invariance laws: translation invariance, scaling invariance, additive invariance, and multiplicative invariance. Since additive and multiplicative invariance encompass translation and scaling invariance, our primary focus is on the latter two. Our interest lies not only in understanding when a dominance relation remains dominant after the addition or multiplication of an independent random variable but also in exploring the conditions under which a non-dominance relation continues to be non-dominant under similar transformations. While the literature extensively addresses invariance properties for dominance relations, the study of these properties concerning non-dominance relations is only beginning to receive attention (e.g., Pomatto et al. 2020). As a non-dominance relation may become dominant after a transformation, overlooking the effects of transformations on non-dominance relations could lead to incomplete conclusions about their impact.

To define fractional degree stochastic dominance, an intermediate preference condition is needed to bridge the gap between the preference conditions for two adjacent integer degrees. In this paper, we introduce a generic interpolating function that specifies the tight lower bound on the ratios of marginal utilities. This preference condition is broadly applicable, covering all intermediate utility sets between those underlying adjacent integer-degree stochastic dominance, and it has a clear economic interpretation related to comparative risk aversion. We review the invariance laws applicable to integer-degree stochastic dominance, examine their mutual compatibility for fractional degree stochastic dominance, and identify the unique dominance rule that adheres to a set of mutually compatible laws. We find that additive and multiplicative invariance for dominance relations uniquely pin down a constant interpolating function, thereby characterizing the rule proposed by Müller et al. (2017). Additive invariance for dominance relations, and for non-dominance relations under the single-crossing condition, uniquely pin down an exponential interpolating function, thereby characterizing the rule proposed by Huang et al. (2020). Multiplicative invariance for dominance relations, and for non-dominance relations under the single-crossing condition, uniquely pin down a power interpolating function.

To make these theoretical results accessible to practitioners in applied decision analysis, we demonstrate how they can be used to guide the choice of an appropriate dominance rule. To adopt our approach, a practitioner only needs to agree that the tight lower bound on ratios of marginal utilities (which must be present in all rules) is continuous, and then evaluate which invariance properties are most acceptable, considering their mutual compatibility and exclusivity. If she accepts that the dominance relations determined by the rule are invariant with respect to both addition and multiplication of independent background risks, then the rule with a constant interpolating function is uniquely appropriate for her. If she values that both dominance relations, and non-dominance relations under the single-crossing condition, are maintained for additive (multiplicative) background risks, then the rule with an exponential (power) interpolating function best suits her preferences. Throughout this process, the practitioner does not need to rely on any specific information about her non-visible utility functions to compare different rules.

To illustrate how studied invariance properties facilitate the understanding and comparison of different dominance rules, we revisit the problem of mutual fund selection using stochastic dominance (Joy and Porter 1974, Dentcheva and Ruszczynski 2003). Suppose an analyst aims to identify superior funds in a financial market that stochastically dominate the market index at fractional degrees. In this context, it is crucial that the identification of superior funds remains invariant to common transformations, ensuring its applicability to a wide range of potential investors who may face unobservable background risks. This consideration necessitates examining the invariance properties of both dominance and non-dominance relations. However, it is important to note that no single rule can perfectly preserve both relations under the addition and multiplication of background risks. Analysts must, therefore, understand the specific implications of each rule: whether superior funds remain superior under these transformations and whether non-superior funds might become superior. By comparing all value mutual funds registered in the US that were active throughout the period from January 2014 to January 2024 with the Dow Jones Index, both with and without background risks, we find that a noticeable number of funds, initially not dominating the index in the absence of background risks, become dominant after the addition or multiplication of such risks under each rule. Moreover, the computational burden of identifying new dominant funds under these transformations can be significantly reduced by leveraging the specific invariance properties of dominance and non-dominance relations for each rule.

Typically, verifying the invariance properties of a fractional degree stochastic dominance rule with a given interpolating function is straightforward. The difficulty lies in clarifying the mutual compatibility and exclusivity of various invariance properties, as well as in excluding all interpolating functions that do not meet certain invariance requirements, thereby establishing the uniqueness of the interpolating function for compatible invariance laws. Our findings, in conjunction with previous studies, provide a more comprehensive foundation for fractional degree stochastic dominance. When information about the characteristics of preferences is sufficiently accessible, users can leverage insights from previous research to formulate a fractional degree stochastic dominance rule. Conversely, in situations where information on preferences is limited or users exhibit indifference towards varying conditions of preferences, our research provides a framework for deriving rules based on invariance properties. Overall, our study contributes a unique complementary perspective to the understanding of fractional degree stochastic dominance.

2. Preliminaries

In this section, we introduce the fundamental concepts and results that form the basis of our subsequent analysis. Let $\tilde{x}$, $\tilde{y}$, $\tilde{z}$, and so forth, denote random variables. We denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}_+$ the set $(0, \infty)$ of positive real numbers.

2.1. Transformations

Translation and scaling are perhaps the simplest transformations. Translation, denoted as

$$\tilde{x} \mapsto \tilde{x} + k,$$

shifts a random variable $\tilde{x}$ by a fixed amount $k \in \mathbb{R}$. Scaling, expressed as

$$\tilde{x} \mapsto \lambda \tilde{x},$$

multiplies the variable by a constant $\lambda \neq 0$. For economic studies, it is convenient to assume $\lambda > 0$ in scaling. Extending these concepts,

$$\tilde{x} \mapsto \tilde{x} + \tilde{z}$$

adds an independent random amount to a random variable, and

$$\tilde{x} \mapsto \tilde{z}\tilde{x}$$

multiplies the variable by an independent random factor. We typically assume the multiplier $\tilde{z}$ is supported on $\mathbb{R}_+$ when discussing multiplication.

Translation and scaling are commonly observed in practice. Translation, for instance, occurs when an individual's wealth experiences a constant increase or decrease, while scaling is seen when the proportion of a risky asset is adjusted. The addition and multiplication of independent random variables naturally arise in scenarios involving multiple sources of uncertainty. For instance, in portfolio choice, the fluctuating income from an alternative asset can be added to the return on a risky asset as an independent random variable (Kihlstrom et al. 1981, Gollier and Pratt 1996), while a price deflator or exchange rate with uncertainty can serve as an independent random multiplier when profits are nominal or denominated in foreign currency (Franke et al. 2006, 2011).

2.2. Invariance Laws

Let $\succeq$ denote a generic partial ordering on random variables. The invariance laws we are concerned with are formally stated as follows.

- *Translation Invariance for Dominance.* If $\tilde{x} \succeq \tilde{y}$ and $k \in \mathbb{R}$, then $\tilde{x} + k \succeq \tilde{y} + k$.
- *Translation Invariance for Non-Dominance.* If $\tilde{x} \not\succeq \tilde{y}$ and $k \in \mathbb{R}$, then $\tilde{x} + k \not\succeq \tilde{y} + k$.
- *Scaling Invariance for Dominance.* If $\tilde{x} \succeq \tilde{y}$ and $\lambda > 0$, then $\lambda\tilde{x} \succeq \lambda\tilde{y}$.
- *Scaling Invariance for Non-Dominance.* If $\tilde{x} \not\succeq \tilde{y}$ and $\lambda > 0$, then $\lambda\tilde{x} \not\succeq \lambda\tilde{y}$.
- *Additive Invariance for Dominance.* If $\tilde{x} \succeq \tilde{y}$ and $\tilde{z}$ is independent of $\tilde{x}$ and $\tilde{y}$, then $\tilde{x} + \tilde{z} \succeq \tilde{y} + \tilde{z}$.
- *Additive Invariance for Non-Dominance.* If $\tilde{x} \not\succeq \tilde{y}$ and $\tilde{z}$ is independent of $\tilde{x}$ and $\tilde{y}$, then $\tilde{x} + \tilde{z} \not\succeq \tilde{y} + \tilde{z}$.
- *Multiplicative Invariance for Dominance.* If $\tilde{x} \succeq \tilde{y}$ and $\tilde{z} > 0$ is independent of $\tilde{x}$ and $\tilde{y}$, then $\tilde{x}\tilde{z} \succeq \tilde{y}\tilde{z}$.
- *Multiplicative Invariance for Non-Dominance.* If $\tilde{x} \not\succeq \tilde{y}$ and $\tilde{z} \in \mathbb{R}_+$ is independent of $\tilde{x}$ and $\tilde{y}$, then $\tilde{x}\tilde{z} \not\succeq \tilde{y}\tilde{z}$.

Some invariance laws above, in particular scaling invariance and additive invariance for dominance relations, have been studied by Müller (1997). Later, we will consider stochastic dominance rules on the set $\mathcal{X}^{[a,b]}$ of random variables supported on a fixed bounded interval $[a, b]$. In this context, the application of invariance laws is restricted to $\mathcal{X}^{[a,b]}$, meaning that they apply only to relations where random variables, both before and after transformations (e.g., $\tilde{x}$, $\tilde{y}$, $\tilde{x} + \tilde{z}$ and $\tilde{y} + \tilde{z}$ for additive invariance), are elements of $\mathcal{X}^{[a,b]}$.

The reverse operations to translation and scaling, represented as $\tilde{x} + k \mapsto \tilde{x}$ and $\lambda\tilde{x} \mapsto \tilde{x}$, are essentially part of the same transformation as translation and scaling. Specifically, the former can be seen as translating $\tilde{x} + k$ by $-k$, and the latter as scaling $\lambda\tilde{x}$ by $1/\lambda$. This fact implies that for translation and scaling, invariance for dominance relations and non-dominance relations is equivalent. In contrast, the reverse operations for addition and multiplication of an independent random variable,

$$\tilde{x} + \tilde{z} \mapsto \tilde{x} \quad \text{and} \quad \tilde{z}\tilde{x} \mapsto \tilde{x},$$

cannot be reproduced by adding another independent variable to $\tilde{x} + \tilde{z}$ or multiplying another independent variable with $\tilde{z}\tilde{x}$. This distinction indicates that for addition and multiplication of independent random variables, invariance for dominance relations does not imply invariance for non-dominance relations, and vice versa.

Since additive and multiplicative invariance include translation and scaling invariance as special cases when the added or multiplied random variable is a constant, we will henceforth focus mainly on additive and multiplicative invariance, for both dominance and non-dominance relations.

2.3. Integer-Degree Stochastic Dominance

We consider stochastic dominance rules on a bounded interval $[a,b] \subset \mathbb{R}$. The assumption of a bounded support is advantageous for two reasons. First, it ensures that expected utilities remain finite, thus avoiding the issue of them approaching infinity, as seen in the St. Petersburg paradox. Second, it allows for a clear evaluation of the effects of higher-degree derivatives on expected utility, while keeping lower-degree effects separable. These points, along with the fact that real-world random variables can only assume finite values, justify the widespread use of bounded supports in studies of higher-degree stochastic dominance (Ekern 1980, Eeckhoudt et al. 2009).

Assume that von Neumann-Morgenstern utility functions are defined on $[a,b]$ and are differentiable up to the desired order. Also, all transformations under consideration produce supports that are contained in $[a,b]$. This means that all random variables under transformation, whether standalone, like $\tilde{x}$, or after transformations such as $\tilde{x}+\tilde{z}$ or $\tilde{x}\tilde{z}$, remain supported within $[a,b]$. While this excludes transformations that extend the supports beyond $[a,b]$, it allows us to define stochastic dominance relations and study invariance properties on the same interval. In short, all stochastic dominance rules are binary relations on $\mathcal{X}^{[a,b]}$, and all transformations are constrained to $\mathcal{X}^{[a,b]}$.

Given $N \in \mathbb{N}$ and $n = 1, \ldots, N$, we use $u^{(n)}$ to denote the nth-degree derivative of u. Here and throughout, when $u^{(n)}$ appears, it implies that u is nth-degree differentiable, with the boundary values $u^{(n)}(a)$ and $u^{(n)}(b)$ representing the corresponding right and left derivatives. Let

$$U_N = \left\{ u: \ (-1)^{n+1} u^{(n)}(x) \geq 0 \text{ for all } x \in [a,b],\ n = 1, \ldots, N \right\},$$

that is, the set of utility functions whose odd derivatives are nonnegative and even derivatives are nonpositive up to order N. The property of alternating signs in successive derivatives is called *mixed risk aversion*, indicating that utility functions with such property can be expressed as mixtures of exponential utilities (Caballé and Pomansky 1996). Let $\mathbb{E}$ be the expectation operator.

Definition 1. Given $N \in \mathbb{N}$, $\tilde{x}$ dominates $\tilde{y}$ by *Nth-degree stochastic dominance*, written as $\tilde{x} \succeq_N \tilde{y}$, if $\mathbb{E}u(\tilde{x}) \geq \mathbb{E}u(\tilde{y})$ for all $u \in U_N$.

The Nth-degree stochastic dominance formalizes the consensus of all utility functions within U_N in comparing risk prospects. Note that whether $\tilde{x} \succeq_N \tilde{y}$ depends not only on $\tilde{x}$, $\tilde{y}$ but also on the choice of $[a,b]$ for $N \geq 4$ (see Lemma 1 below for the case $N \leq 3$), but we omit $[a,b]$ as is commonly done in the literature (Eeckhoudt et al. 2009). The cases of $N=1$ and $N=2$ correspond to the widely used first-degree and second-degree stochastic dominance, which respectively characterize the consensus of all nondecreasing utility functions and all nondecreasing concave utility functions (Rothschild and Stiglitz 1970).

For a generic cumulative distribution function (cdf) H, write $H^{(0)}(x) = H(x)$ and define the *nth-degree integral of H*, $H^{(n)}(x) = \int_a^x H^{(n-1)}(t)dt$ $(n \geq 1)$, iteratively. When talking about distribution conditions, we will always use F and G to denote the cdfs of $\tilde{x}$ and $\tilde{y}$ throughout the paper. It is a well-established result (Eeckhoudt et al. 2009) that $\tilde{x} \succeq_N \tilde{y}$ if and only if

$$G^{(n)}(b) \geq F^{(n)}(b), \quad \text{for } n = 0, 1, \ldots, N-2, \quad \text{and} \tag{1}$$

$$G^{(N-1)}(x) \geq F^{(N-1)}(x) \quad \text{for all } x \in [a,b]. \tag{2}$$

While conditions (1) and (2) depend heavily on $[a,b]$ in general, the specific choice of this interval is irrelevant for determining whether $\tilde{x} \succeq_N \tilde{y}$ when $N \leq 3$.

Lemma 1. *For $N = 1, 2, 3$ and random variables $\tilde{x}$ and $\tilde{y}$ supported on both $[a,b]$ and $[c,d]$, the following statements are equivalent:*

(i) Conditions (1) *and* (2) *hold for $[a,b]$;*

(ii) Conditions (1) *and* (2) *hold for $[c,d]$, that is, $G^{(n)}(d) \geq F^{(n)}(d)$ for $n = 0, 1, \ldots, N-2$ and $G^{(N-1)}(x) \geq F^{(N-1)}(x)$ for all $x \in [c,d]$.*

In other words, evaluating conditions (1) and (2) using any interval covering the supports of $\tilde{x}$ and $\tilde{y}$ consistently yields the same conclusion regarding whether $\tilde{x} \succeq_N \tilde{y}$ for $N \leq 3$. All omitted proofs are provided in the Appendix.

The fact that the dominance relation $\tilde{x} \succeq_N \tilde{y}$ is translation and scaling invariant has been pointed out by Hadar and Russell (1971). By the law of iterated expectations, $\tilde{x} \succeq_N \tilde{y}$ is also additive and multiplicative invariant. The question whether the non-dominance relation $\tilde{x} \not\succeq_N \tilde{y}$ is additive or multiplicative invariant is not as trivial as it might seem at first sight. Pomatto et al. (2020) investigate the conditions under which two random variables, say $\tilde{x}$ and $\tilde{y}$, can be ranked in terms of $\succeq_1$ or $\succeq_2$ after the addition of a suitable independent random variable $\tilde{z}$. The conditions are found to be very weak, so $\tilde{x}$ and $\tilde{y}$ need not be ranked ex ante. For completeness, we provide explicit counterexamples to additive and multiplicative invariance for the non-dominance relations $\tilde{x} \not\succeq_1 \tilde{y}$ and $\tilde{x} \not\succeq_2 \tilde{y}$, respectively. Denote by $\tilde{x} \stackrel{d}{=} [x_1; \ldots; x_n]$ the random variable $\tilde{x}$ that takes on the values x_i $(i = 1, \ldots, n)$ with an equal probability of $1/n$. Thanks to Lemma 1, we will consistently use the minimal interval $[a, b]$ to cover the relevant random variables when discussing stochastic dominance with degrees up to three, without further specifying the interval explicitly.

Example 1. Let $\tilde{x} \stackrel{d}{=} [1; 2; 5]$, $\tilde{y} \stackrel{d}{=} [0; 3; 4]$ and $\tilde{z} \stackrel{d}{=} [0; 2]$. It follows that $\tilde{x} + \tilde{z} \stackrel{d}{=} [1; 2; 3; 4; 5; 7]$ and $\tilde{y} + \tilde{z} \stackrel{d}{=} [0; 2; 3; 4; 5; 6]$. It is standard to verify that $\tilde{x} \not\succeq_1 \tilde{y}$ but $\tilde{x} + \tilde{z} \succeq_1 \tilde{y} + \tilde{z}$. Let $\tilde{x}' = e^{\tilde{x}}$, $\tilde{y}' = e^{\tilde{y}}$ and $\tilde{z}' = e^{\tilde{z}}$. Then, $\tilde{x}' \not\succeq_1 \tilde{y}'$ but $\tilde{x}'\tilde{z}' \succeq_1 \tilde{y}'\tilde{z}'$.

Example 2. Let $\tilde{x} \stackrel{d}{=} [1; 2; 7; 8]$, $\tilde{y} \stackrel{d}{=} [0; 4; 4; 10]$ and $\tilde{z} \stackrel{d}{=} [0; 2]$. It follows that $\mathbb{E}[\tilde{x}] = \mathbb{E}[\tilde{y}]$, $\tilde{x} + \tilde{z} \stackrel{d}{=} [1; 2; 3; 4; 7; 8; 9; 10]$ and $\tilde{y} + \tilde{z} \stackrel{d}{=} [0; 2; 4; 4; 6; 6; 10; 12]$. It is standard to verify that $\tilde{x} \not\succeq_2 \tilde{y}$ but $\tilde{x} + \tilde{z} \succeq_2 \tilde{y} + \tilde{z}$. Let $\tilde{z}' \stackrel{d}{=} [1; 2]$. Then $\tilde{x} \not\succeq_2 \tilde{y}$ but $\tilde{x}\tilde{z}' \succeq_2 \tilde{y}\tilde{z}'$.

The intuition behind Examples 1 and 2 is that by introducing a new random source, the distributions being compared become more diffuse, thereby eliminating local violations of stochastic dominance present in the original distributions. Crucially, the original cdfs in these examples do not satisfy the single-crossing condition. This fact disproves the general applicability of additive and multiplicative invariance for $\tilde{x} \not\succeq_N \tilde{y}$. However, these invariance properties do hold in the special but important case where cdfs are single-crossing.

Definition 2. Given $n \in \{0\} \cup \mathbb{N}$, two cdfs F and G are called *nth-degree single-crossing* if there exists $z \in [a,b]$ such that $F^{(n)}(x) \leq (\geq) G^{(n)}(x)$ on $x \leq z$ and $F^{(n)}(x) \geq (\leq) G^{(n)}(x)$ on $x \geq z$.

The 0th-degree single-crossing condition holds under several circumstances, such as: i) F or G is degenerate; ii) $F(x) \geq G(x)$ or $F(x) \leq G(x)$ for all $x \in [a,b]$; iii) F (or G) is a mean-preserving transfer of F (or G) (Rothschild and Stiglitz 1970), where probability mass is symmetrically shifted from one point to two others, leaving the mean unchanged; iv) F and G belong to the same location-scale family, such that $F(x) = H((x-\mu_F)/\sigma_F)$ and $G(x) = H((x-\mu_G)/\sigma_G)$ $(\sigma_F \neq \sigma_G)$ for some cdf function H, with the crossing point given by $x = (\mu_G \sigma_F - \mu_F \sigma_G)/(\sigma_F - \sigma_G)$ (Müller et al. 2017). Since a lower-degree single-crossing condition implies a higher-degree one, all these cases satisfy the n-th degree single-crossing condition for all $n \geq 0$. A typical instance where the 1st-degree condition holds without satisfying the 0th-degree condition involves F (or G) being a mean-variance-preserving transformation of F (or G), which includes a mean-preserving spread followed by a mean-preserving contraction to capture increased downside risks (Menezes et al. 1980). Generally, higher-degree conditions are easier to satisfy, as the additional integration smooths out fluctuations in $F^{(n)}$ and $G^{(n)}$.

Theorem 1. *For all $N \in \mathbb{N}$ and all random variables $\tilde{x}, \tilde{y} \in \mathcal{X}^{[a,b]}$, the dominance relation $\tilde{x} \succeq_N \tilde{y}$ always obeys additive and multiplicative invariance. The non-dominance relation $\tilde{x} \not\succeq_N \tilde{y}$ also obeys additive and multiplicative invariance when the cdfs of $\tilde{x}$ and $\tilde{y}$ are $(N-1)$th-degree single-crossing and satisfy* (1).

As explained above, the first statement of Theorem 1 is known in the literature. Imposing condition (1) in the second statement enables us to focus on how an independent variable influences condition (2) when examining the invariance properties of non-dominance relations. Condition (1) automatically holds for $N=1$, equates to $\mathbb{E}\tilde{x} \geq \mathbb{E}\tilde{y}$ for $N=2$, and simplifies to comparing moments of distributions for $N \geq 3$, making it straightforward to verify.

3. Fractional Degree Stochastic Dominance

In this section, we present a general formulation of fractional degree stochastic dominance and explain why it is well-suited to our goal of determining specific rules based on invariance properties.

3.1. Formulation

In analogy to the fact that Nth-degree stochastic dominance represents the consensus of utility set U_N, a fractional degree stochastic dominance rule captures the consensus of an intermediate utility set situated between U_N and U_{N+1}. To define fractional degree stochastic dominance, we need to specify this intermediate utility set. The sole additional restriction imposed by U_{N+1} relative to U_N is the decreasing monotonicity of $(-1)^{N+1}u^{(N)}$. Therefore, a natural preference condition for characterizing intermediate utility sets involves controlling the increasing rate of $(-1)^{N+1}u^{(N)}$.

Let us keep in mind that our starting set is U_N, where all utility functions satisfy $(-1)^{N+1}u^{(N)} \geq 0$ on $[a,b]$. For u with $(-1)^{N+1}u^{(N)}(y) > 0$, the ratio $u^{(N)}(x)/u^{(N)}(y)$ for $x < y$ on $[a,b]$ is smaller if $(-1)^{N+1}u^{(N)}$ increases more rapidly on $[x,y]$. Consequently, an upper limit on the increasing rate of $(-1)^{N+1}u^{(N)}$ can be established by setting a lower bound on the ratio $u^{(N)}(x)/u^{(N)}(y)$, using a generic function $\phi : \{(x,y) \in [a,b]^2 : x < y\} \to [0,1]$, that is

$$\frac{u^{(N)}(x)}{u^{(N)}(y)} \geq \phi(x,y) \quad \text{for all } x < y \text{ in } [a,b] \text{ with } (-1)^{N+1}u^{(N)}(y) > 0. \tag{3}$$

This condition can alternatively be expressed as $0 \leq \phi(x,y)(-1)^{(N+1)}u^{(N)}(y) \leq (-1)^{N+1}u^{(N)}(x)$ for all $x < y$ on $[a,b]$ and all $u \in U_N$. Building on this procedure, for a given $N \in \mathbb{N}$ and $\phi(x,y) \in [0,1]$, we define the intermediate utility set as

$$U_{N,\phi} = \{u \in U_N : \ u \text{ satisfies condition (3)}\}. \tag{4}$$

Obviously, $U_{N,\phi}$ coincides with U_N when $\phi \equiv 0$, and includes U_{N+1} as a subset when $\phi \equiv 1$. Moreover, $U_{N,1}$ and U_{N+1} generate the same $(N+1)$th-degree stochastic dominance rule. The function ϕ is referred to as an *interpolating function*.

Different specifications of ϕ can result in the same utility set. For instance, if $\phi(x,y) = \mathbb{1}_{\{y-x<\delta\}}$ for some $\delta > 0$, then $U_{N,\phi}$ always coincides with $U_{N,1}$, regardless of the value of δ. To uniquely associate each set $U_{N,\phi}$ with a specific ϕ, we assume that ϕ is the maximal interpolator, in the sense that any $\tilde{\phi} \geq \phi$ with strict inequality somewhere would lead to a strictly smaller set than $U_{N,\phi}$ (i.e., $U_{N,\tilde{\phi}} \subseteq U_{N,\phi}$ and there exists $u \in U_{N,\phi}$ but $u \notin U_{N,\tilde{\phi}}$). The maximal interpolator corresponds to the infimum of the ratio $u^{(N)}(x)/u^{(N)}(y)$ for utility functions with $u^{(N)}(y) \neq 0$ in the utility set.

Lemma 2. *For each $U_{N,\phi}$ in (4), the maximal interpolator ϕ^* exists and is uniquely given by*

$$\phi^*(x,y)=\inf\left\{u^{(N)}(x)/u^{(N)}(y):\ u\in U_{N,\phi}\ \text{and}\ u^{(N)}(y)\neq 0\right\}.$$

Moreover, it satisfies the multiplicative triangular inequality:

$$\phi^*(x,z)\geq\phi^*(x,y)\phi^*(y,z)\quad\text{for any } x<y<z \text{ in } [a,b].$$

It is not hard to verify that $\phi\equiv 0$ and $\phi\equiv 1$ are the maximal interpolators of U_N and U_{N+1}. Henceforth, we will always work with maximal interpolators without further mention, and assume they are *continuous* in the triangular region $x<y$ with $x,y\in[a,b]$ for both intuitive sensibility and analytical tractability.

Definition 3. Let $U_{N,\phi}$ be as in (4). We say that $\tilde{x}$ (N,ϕ)-dominates $\tilde{y}$, written as $\tilde{x}\succeq_{N,\phi}\tilde{y}$, if $\mathbb{E}u(\tilde{x})\geq\mathbb{E}u(\tilde{y})$ for all $u\in U_{N,\phi}$.

As an interpolation between $\succeq_N$ and $\succeq_{N+1}$, $\succeq_{N,\phi}$ simplifies to $\succeq_N$ when $\phi\equiv 0$, to $\succeq_{N+1}$ when $\phi\equiv 1$, and falls between $\succeq_N$ and $\succeq_{N+1}$ for $0\leq\phi\leq 1$. This makes $\succeq_{N,\phi}$ a representation of a stochastic dominance rule with a degree between N and $N+1$. Moreover, (N,ϕ)-dominance shares the hierarchy property of integer-degree stochastic dominance: lower-degree dominance is always stronger than higher-degree dominance. In particular, $\tilde{x}\succeq_{N,\phi_1}\tilde{y}$ always implies $\tilde{x}\succeq_{N,\phi_2}\tilde{y}$ whenever $\phi_1\leq\phi_2$ everywhere.

The (N,ϕ)-dominance framework encompasses the rules of fractional degree stochastic dominance developed in recent literature as special cases. Müller et al. (2017) study the case with $N=1$ and $\phi(x,y)=\gamma\in[0,1]$. Bi and Zhu (2019) consider $\phi(x,y)=\gamma\in[0,1]$ for all $N\in\mathbb{N}$. Huang et al. (2020) require $(-1)^{N+1}u^{(N)}(x)e^{-(1/c-1)x}$ $(c\in(0,1])$ to be nonnegative and decreasing in x, which is equivalent to (3) with $\phi(x,y)=e^{(1/c-1)(x-y)}$ $(c\in(0,1])$.

3.2. Justification

There are three reasons why our formulation of fractional degree stochastic dominance, grounded in condition (3), is well-suited for determining the specific rule based on invariance properties.

First, condition (3) is broadly applicable to any intermediate utility set situated between U_N and U_{N+1}. For a utility set V with $U_{N+1} \subseteq V \subseteq U_N$, consider

$$\phi_V(x,y) = \inf\left\{u^{(N)}(x)/u^{(N)}(y) : \ u \in V \text{ and } u^{(N)}(y) \neq 0\right\}$$

for $x < y$ on $[a,b]$. We claim that ϕ_V takes values in $[0,1]$. Indeed, $U_{N+1} \subseteq V \subseteq U_N$ implies $(-1)^{(N+1)} u^{(N)} \geq 0$ for any $u \in V$, ensuring $\phi_V(x,y) \geq 0$. Additionally, the set $\{u \in V : u^{(N)}(y) \neq 0\}$ over which the infimum is taken is nonempty, because $U_{N+1} \subseteq V$ contains infinitely many utility functions with $u^{(N)}(y) \neq 0$. Since V is larger than U_{N+1}, we also conclude that $\phi_V(x,y) \leq 1$, as the infimum is bounded above by $\inf\{u^{(N)}(x)/u^{(N)}(y) : u \in U_{N+1} \text{ and } u^{(N)}(y) \neq 0\} = 1$. With this function ϕ_V, it holds

$$\frac{u^{(N)}(x)}{u^{(N)}(y)} \geq \phi_V(x,y) \quad \text{for all } x < y \text{ on } [a,b] \text{ and all } u \in V \text{ with } (-1)^{N+1} u^{(N)}(y) > 0,$$

which exactly matches (3), making (3) a necessary and binding condition for any intermediate utility set. Thus, condition (3) itself is innocuous. The only extra assumption we impose is that $\phi_V(x,y)$ is continuous, which is intuitively sensible and sufficiently weak.

Second, condition (3) does not impose any parametric structure or functional form on utility functions, aligning with our goal of deriving dominance rules solely from invariance properties. Specifying a particular functional form would require additional theoretical support from related fields, which would undermine our objective of basing the rules purely on invariance laws. Starting with a general condition like (3) showcases our strength in obtaining the rules without specific assumptions about utility functions.

Third, condition (3) has a clear economic interpretation related to comparative risk aversion. In the expected utility framework, risk attitudes are reflected in the changes in marginal utilities with respect to wealth. Pratt (1964, Theorem 1) introduced the index of absolute risk aversion, $-u^{(2)}/u^{(1)}$, to quantify an individual's willingness to take risks. He demonstrated that, given $u_1^{(1)} > 0$ and $u_2^{(1)} > 0$, u_2 is more risk averse than u_1, i.e., $-u_2^{(2)}(x)/u_2^{(1)}(x) \geq -u_1^{(2)}(x)/u_1^{(1)}(x)$ for all x, if and only if

$$\frac{u_2^{(1)}(x)}{u_2^{(1)}(y)} \geq \frac{u_1^{(1)}(x)}{u_1^{(1)}(y)} \quad \text{for all } x < y.$$

To assess the motive of precautionary saving, Kimball (1990) introduced the index of absolute prudence, $-u^{(3)}/u^{(2)}$, which extends the index of absolute risk aversion by shifting up a derivative. This approach naturally extends to higher orders, with the Nth-order index of absolute risk aversion, $-u^{(N)}/u^{(N-1)}$, being crucial in various economic applications (Denuit and Eeckhoudt 2010). For example, Eeckhoudt and Schlesinger (2008) show that whether a decision maker exhibits $-u^{(2)}(x)/u^{(1)}(x) \geq 1/x$ and $-u^{(3)}(x)/u^{(2)}(x) \geq 2/x$ for $x > 0$ and so on determines how that persons savings decisions will respond to increases in different orders of interest rate risk. Building on Pratt's analysis, it is straightforward to show that, under $(-1)^{N+1}u_1^{(N)} > 0$ and $(-1)^{(N+1)}u_2^{(N)} > 0$, u_2 is more risk averse than u_1 in terms of the $(N+1)$th-order index of absolute risk aversion, i.e., $-u_2^{(N+1)}(x)/u_2^{(N)}(x) \geq -u_1^{(N+1)}(x)/u_1^{(N)}(x)$ for all x, if and only if

$$\frac{u_2^{(N)}(x)}{u_2^{(N)}(y)} \geq \frac{u_1^{(N)}(x)}{u_1^{(N)}(y)} \quad \text{for all } x < y.$$

From this perspective, condition (3) effectively specifies a minimal level of risk aversion that a utility set must accommodate. Consequently, $U_{N,\phi}$ possesses the sensible property of being closed under the operation of reinforcing risk aversion.

LEMMA 3. *For $u_1, u_2 \in U_N$, if u_2 is more risk averse than u_1 according to the $(N+1)$th-order index of absolute risk aversion, then $u_1 \in U_{N,\phi}$ implies $u_2 \in U_{N,\phi}$.*

In the literature, an alternative approach to defining fractional degree stochastic dominance was proposed by Fishburn (1976). This method focuses on random variables supported on a bounded interval $[0, b]$ and constructs the rule of degree α ($\alpha \geq 1$) based on the utility set

$$F_\alpha = \overline{\text{cone}}\left\{u(x) = -(t-x)_+^{\alpha-1} : \ t \in [0, b]\right\}.$$

Compared to our approach, Fishburn employs a more specific specification of the intermediate utility set, which does not have a behavioral foundation, because utility functions in F_α are not normatively appealing. This rule does not belong to the class of $\succeq_{N,\phi}$ in our formulation, because it is not closed under the operation of reinforcing risk aversion, as illustrated in the following example.

Example 3. Let $b=1$ and $u_1(x) = -(1-x)_+^{1.5} \in F_{2.5}$. Then, $u_2(x) = -(1-x)_+^{1.8} + x$ satisfies $u_2^{(1)} \geq 0$, $u_2^{(2)} \leq 0$ and is more risk averse than u_1 according to the third-order index of absolute risk aversion. However, $u_2 \notin F_{2.5}$ because $u_2'(b) = 1 \neq 0$.

The lack of a clear connection with comparative risk attitudes makes the stochastic dominance rule generated by F_α difficult to interpret from a behavioral perspective. Nevertheless, Fishburn (1976)'s rule remains valuable in settings that prioritize analytical tractability, because its cone-based representation provides a computationally friendly extension of integer-degree dominance, even though it lacks closure under reinforcement of risk aversion. Our stochastic dominance rules based on interpolating functions serve as an economically interpretable complement to the rule of Fishburn (1976).

4. Determining Interpolating Functions with Invariance Laws

We determine the functional form of the interpolating function $\phi(x,y)$ with the goal of ensuring that fractional degree stochastic dominance inherits as many invariance properties as possible from integer-degree stochastic dominance. To achieve this, we first examine the inclusion relations among various invariance laws and then identify $\phi(x,y)$ based on mutually compatible laws. For simplicity, we assume $\phi(x,y) > 0$.

The seven equivalence relations, presented in Figure 1, serve as the building blocks for the main theorem (Theorem 2). They are explained and discussed separately in Results 1 to 7 below.

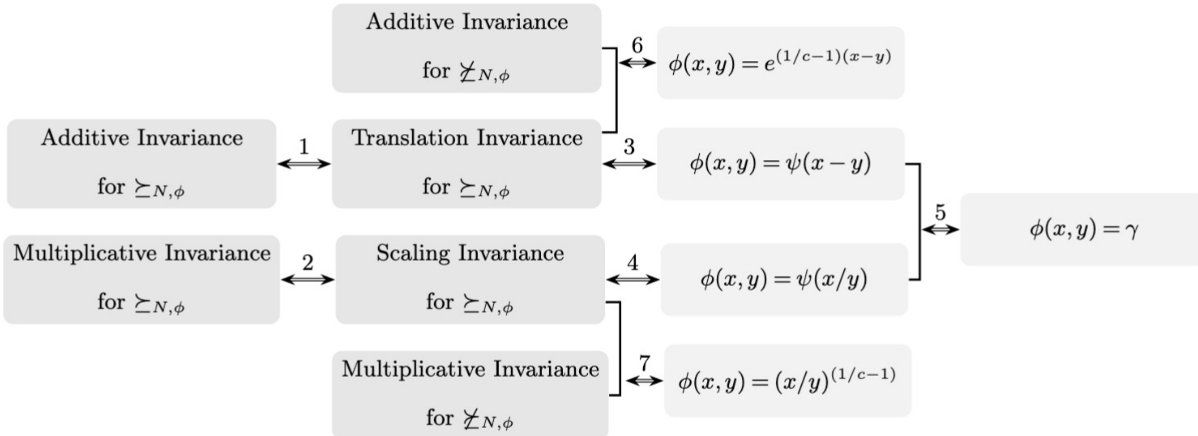
Result 1. For $\succeq_{N,\phi}$, additive invariance and translation invariance are equivalent.

Result 2. For $\succeq_{N,\phi}$, multiplicative invariance and scaling invariance are equivalent.

In Results 1 and 2, it is evident that additive and multiplicative invariance encompass translation and scaling invariance as special cases. The fact that translation and scaling invariance also imply additive and multiplicative invariance for dominance relations is a consequence of the law of iterated expectations. This equivalence extends to any stochastic orders generated by comparing $\mathbb{E}u(\tilde{x})$ and $\mathbb{E}u(\tilde{y})$ over a class of functions u, known as integral stochastic orders (Müller 1997).

Result 3. For $\succeq_{N,\phi}$, translation invariance holds if and only if $\phi(x,y)$ takes the form of $\psi(x-y)$.

Figure 1 Invariance properties and corresponding ϕ.



Notes: The equivalence relations denoted by $\Leftrightarrow$ are numbered consecutively in the order of their appearance in the main text, where invariance for non-dominance relations is considered under the single-crossing condition.

Result 4. For $\succeq_{N,\phi}$, scaling invariance holds if and only if $\phi(x,y)$ takes the form of $\psi(x/y)$.

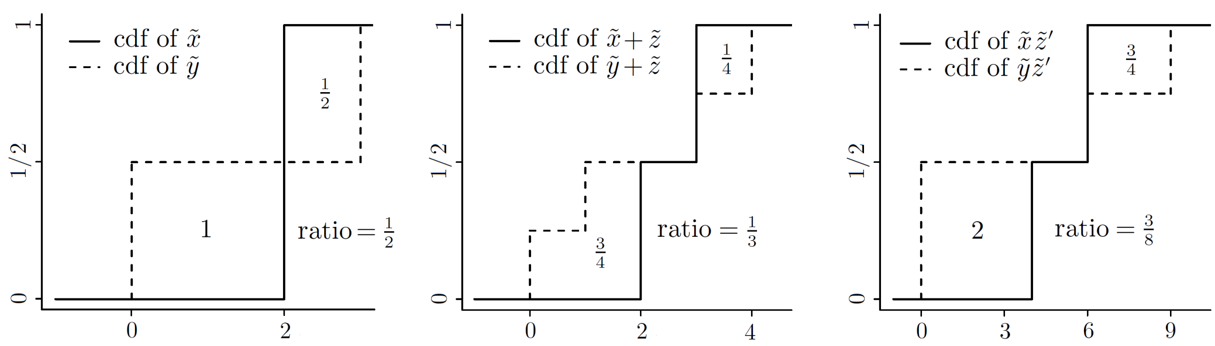
Results 3 and 4 confirm the intuition that if $\phi(x,y)$ is translation or scaling invariant, then $\tilde{x} \succeq_{N,\phi} \tilde{y}$ also obeys translation or scaling invariance, and the converse holds as well. Formally, $\phi(x,y)$ is *translation invariant* if $\phi(x,y) = \phi(x+k, y+k)$ for all $x < y$ on $[a,b]$ and all $k \in [a-x, b-y]$. This condition is satisfied if and only if $\phi(x,y) = \psi(x-y)$ for $x < y$ on $[a,b]$ for some function ψ. Similarly, $\phi(x,y)$ is *scaling invariant* if $\phi(x,y) = \phi(\lambda x, \lambda y)$ for all $x < y$ on $[a,b]$ and all $\lambda > 0$ such that $\lambda x, \lambda y \in [a,b]$. This holds if and only if $\phi(x,y) = \psi(x/y)$ for $x < y$ on $[a,b]$ when $y \neq 0$, or $\phi(x,y) = \psi(y/x)$ for $x < y$ on $[a,b]$ when $x \neq 0$. As a related result, Theorem 4.2 of Müller (1997) characterizes translation invariance and scaling invariance using properties of the set of utility functions underlying the stochastic dominance rule, which is different from our characterization based on the maximal interpolator.

Result 5. For $\succeq_{N,\phi}$, both translation invariance and scaling invariance hold, if and only if $\phi(x,y) \equiv \gamma$, where $\gamma \in (0,1]$ is a constant.

Result 5 follows directly from Results 3 and 4, since the constant function is the unique function invariant with respect to both translation and scaling.

We now turn our attention to invariance laws for non-dominance relations. Under the assumption that the random variables being compared have single-crossing cdfs, Theorem 1 establishes that

Figure 2 The cdfs of $\tilde{x}$, $\tilde{y}$, $\tilde{x}+\tilde{z}$, $\tilde{y}+\tilde{z}$, $\tilde{x}\tilde{z}'$ and $\tilde{y}\tilde{z}'$.



Notes: Dominance according to $\succeq_{1,\gamma}$ with $\gamma = 3/8$ holds if and only if the ratio of the area where the cdf of the dominating random variable lies above the cdf of the dominated variable to the area on the opposite side is no greater than 3/8. Based on this criterion, $\tilde{x} \not\succeq_{1,\gamma} \tilde{y}$, but $\tilde{x}+\tilde{z} \succeq_{1,\gamma} \tilde{y}+\tilde{z}$ and $\tilde{x}\tilde{z}' \succeq_{1,\gamma} \tilde{y}\tilde{z}'$.

these invariance properties hold for $\tilde{x} \not\succeq_{N,\phi} \tilde{y}$ when $\phi \equiv 1$. Somewhat surprisingly, however, when $\phi \equiv \gamma \in (0,1)$, neither additive nor multiplicative invariance remains valid for non-dominance relations, even in the context of single-crossing cdfs. The following example demonstrates that $\tilde{x} \not\succeq_{1,\gamma} \tilde{y}$ with $\gamma \in (0,1)$ fails to preserve these invariance properties for single-crossing cdfs.

Example 4. Let $[a,b] = [0,10]$, $\gamma = 3/8$, $\tilde{x} \stackrel{d}{=} 2$ and $\tilde{y} \stackrel{d}{=} [0;3]$. The cdfs of $\tilde{x}$ and $\tilde{y}$ are single-crossing. Now, let $\tilde{z} \stackrel{d}{=} [0;1]$ and $\tilde{z}' \stackrel{d}{=} [2;3]$ be independent of $\tilde{x}$ and $\tilde{y}$. Using the integral condition, it can be verified that $\tilde{x} \not\succeq_{1,\gamma} \tilde{y}$ but $\tilde{x}+\tilde{z} \succeq_{1,\gamma} \tilde{y}+\tilde{z}$ and $\tilde{x}\tilde{z}' \succeq_{1,\gamma} \tilde{y}\tilde{z}'$. Figure 2 provides a graphical illustration of the dominance relations involved.

Taken together, Result 5 and Example 4 present a key impossibility result: It is impossible to find a stochastic dominance rule within the class $\succeq_{N,\phi}$ that lies strictly between $\succeq_N$ and $\succeq_{N+1}$ while satisfying both additive and multiplicative invariance for dominance relations, as well as for non-dominance relations with single-crossing cdfs. If we wish to preserve additive and multiplicative invariance for non-dominance relations in fractional degree stochastic dominance, we must forgo either additive invariance or multiplicative invariance for dominance relations.

To proceed, we consider additive invariance for both dominance and non-dominance relations in combination, as well as multiplicative invariance for both types of relations in combination. For simplicity, when considering multiplicative invariance, we always assume $[a,b] \subset \mathbb{R}_+ = (0,\infty)$ from this point onward.

Result 6. Additive invariance holds for $\succeq_{N,\phi}$, and conditionally holds for $\not\succeq_{N,\phi}$ provided that the cdfs of $\tilde{x}$ and $\tilde{y}$ are $(N-1)$th-degree single-crossing and satisfy (1), if and only if $\phi(x,y) = e^{(1/c-1)(x-y)}$, where $c \in (0,1]$ is a constant.

Result 7. For positive random variables, multiplicative invariance holds for $\succeq_{N,\phi}$, and conditionally holds for $\not\succeq_{N,\phi}$ provided that the cdfs of $\tilde{x}$ and $\tilde{y}$ are $(N-1)$th-degree single-crossing and satisfy (1), if and only if $\phi(x,y) = (x/y)^{(1/c-1)}$, where $c \in (0,1]$ is a constant.

Results 6 and 7 identify the unique functional form of $\phi > 0$ that allows additive invariance and multiplicative invariance to hold for both dominance and non-dominance relations, respectively. The parameter c in ϕ is intentionally restricted to $(0,1]$ to represent the fractional degree. The "if" part of the theorem can be verified using the equivalent distribution conditions (see below). The novel contribution of the theorem is the "only if" part, which is also the most challenging. To provide intuition, consider a local equation $\phi(x,y) = e^{\beta_{x,y}(x-y)}$, which can be introduced at every feasible point (x,y), where $\beta_{x,y}$ is determined point by point. Under additive invariance for both dominance and non-dominance relations, we demonstrate that by adding and subtracting a suitably chosen random variable, the dominance relation implied by the local equation holds globally under $\succeq_{N,\phi}$. Consequently, $\beta_{x,y}$ must be constant across all (x,y), leading to the exponential form of ϕ. A similar approach establishes the result for multiplicative invariance.

After these intermediate steps, we obtain the main theorem below.

Theorem 2 **(Main Theorem)**. *Given that $N \in \mathbb{N}$ and $\phi > 0$ is continuous, the following results hold for $\succeq_{N,\phi}$ blue on $\mathcal{X}^{[a,b]}$:*

i) Both additive invariance and multiplicative invariance hold for dominance relations if and only if $\phi(x,y) \equiv \gamma$, where $\gamma \in (0,1]$ is a constant.

ii) Additive invariance holds for dominance relations, and conditionally holds for non-dominance relations provided that the cdfs of $\tilde{x}$ and $\tilde{y}$ are $(N-1)$th-degree single-crossing and satisfy (1), if and only if $\phi(x,y) = e^{(1/c-1)(x-y)}$, where $c \in (0,1]$ is a constant.

iii) For positive random variables, multiplicative invariance holds for dominance relations, and conditionally holds for non-dominance relations provided that the cdfs of $\tilde{x}$ and $\tilde{y}$ are $(N-1)$th-degree single-crossing and satisfy (1)*, if and only if $\phi(x,y)=(x/y)^{(1/c-1)}$, where $c\in(0,1]$ is a constant.*

Some remarks are in order. Part (i) enhances the understanding of the rule proposed by Müller et al. (2017), who introduce $\gamma u'(y)\leq u'(x)$ for $x\leq y$ to interpolate between first-degree and second-degree stochastic dominance. Economically, this condition can be interpreted as imposing an upper bound on the index of greediness (Chateauneuf et al. 2005) of a utility function. Bi and Zhu (2019) extend this approach to higher degrees by limiting the ratio of higher-degree derivatives

$$\gamma(-1)^{(N+1)}u^{(N)}(y)\leq(-1)^{N+1}u^{(N)}(x)\ \text{ for all } x<y. \tag{5}$$

By showing that (5) is the unique condition guaranteeing both additive and multiplicative invariance of $\tilde{x}\succeq_{N,\phi}\tilde{y}$, part (i) justifies the approach of Müller et al. (2017) from an operational perspective. The integral condition for $\tilde{x}\succeq_{N,\phi}\tilde{y}$ under $\phi(x,y)\equiv\gamma$ was provided by Bi and Zhu (2019) as follows: It is (1) together with

$$\gamma\int_a^x\left[G^{(N-1)}(t)-F^{(N-1)}(t)\right]_+dt\geq\int_a^x\left[F^{(N-1)}(t)-G^{(N-1)}(t)\right]_+dt\ \text{ for all } x\in[a,b]. \tag{6}$$

Part (ii) relates to Huang et al. (2020), who construct fractional degree stochastic dominance by taking $\phi(x,y)=e^{(1/c-1)(x-y)}$ in condition (4). They rewrite this condition as $-u^{(N+1)}(x)/u^{(N)}(x)\geq-(1/c-1)$ for u differentiable up to order $N+1$, interpreting it as imposing a constant lower bound on the $(N+1)$th-degree index of absolute risk aversion. By showing that $\phi(x,y)=e^{(1/c-1)(x-y)}$ is the unique interpolating function that guarantees additive invariance of $\succeq_{N,\phi}$ for both dominance and non-dominance relations, part (ii) provides a new justification for the rule proposed by Huang et al. (2020). The integral condition for $\tilde{x}\succeq_{N,\phi}\tilde{y}$ under $\phi(x,y)=e^{(1/c-1)(x-y)}$ was provided by Huang et al. (2020) as follows: It is (1) together with

$$\int_a^x\left[G^{(N-1)}(t)-F^{(N-1)}(t)\right]e^{(1/c-1)t}dt\geq 0\ \text{ for all } x\in[a,b]. \tag{7}$$

Table 1 Summary of the three dominance rules.

ϕ	Additive or multiplicative invariance	Risk attitude bound	Integral condition
γ	both for $\succeq_{N,\phi}$	Greediness index	Eq. (6)
$e^{(1/c-1)(x-y)}$	additive for $\succeq_{N,\phi}$ and $\not\succeq_{N,\phi}$	Absolute risk aversion	Eq. (7)
$(x/y)^{(1/c-1)}$	multiplicative for $\succeq_{N,\phi}$ and $\not\succeq_{N,\phi}$	Relative risk aversion	Eq. (8)

Notes: The invariance of non-dominance requires the single-crossing condition stated in Theorem 2.

Part (iii) provides an operational reason to use $\phi(x,y)=(x/y)^{(1/c-1)}$ instead. With this specification, the $(N+1)$th-degree index of relative risk aversion $-xu^{(N+1)}(x)/u^{(N)}(x)$ has a lower bound $-(1/c-1)$, and the integral condition for $\tilde{x}\succeq_{N,\phi}\tilde{y}$ can be derived analogously to the work of Huang et al. (2020): It is (1) together with

$$\int_a^x \left[G^{(N-1)}(t)-F^{(N-1)}(t)\right] t^{(1/c-1)}dt \geq 0 \quad \text{for all } x\in[a,b]. \tag{8}$$

See Table 1 for a summary of the three dominance rules.

Theorem 2 (also illustrated in Figure 1) establishes a bidirectional mapping between invariance properties and interpolating functions. This not only helps dispel doubts about whether alternative forms of interpolation adhere to the studied invariance laws but also provides insights into how to revise these laws when developing new dominance rules. To illustrate it, consider the "intermediate equivalence" concept introduced by Bossert and Pfingsten (1990) who postulates that degrees of inequality of different distributions remain unchanged when random variables undergo the so-called μ-transform

$$\tilde{x}\mapsto\tilde{x}+\lambda[\mu\tilde{x}+(1-\mu)],$$

where $\lambda>0$ is arbitrary and $\mu\in[0,1]$ is a pre-specified parameter. Obviously, the μ-transform collapses to translation when $\mu=0$, to scaling when $\mu=1$, and formulates an intermediate value judgement between relative and absolute views when $0<\mu<1$. For $\mu\in(0,1)$, researchers can quickly conclude from our results that among the three interpolating functions studied above, only

$\phi(x,y) \equiv \gamma$ guarantees the fractional degree stochastic dominance rule to be invariant with respect to the μ-transform. Taking Figure 1 as a roadmap, one can readily show that

$$\phi(x,y) = \left[\frac{\mu x + (1-\mu)}{\mu y + (1-\mu)}\right]^{(1/c-1)/\mu} \tag{9}$$

is unique in guaranteeing the rule to obey μ-additive invariance for both dominance and non-dominance relations. Assuming $\mu a + (1-\mu) \geq 0$, the integral condition for $\tilde{x} \succeq_{N,\phi} \tilde{y}$ under (9) is (1) together with the condition

$$\int_a^x \left[G^{(N-1)}(t) - F^{(N-1)}(t)\right] [\mu t + (1-\mu)]^{(1/c-1)/\mu} dt \geq 0 \quad \text{for all } x \in [a,b].$$

5. Extension to Unbounded Supports

Until now, we have required that random variables are supported within a bounded interval. To extend the analysis to accommodate cases with unbounded supports, additional assumptions about utility functions and random variables are necessary. Focusing on the first two degrees, Hadar and Russell (1971) assumed utility functions are bounded by a constant with random variables bounded from below to maintain concavity. In contrast, Pomatto et al. (2020) assumed utility functions are bounded by a polynomial while allowing random variables to be unbounded in both directions. For higher degrees, Shaked and Shanthikumar (2007, Section 4.A.7) considered random variables that are unbounded in only one direction, with derivatives of utility functions approaching zero in that direction. As far as invariance properties are concerned, we assume

$$|u(x)| \leq k_u |x|^N + d_u \quad \text{and} \quad \lim_{x \to +\infty} x^{n-1} u^{(n)}(x) = 0 \quad \text{for } n = 2, \ldots, N-1, \tag{10}$$

where $k_u > 0$ and $d_u > 0$ are constants depending on u. We do not impose any requirement on $u^{(1)}$, ensuring that risk-neutral functions, where $u^{(1)}$ is constant, are always included in our utility set.

After introducing condition (10) to utility functions, the definition of integer-degree stochastic dominance reviewed in Section 2 extends directly to random variables supported on $\mathbb{R}$, covering all random variables whose values may lie anywhere on the real line—whether bounded, semi-bounded or unbounded—provided that they have finite first N moments. Moment conditions are important for higher-order stochastic dominance (Fishburn 1980). To characterize dominance relations via integral conditions, we extend the nth-degree integral of a cdf H as follows: $H^{(0)}(x) = H(x)$ and $H^{(n)}(x) = \int_{-\infty}^{x} H^{(n-1)}(t)dt$ iteratively. Notably, assuming H is the cdf of a random variable $\tilde{z}$, for $n = 1, \ldots, N-1$, $H^{(n)}(x) = \mathbb{E}(x-\tilde{z})_{+}^{n}/n!$ remains finite for each x, but $\lim_{x\to+\infty} H^{(n)}(x) = +\infty$.

Lemma 4. *Given $N \in \mathbb{N}$ and with condition (10) imposed on utility functions, for $\tilde{x}$ and $\tilde{y}$ supported on $\mathbb{R}$ with finite first N moments, $\tilde{x} \succeq_N \tilde{y}$ if and only if*

$$\mathbb{E}\tilde{x} \geq \mathbb{E}\tilde{y} \quad \text{and} \quad G^{(N-1)}(x) \geq F^{(N-1)}(x) \quad \text{for all } x \in \mathbb{R}. \tag{11}$$

We are now prepared to examine invariance properties. We allow random variables to be unbounded over $\mathbb{R}$ when studying additive invariance, while focusing on random variables unbounded over $\mathbb{R}_+ = (0, \infty)$ for the study of only multiplicative invariance. Clearly, for dominance relations, both additive and multiplicative invariance directly extend to random variables with unbounded supports. However, for non-dominance relations, the situation is more nuanced and depends on whether the independent random variable being added or multiplied is bounded. When this random variable is unbounded, additive and multiplicative invariance may not hold, even if the single-crossing condition is satisfied. The following examples illustrate this point explicitly.

Example 5. Let $\tilde{x}$ and $\tilde{y}$ follow normal distributions $N(2,2)$ and $N(1,1)$, respectively. Their cdfs exhibit 0th-degree (and thus 1st-degree) single-crossing. It holds that $\mathbb{E}\tilde{x} > \mathbb{E}\tilde{y}$ while $\tilde{x} \not\succeq_1 \tilde{y}$ and $\tilde{x} \not\succeq_2 \tilde{y}$. According to Theorem 1 in Pomatto et al. (2020), there exists an independent random variable $\tilde{z}$ such that $\tilde{x} + \tilde{z} \succeq_1 \tilde{y} + \tilde{z}$, and hence $\tilde{x} + \tilde{z} \succeq_2 \tilde{y} + \tilde{z}$.

Example 6. Let $\tilde{x}$ and $\tilde{y}$ follow lognormal distributions $\log N(2,2)$ and $\log N(1,1)$, respectively. Their cdfs exhibit 0th-degree (and thus 1st-degree) single-crossing. It holds that $\mathbb{E}\log\tilde{x} > \mathbb{E}\log\tilde{y}$ while $\tilde{x} \not\succeq_1 \tilde{y}$ and $\tilde{x} \not\succeq_2 \tilde{y}$. According to Theorem 1 in Pomatto et al. (2020), there exists an independent random variable $\tilde{w}$ such that $\log\tilde{x} + \tilde{w} \succeq_1 \log\tilde{y} + \tilde{w}$. Letting $\tilde{z} = e^{\tilde{w}}$, we have $\tilde{x}\tilde{z} \succeq_1 \tilde{y}\tilde{z}$ and hence $\tilde{x}\tilde{z} \succeq_2 \tilde{y}\tilde{z}$.

In constructing the desired random variable $\tilde{z}$ in the examples above, Pomatto et al. (2020) emphasize the importance of $\tilde{z}$ exhibiting non-negligible risk at unbounded tails. However, if $\tilde{z}$

is constrained to bounded support (away from 0 for multiplication), its ability to convert a non-dominance relation into a dominance one is limited, ensuring that additive and multiplicative invariance for non-dominance relations continue to hold under the single-crossing condition. Our approach to determining interpolating functions with invariance properties remains viable when $\tilde{z}$ has bounded support.

We formally state our main results in this unbounded setting. Let $\mathbb{D} = \mathbb{R}$ or $\mathbb{D} = \mathbb{R}_+$ and $\phi : \{(x,y) \in \mathbb{D}^2 : x < y\} \to [0,1]$ be a continuous function. The condition corresponding to (3) is now

$$\frac{u^{(N)}(x)}{u^{(N)}(y)} \geq \phi(x,y) \quad \text{for all } x < y \text{ in } \mathbb{D} \text{ with } (-1)^{N+1} u^{(N)}(y) > 0. \tag{12}$$

For $\tilde{x}$ and $\tilde{y}$ taking values on $\mathbb{D}$, the stochastic dominance relation $\tilde{x} \succeq_{N,\phi} \tilde{y}$ on $\mathbb{D}$ is defined as $\mathbb{E}u(\tilde{x}) \geq \mathbb{E}u(\tilde{y})$ for all $u \in U^*_{N,\phi}$, where

$$U^*_{N,\phi} = \left\{ u:\ (-1)^{n+1} u^{(n)}(x) \geq 0 \text{ for } x \in \mathbb{D} \text{ and } n = 1, \ldots, N;\ u \text{ satisfies (10) and (12)} \right\}.$$

THEOREM 3. *The statements in Theorem 2 extend to the comparison of random variables supported on $\mathbb{R}$ or $\mathbb{R}_+$, provided that $\tilde{x}$ and $\tilde{y}$, the random variables being ranked, have finite first N moments, and that $\tilde{z}$, the random variable involved in the transformations, has bounded support.*

The integral condition for $\tilde{x} \succeq_{N,\phi} \tilde{y}$ defined on $\mathbb{R}$ or $\mathbb{R}_+$ is $\mathbb{E}\tilde{x} \geq \mathbb{E}\tilde{y}$, along with the inequalities in (6), (7), and (8), with the integration interval $[a,x]$ replaced by $(-\infty, x]$ for $\phi(x,y) \equiv \gamma$ and $\phi(x,y) = e^{(1/c-1)(x-y)}$, and by $(0,x]$ for $\phi(x,y) = (x/y)^{(1/c-1)}$.

We conclude this section by highlighting a subtle but important distinction between stochastic dominance relations defined on $\mathbb{R}$ or $\mathbb{R}_+$ and those on a bounded interval $[a,b] \subset \mathbb{R}$ or $[a,b] \subset \mathbb{R}_+$. For the former, conditions on utility functions apply across the entire $\mathbb{R}$ or $\mathbb{R}_+$, while for the latter, they are restricted to $[a,b]$. When ranking random variables supported on $[a,b]$ using stochastic dominance defined on $\mathbb{R}$ or $\mathbb{R}_+$, conditions outside $[a,b]$ can reshape the role of lower-degree derivatives. In the next result, we show that up to degree three, the restriction to $[a,b]$ does not change the stochastic dominance rule, extending Lemma 1 to fractional degrees. The case for higher fractional degrees remains open for future research.

Proposition 1. *For degrees up to and including three, and with condition (10) imposed on utility functions, the restriction of $\succeq_{N,\phi}$ defined on $\mathbb{R}$ or $\mathbb{R}_+$ to random variables supported on $[a,b] \subset \mathbb{R}$ or $[a,b] \subset \mathbb{R}_+$ is identical to $\succeq_{N,\phi}$ defined directly over $[a,b]$ for all $\phi(x,y) \equiv \gamma$, $\phi(x,y) = e^{(1/c-1)(x-y)}$ and $\phi(x,y) = (x/y)^{(1/c-1)}$.*

6. Application to Mutual Fund Selection

Integer-degree stochastic dominance, a nonparametric method based on expected utility theory, has long served as an appealing alternative to moment-based methods for evaluating risky asset performance (Bawa et al. 1979) and remains vital in financial investment (Peng and Delage 2024). Fractional degree stochastic dominance, being more refined, better aligns evaluations with investors' preferences. To demonstrate the applicability of our characterization results for fractional degree stochastic dominance, we revisit the problem of assessing mutual fund performance relative to market performance (Joy and Porter 1974). Specifically, we compare mutual funds with the Dow Jones Industrial Average (DJIA) index over the same period to identify the funds that outperform the index. This analysis provides a quantitative assessment of the fund market and reveals investment opportunities for investors who seek to optimize portfolios with stochastic dominance constraints (Dentcheva and Ruszczynski 2003, Armbruster and Delage 2015, Post and Kopa 2017).

6.1. Data

Mutual funds can be classified into various types based on their investment style. We illustrate the usefulness of our results using value-oriented funds, which invest in undervalued stocks likely to attract investors with their sustainability, longevity, and typically stable returns (Cronqvist et al. 2015). We retrieved data on US mutual funds that were active throughout the period from January 2014 to January 2024 from the Bloomberg database, including fund names, ISIN codes, and investment styles. Monthly prices were obtained from the Datastream database, and funds with missing price values were excluded. Using ISIN codes to merge the datasets, we successfully matched 91% of the value funds with their monthly prices, resulting in a final sample of 871 value mutual funds. Price data were also collected for the DJIA index over the same period.

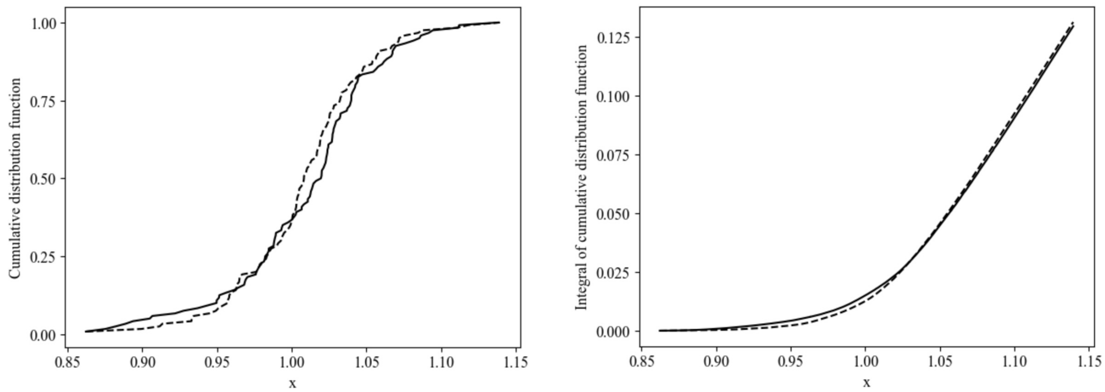
To examine both additive and multiplicative invariance with the same data, we employ the monthly price relative—defined as the ratio of prices between two consecutive months—to measure monthly return. Unlike logarithmic returns, which indicate gains or losses relative to a base value of zero, the price relative is always positive and indicates gains or losses relative to a base value of one. Using historical price data on the 871 funds and the index over 121 months, we obtain 872 empirical distributions, each comprising 120 observations of price relatives, with equal probabilities assigned to each observation.

6.2. Single-Crossing Condition

Empirical cdfs are stepwise functions, while their first-degree integrals, starting from the lower bound of the price relatives, are piecewise linear functions. Both connect adjacent observation points with straight lines, allowing the single-crossing condition to be verified using only the observation points. Specifically, the 0th-degree or 1st-degree single-crossing condition is met if and only if the difference between the empirical cdfs or their first-degree integrals at the merged observation points crosses zero only once as the observation point increases. Using this approach to assess whether the empirical cdf of a mutual fund meets the single-crossing condition with the DJIA index, we find that among 871 funds, 112 (13%) satisfy the 0th-degree single-crossing condition and 583 (67%) meet the 1st-degree condition. The substantially higher number of funds satisfying the 1st-degree single-crossing condition compared to the 0th-degree single-crossing condition is attributable to the greater regularity of the first-degree integral curves. This is clearly illustrated in Figure 3, which shows the non-single-crossing empirical cdfs of a mutual fund and the index, as well as the first-degree integrals of these cdfs that exhibit single-crossing behavior.

In practice, verifying the nth-degree single-crossing condition requires checking the sign changes of the integrated difference $F^{(n)} - G^{(n)}$, which, for empirical distributions with T observations, can be done in $O(T)$ time after sorting (since integration reduces to cumulative sums). For pairwise comparisons across M assets, this yields $O(M^2T)$ complexity, which is manageable for moderate M (e.g., hundreds of funds) but may become costly for very large portfolios. Several strategies

Figure 3 Verifying the single-crossing condition for a mutual fund and the index.



Notes: The left panel shows non-single-crossing empirical cdfs of a mutual fund and the DJIA index, while the right displays their single-crossing first-degree integrals.

can alleviate the burden: (i) pre-screening using moment-based necessary conditions (e.g., mean, variance, skewness) to rule out non-candidates (i.e., to reduce M); (ii) leveraging the fact that single-crossing is often implied by structural assumptions (e.g., location-scale families, as noted in Section 2.3) when empirical cdfs can be approximated by parametric distributions; and (iii) employing grid-based or spline-smoothed cdfs, where the original T-point empirical distribution is approximated on a much coarser grid of size $K \ll T$, thereby reducing computational cost while preserving the essential crossing structure.

6.3. Implementation

Applying fractional degree stochastic dominance requires only two key pieces of information: the invariance laws favored by the decision maker and the specific value of the fractional degree. This information can be gathered through well-designed questionnaires or simple lottery-choice experiments, which is often much simpler than the task of determining a specific utility function for investors. We categorize mutual funds into two types: funds that stochastically dominate the index (superior funds) and those that do not (non-superior funds). The latter group is further divided into two subgroups: funds that are dominated by the index (inferior funds) and funds that neither dominate nor are dominated by the index (mediocre funds). Under this classification, non-superior funds consist of both inferior and mediocre funds. Investors may consider two criteria for selecting investment opportunities. A stricter approach focuses solely on superior funds, excluding both inferior and mediocre ones, while a looser approach excludes only inferior funds, keeping mediocre funds as potential investment opportunities.

To avoid getting bogged down in unnecessary detail, we follow earlier work (Joy and Porter 1974, Bawa et al. 1979, Levy and Sarnat 1984) to identify stochastic dominance relations through directly examining empirical distribution functions. Compared with statistical methods, empirical cdfs are purely data-driven, allowing for a flexible and unbiased representation of the underlying distribution without requiring parametric assumptions or asymptotic analysis. We begin by evaluating whether a fund's mean price relative, derived from its empirical cdf, is greater or smaller than that of the index. For the 166 funds with a mean price relative higher than the index, we assess if they dominate the index; for those with a lower mean price relative, we evaluate if the index dominates them. Verifying stochastic dominance involves checking the integral condition across the entire common support. For degrees within $[1, 2]$, since empirical cdfs are stepwise functions, it is sufficient to check the integral condition at the merged observation points of the fund and the index, and dominance is confirmed if it holds at all points. For degrees within $(2, 3)$, if the 1st-degree single-crossing condition is met, checking the integral condition once at the right endpoint of the common support is sufficient. If the single-crossing condition is not met, the integral condition must be checked at all interior points of the 120 equal subintervals dividing the common support, and dominance is confirmed only if it holds at all points. Further partitioning the interval into more than 120 subintervals yields similar dominance results, indicating that our step size for evaluating the integral condition is sufficiently small to achieve convergence.

6.4. Results

We evaluate the relative performance of mutual funds using fractional degree stochastic dominance rules of degrees 1, 1.5, 2, and 2.5 with three interpolating functions. The selection of $\gamma = c = 0.5$ is motivated by both conceptual representativeness and empirical relevance. First, $\gamma = c = 0.5$ lies exactly midway between adjacent integer degrees. This neutral positioning ensures that the resulting fractional-degree rule does not disproportionately resemble either the lower or the higher integer-degree benchmark, thereby offering a balanced illustration of intermediate risk attitudes. Second, $c = 0.5$ is well supported by experimental evidence on higher-order risk attitudes, and we set $\gamma = 0.5$ to ensure that comparisons are made at the same fractional degree. Specifically, under the exponential form of ϕ, $c = 0.5$ implies that the decision maker's second-order (when $N = 1$) and third-order (when $N = 2$) index of absolute risk aversion are both bounded below by -1. In the experiment reported in Deck et al. (2025), more than 96% of subjects satisfy this bound for both values of N (see Tables 3 and 4 therein). Under the power form of ϕ, $c = 0.5$ implies a lower bound of -1 for both the second- and third-order indices of relative risk aversion, a condition satisfied by more than 60% of subjects when $N = 1$ and more than 80% of subjects when $N = 2$ (see Tables 3 and 4 therein). The main pattern observed in our numerical illustration remains robust to alternative choices of the degree parameters.

The comparison results without background risks are shown in Table 2, where the subscripts Co, Ex, and Pw in $\succeq$ indicate that ϕ is the constant, exponential, and power functions, respectively. For mediocre funds, we report the numbers of funds meeting the 0th-degree single-crossing condition for $\succeq_{1,\phi}$ and the 1st-degree single-crossing condition for $\succeq_{2,\phi}$ with the index. Consistent with the hierarchy property of stochastic dominance, the number of superior and inferior funds increases as the degree rises. Funds that dominate the index are rare, with only 3% becoming dominant at degree 2.5, aligning with previous literature (Joy and Porter 1974). In contrast, funds dominated by the index are more common, increasing from 160 to 566 as the degree rises from 1.5 to 2.5.

To assess the impact of background risks, both invariance properties for dominance and non-dominance relations must be considered. Invariance for dominance relations ensures superior or inferior funds retain their status after transformations, while for non-dominance relations ensures mediocre funds remain unchanged. Focusing only on dominance relations is insufficient, as transformations might reclassify mediocre funds as superior or inferior. Our characterization of fractional degree stochastic dominance clarifies potential reclassification, reducing the computational burden in identifying where this might occur.

Table 2 Stochastic dominance results without background risks.

Rule of dominance	Superior funds	Non-superior funds	
	Fund $\succeq$ Index	Index $\succeq$ Fund	Neither
$\succeq_1$	0	0	871 (112)
$\succeq_{1,\text{Co}}$ with $\gamma = 0.5$	0	160	711 (107)
$\succeq_{1,\text{Ex}}$ with $c = 0.5$	17	432	422 (63)
$\succeq_{1,\text{Pw}}$ with $c = 0.5$	17	432	422 (63)
$\succeq_2$	19	507	345 (160)
$\succeq_{2,\text{Co}}$ with $\gamma = 0.5$	26	532	313 (143)
$\succeq_{2,\text{Ex}}$ with $c = 0.5$	24	565	282 (156)
$\succeq_{2,\text{Pw}}$ with $c = 0.5$	26	566	279 (154)

Notes: This table reports the number of superior, inferior, and mediocre funds. The numbers in brackets in the last column indicate how many funds in each group meet the single-crossing condition with the index.

Table 3 shows the comparison results after introducing an independent background risk $\tilde{z} \overset{\text{d}}{=} [0; 0.1; 0.2]$ to the returns of both funds and the index. The magnitude of $\tilde{z}$ is of the same order as the magnitude of investment returns. The number of funds in each group is expressed as a sum: the first number reflects the original count without background risks, and the second indicates the additional funds. A notable portion of mediocre funds shift to either superior or inferior status, leading to an approximate 10% increase in superior funds and an approximate 5% increase in inferior ones. To re-identify superior funds in the presence of background risk based on Table 2, the required number of additional comparisons is calculated as follows, where 166 represents the funds with a mean price relative greater than the index, the bracketed statements describe funds that do not need re-evaluation, and “SC” stands for “single-crossing condition”:

$$\text{for } \succeq_{1,\text{Co}}\text{: } 166 - 0 \text{ (fund} \succeq_{2,\text{Co}} \text{index)} = 160,$$

$$\text{for } \succeq_{2,\text{Co}}\text{: } 166 - 26 \text{ (fund} \succeq_{2,\text{Co}} \text{index)} = 140,$$

$$\text{for } \succeq_{1,\text{Ex}}\text{: } 166 - 17 \text{ (fund} \succeq_{1,\text{Ex}} \text{index)} - 29 \text{ (fund} \not\succeq_{1,\text{Ex}} \text{index, SC)} = 120,$$

Table 3 Stochastic dominance results with an additive background risk $\tilde{z}$.

Rule of dominance	Superior funds	Non-superior funds	
	$\text{Fund}+\tilde{z} \succeq \text{Index}+\tilde{z}$	$\text{Index}+\tilde{z} \succeq \text{Fund}+\tilde{z}$	Neither
$\succeq_1$	0	0	871 (112)
$\succeq_{1,\text{Co}}$ with $\gamma = 0.5$	$5 = 0 + 5$	$261 = 160 + 101$	605 (90)
$\succeq_{1,\text{Ex}}$ with $c = 0.5$	$17 = 17 + 0$	$452 = 432 + 20$	402 (63)
$\succeq_{1,\text{Pw}}$ with $c = 0.5$	$18 = 17 + 1$	$459 = 432 + 27$	394 (60)
$\succeq_2$	$19 = 19 + 0$	$536 = 507 + 29$	316 (160)
$\succeq_{2,\text{Co}}$ with $\gamma = 0.5$	$29 = 26 + 3$	$557 = 532 + 25$	285 (140)
$\succeq_{2,\text{Ex}}$ with $c = 0.5$	$27 = 24 + 3$	$573 = 565 + 8$	271 (156)
$\succeq_{2,\text{Pw}}$ with $c = 0.5$	$29 = 26 + 3$	$577 = 566 + 11$	265 (149)

Notes: This table reports the number of superior, inferior, and mediocre funds. The numbers in brackets in the last column indicate how many funds in each group meet the single-crossing condition with the index.

$$\text{for } \succeq_{2,\text{Ex}}\text{: } 166 - 24 \text{ (fund} \succeq_{2,\text{Ex}} \text{index)} - 89 \text{ (fund} \not\succeq_{2,\text{Ex}} \text{index, SC)} = 53.$$

For the rules $\succeq_{2,\text{Co}}$, $\succeq_{1,\text{Ex}}$, and $\succeq_{2,\text{Ex}}$, additive invariance for dominance relations ensures originally superior funds remain so, reducing comparisons by 15.7% (= 26/166), 10.2% (= 17/166), and 14.5% (= 24/166), respectively. For $\succeq_{1,\text{Ex}}$ and $\succeq_{2,\text{Ex}}$, additive invariance for non-dominance relations ensures originally non-superior funds remain non-superior under the single-crossing condition, further reducing comparisons by 17.5% (= 29/166) and 53.6% (= 89/166), respectively. This reduction works independently of the specification of $\tilde{z}$ and is equally effective in re-identifying inferior funds. These calculations demonstrate the practical value of invariance properties in reducing the computational burden when assessing the impact of background risks on fund classification, particularly for degrees higher than 2, where the single-crossing condition is largely met. For $\succeq_{1,\text{Pw}}$ and $\succeq_{2,\text{Pw}}$, where neither relation is additively invariant, all funds must be re-evaluated.

Table 4 presents the comparison results after multiplying an independent background risk $\tilde{w} \overset{\text{d}}{=} [1; 1.2; 1.4]$ with the returns of both the funds and the index. The magnitude of $\tilde{w}$ mirrors the

Table 4 Stochastic dominance results with a multiplicative background risk $\tilde{w}$.

Rule of dominance	Superior funds	Non-superior funds	
	$\text{Fund}\times\tilde{w}\succeq\text{Index}\times\tilde{w}$	$\text{Index}\times\tilde{w}\succeq\text{Fund}\times\tilde{w}$	Neither
$\succeq_1$	0	0	871 (112)
$\succeq_{1,\text{Co}}$ with $\gamma=0.5$	$2=0+2$	$202=160+42$	667 (99)
$\succeq_{1,\text{Ex}}$ with $c=0.5$	$14=17-3$	$417=432-15$	440 (69)
$\succeq_{1,\text{Pw}}$ with $c=0.5$	$17=17+0$	$445=432+13$	409 (63)
$\succeq_2$	$20=19+1$	$515=507+8$	336 (160)
$\succeq_{2,\text{Co}}$ with $\gamma=0.5$	$28=26+2$	$543=532+11$	300 (138)
$\succeq_{2,\text{Ex}}$ with $c=0.5$	$27=24+3$	$569=565+4$	275 (154)
$\succeq_{2,\text{Pw}}$ with $c=0.5$	$27=26+1$	$570=566+4$	274 (154)

Notes: This table reports the number of superior, inferior, and mediocre funds. The numbers in brackets in the last column indicate how many funds in each group meet the single-crossing condition with the index.

magnitude of the exchange rate. The number of superior funds increases by around 10% and inferior funds by around 4%. Following the same procedure as in Table 3, the additional comparisons required to re-identify superior funds are:

$$\text{for } \succeq_{1,\text{Co}}\text{: } 166-0 \text{ (fund} \succeq_{2,\text{Co}} \text{index)} = 160,$$

$$\text{for } \succeq_{2,\text{Co}}\text{: } 166-26 \text{ (fund} \succeq_{2,\text{Co}} \text{index)} = 140,$$

$$\text{for } \succeq_{1,\text{Pw}}\text{: } 166-17 \text{ (fund} \succeq_{1,\text{Ex}} \text{index)} - 29 \text{ (fund} \not\succeq_{1,\text{Ex}} \text{index, SC)} = 120,$$

$$\text{for } \succeq_{2,\text{Pw}}\text{: } 166-26 \text{ (fund} \succeq_{2,\text{Ex}} \text{index)} - 87 \text{ (fund} \not\succeq_{2,\text{Ex}} \text{index, SC)} = 53,$$

which are substantially reduced by the invariance properties. However, for $\succeq_{1,\text{Ex}}$ and $\succeq_{2,\text{Ex}}$, where neither relation is multiplicative invariant, all funds must be re-evaluated.

7. Conclusion

This paper establishes fractional degree stochastic dominance rules based solely on invariance laws under a general condition broadly applicable to all intermediate utility sets. We review an array of invariance laws associated with integer-degree stochastic dominance and identify the unique fractional degree stochastic dominance rule that obeys a subset of mutually compatible laws. The results offer clear guidelines for practitioners to compare and select the appropriate rules, provide a precise classification of previously studied fractional degree dominance rules that encompasses Müller et al. (2017) and Huang et al. (2020) as special cases, and offer potential insights for developing new rules. By applying fractional degree stochastic dominance to identify mutual funds with superior relative performance, we demonstrate that our findings effectively facilitate the assessment of the impacts of transformations on dominance relations.

Given that integer-degree stochastic dominance rules have been extensively applied in various decision-making contexts, we hope that our construction of fractional degree stochastic dominance based on operational properties will make it more accessible and contribute to the further proliferation of the stochastic dominance methodology. Our current framework is limited to a static univariate setting under expected utility. Extending the analysis to multi-period settings (Levy and Paroush 1974, Constantinides and Perrakis 2002) would require reformulating invariance properties over time and thus represents a natural avenue for future research. Generalizations to multivariate utility frameworks (Tsetlin and Winkler 2018) or non-expected utility models (Cerreia-Vioglio et al. 2016), while challenging, also offer promising directions.

Appendix: Proofs

Proof of Lemma 1. It suffices to prove the result when $[a,b]$ is the minimal interval covering the supports of $\tilde{x}$ and $\tilde{y}$ and $[c,d] \supseteq [a,b]$. Note that (1) for $N \leq 3$ is independent of the choice of intervals. It is also clear that condition (2), when applied to $[c,d]$, implies (2) using $[a,b]$. The converse holds because $G^{(n)}(x) - F^{(n)}(x) = 0$ for $x < a$, $G^{(1)}(x) - F^{(1)}(x) = G^{(1)}(b) - F^{(1)}(b) = \mathbb{E}\tilde{x} - \mathbb{E}\tilde{y}$ and $G^{(2)}(x) - F^{(2)}(x) = G^{(2)}(b) - F^{(2)}(b) + [G^{(1)}(b) - F^{(1)}(b)](x-b)$ for $x > b$. □

Proof of Theorem 1. We only prove the statement for non-dominance relations, which is new to the literature. Denote by H the cdf of $\tilde{z}$. Assume $\tilde{x} + \tilde{z} \succeq_N \tilde{y} + \tilde{z}$, which implies

$$\int_c^d [G^{(N-1)}(x-y) - F^{(N-1)}(x-y)]dH(y) \geq 0 \quad \text{for all } x \in \mathbb{R}, \tag{A1}$$

where $c = \text{essinf } \tilde{z}$ and $d = \text{esssup } \tilde{z}$. We show $\tilde{x} \succeq_N \tilde{y}$ by contradiction. Suppose otherwise. Since the cdfs of $\tilde{x}$ and $\tilde{y}$ satisfy (1), we have $G^{(N-1)}(x) < F^{(N-1)}(x)$ on some $[x_1, x_2]$. By single-crossing, we have either $G^{(N-1)}(x) \le F^{(N-1)}(x)$ for $x \le x_2$ or $G^{(N-1)}(x) \le F^{(N-1)}(x)$ for $x \ge x_1$. In the former case, we have $G^{(N-1)}(c+x_2-y) - F^{(N-1)}(c+x_2-y) < 0$ for $y \in [c, c+x_2-x_1]$. Due to $c = \text{essinf } \tilde{z}$, H has positive mass on $[c, c+x_2-x_1]$ and hence $\int_c^d [G^{(N-1)}(c+x_2-y) - F^{(N-1)}(c+x_2-y)] dH(y) < 0$, a contradiction to (A1). A similar contradiction arises in the latter case. The result for multiplicative invariance in non-dominance relations can be proved similarly. □

Proof of Lemma 2. The infimum introduced by this lemma is well-defined, because the set $\{u \in U_{N,\phi} : u^{(N)}(y) \neq 0\} \supseteq \{u \in U_{N+1} : u^{(N)}(y) \neq 0\}$ is non-empty. By definition, $\phi^*(x,y)(-1)^{N+1}u^{(N)}(y) \le (-1)^{N+1}u^{(N)}(x)$ for any $x < y$ and $u \in U_{N,\phi}$, which implies $U_{N,\phi^*} \supseteq U_{N,\phi}$. Since $\phi(x,y)(-1)^{N+1}u^{(N)}(y) \le (-1)^{N+1}u^{(N)}(x)$ for any $x < y$ and $u \in U_{N,\phi}$, we have $\phi \le \phi^*$, which in turn implies $U_{N,\phi^*} \subseteq U_{N,\phi}$. Thus, $U_{N,\phi} = U_{N,\phi^*}$, and ϕ^* is the unique maximal interpolator. To prove the multiplicative triangular inequality, note that it holds trivially when $\phi^*(y,z) = 0$. When $\phi^*(y,z) \neq 0$, it follows that $u^{(N)}(y) \neq 0$ for any $u \in U_{N,\phi}$ with $u^{(N)}(z) \neq 0$, and hence

$$\phi^*(x,z) = \inf_{\substack{u \in U_{N,\phi} \\ u^{(N)}(z) \neq 0}} \left[\frac{u^{(N)}(x)}{u^{(N)}(y)} \cdot \frac{u^{(N)}(y)}{u^{(N)}(z)} \right] \ge \inf_{\substack{u \in U_{N,\phi} \\ u^{(N)}(y) \neq 0}} \left[\frac{u^{(N)}(x)}{u^{(N)}(y)} \right] \inf_{\substack{u \in U_{N,\phi} \\ u^{(N)}(z) \neq 0}} \left[\frac{u^{(N)}(y)}{u^{(N)}(z)} \right] = \phi^*(x,y)\phi^*(y,z)$$

as desired. □

Proof of Theorem 2. The proof boils down to proving Results 1 to 7 in sequence, with Results 1 and 2 being straightforward. In all proofs below, we use the contrapositive forms of "additive invariance for non-dominance" and "multiplicative invariance for non-dominance," which are "$\tilde{x} + \tilde{z} \succeq \tilde{y} + \tilde{z} \Rightarrow \tilde{x} \succeq \tilde{y}$" and "$\tilde{x}\tilde{z} \succeq \tilde{y}\tilde{z} \Rightarrow \tilde{x} \succeq \tilde{y}$," respectively. Let $[c_1; \ldots; c_n]$ denote a random variable that takes the value c_i (where c_i may be random or constant) for $i = 1, \ldots, n$, each with equal probability $1/n$. Define $F * G : x \mapsto \int_{\mathbb{R}} F(x-y) dG(y)$, and note that

$$(F * G)^{(n)} = F^{(n)} * G \quad \text{for } n \ge 1. \tag{A2}$$

We first present a lemma that will be useful in the proof of Results 3–4. For $x < y$ and $\varepsilon_1, \ldots, \varepsilon_n > 0$, define $\tilde{z}_1(x,y) = [x; y]$, $\tilde{w}_1(x,y) = [x - \varepsilon_1; y + \alpha\varepsilon_1]$, and for $n \ge 2$,

$$\tilde{z}_n(x,y) = [\tilde{z}_{n-1}(x,y); \tilde{w}_{n-1}(x,y) + \varepsilon_n], \quad \tilde{w}_n(x,y) = [\tilde{z}_{n-1}(x,y) + \varepsilon_n; \tilde{w}_{n-1}(x,y)]. \tag{A3}$$

LEMMA A1. *Let $\alpha \geq 0$ be a constant and $a < x < y < b$ be given.*

i) If $\alpha < \phi(x,y)$, then $\tilde{z}_N(x,y) \succeq_{N,\phi} \tilde{w}_N(x,y)$ for $\varepsilon_1,\ldots,\varepsilon_N > 0$ small enough.

ii) If $\tilde{z}_N(x,y) \succeq_{N,\phi} \tilde{w}_N(x,y)$ for $\varepsilon_1,\ldots,\varepsilon_N > 0$ small enough, then $\alpha \leq \phi(x,y)$.

Proof. (i) For $N=1$, by continuity, we have $\alpha < \phi$ on $[x-\varepsilon, x] \times [y, y+\alpha\varepsilon]$ for $\varepsilon > 0$ small enough. For $u \in U_{1,\phi}$, by the mean value theorem, there exists $(\hat{x},\hat{y}) \in (x-\varepsilon, x) \times (y, y+\alpha\varepsilon)$ such that

$$\begin{aligned}\mathbb{E}u(\tilde{z}_1(x,y)) - \mathbb{E}u(\tilde{w}_1(x,y)) &= \frac{1}{2}[u(x)-u(x-\varepsilon)] - \frac{1}{2}[u(y+\alpha\varepsilon)-u(y)] \\ &= \frac{\varepsilon}{2}[u'(\hat{x}) - \alpha u'(\hat{y})] \geq \frac{\varepsilon}{2}u'(\hat{y})[\phi(\hat{x},\hat{y})-\alpha] \geq 0,\end{aligned}$$

where the first inequality follows from $u \in U_{1,\phi}$ and the second inequality follows from $\alpha < \phi(\hat{x},\hat{y})$. Hence $\tilde{z}_1(x,y) \succeq_{1,\phi} \tilde{w}_1(x,y)$ for sufficiently small $\varepsilon > 0$. For $N \geq 2$, by the continuity of ϕ, we have $\alpha < \phi$ on $[x-\varepsilon_1, x+\sum_{i=2}^N \varepsilon_i] \times [y, y+\alpha\varepsilon_1+\sum_{i=2}^N \varepsilon_i]$ for sufficiently small $\varepsilon_1,\ldots,\varepsilon_N > 0$. Let $\tilde{z}_N$ and $\tilde{w}_N$ be shorthand for $\tilde{z}_N(x,y)$ and $\tilde{w}_N(x,y)$, as defined in (A3), respectively. Define $\Delta_\varepsilon^1 u(x) = u(x+\varepsilon)-u(x)$ and $\Delta^n_{\varepsilon_1,\ldots,\varepsilon_n} u(x) = \Delta^{n-1}_{\varepsilon_1,\ldots,\varepsilon_{n-1}} u(x+\varepsilon_n) - \Delta^{n-1}_{\varepsilon_1,\ldots,\varepsilon_{n-1}} u(x)$ for $n \geq 2$. For $u \in U_{N,\phi}$, by the mean value theorem, there exists $(\hat{x},\hat{y}) \in [x-\varepsilon_1, x+\sum_{i=2}^N \varepsilon_i] \times [y, y+\alpha\varepsilon_1+\sum_{i=2}^N \varepsilon_i]$ such that

$$\begin{aligned}\mathbb{E}u(\tilde{z}_N) - \mathbb{E}u(\tilde{w}_N) &= \frac{(-1)^{N+1}}{2^N}\left[\Delta^N_{\varepsilon_1,\ldots,\varepsilon_N} u(x-\varepsilon_1) - \Delta^N_{\alpha\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_N} u(y)\right] \\ &= \frac{\varepsilon_1\cdots\varepsilon_N}{2^N}\left[(-1)^{N+1}u^{(N)}(\hat{x}) - \alpha(-1)^{N+1}u^{(N)}(\hat{y})\right] \\ &\geq \frac{\varepsilon_1\cdots\varepsilon_N}{2^N}(-1)^{N+1}u^{(N)}(\hat{y})[\phi(\hat{x},\hat{y})-\alpha] \geq 0,\end{aligned}$$

where the first inequality follows from $u \in U_{N,\phi}$ and the second inequality follows from $\alpha < \phi(\hat{x},\hat{y})$. Hence, $\mathbb{E}u(\tilde{z}_N) \geq \mathbb{E}u(\tilde{w}_N)$ for all $u \in U_{N,\phi}$, yielding $\tilde{z}_N \succeq_{N,\phi} \tilde{w}_N$.

(ii) We show this by contradiction. Suppose $\alpha > \phi(x,y)$. By the maximality of ϕ, there exists $u \in U_{N,\phi}$ such that $\alpha(-1)^{N-1}u^{(N)}(y) > (-1)^{N-1}u^{(N)}(x)$. Thus,

$$\lim_{\varepsilon_1,\ldots,\varepsilon_N \to 0} \frac{2^N}{\varepsilon_1\cdots\varepsilon_N}[\mathbb{E}u(\tilde{z}_N) - \mathbb{E}u(\tilde{w}_N)] = (-1)^{N+1}\left[u^{(N)}(x) - \alpha u^{(N)}(y)\right] < 0,$$

a contradiction to $\tilde{z}_N \succeq_{N,\phi} \tilde{w}_N$ for $\varepsilon_1,\ldots,\varepsilon_N > 0$ small enough.

Proof of Results 3–4. It suffices to prove the "only if" statement. Since ϕ is continuous, we only need to focus on the case $a < x < y < b$. To prove Result 3, fix $a < x < y < b$ and $\alpha < \phi(x,y)$. Let

$\tilde{z}_N(x,y)$ and $\tilde{w}_N(x,y)$ be as in (A3). By (i) of Lemma A1, $\tilde{z}_N(x,y) \succeq_{N,\phi} \tilde{w}_N(x,y)$ for $\varepsilon_1,\ldots,\varepsilon_N > 0$ small enough. By translation invariance, for any $k \in (a-x, b-y)$, we have $\tilde{z}_N(x+k; y+k) \succeq_{N,\phi} \tilde{w}_N(x+k, y+k)$. By (ii) of Lemma A1, we have $\alpha \le \phi(x+k, y+k)$. Since $\alpha < \phi(x,y)$ is arbitrary, we have $\phi(x,y) \le \phi(x+k, y+k)$ for any k above. Switching the positions of (x,y) and $(x+k, y+k)$, we obtain the equality $\phi(x,y) = \phi(x+k, y+k)$. The proof of Result 4 is in parallel.

With Results 3–4 established, Result 5 follows directly. We now move to Results 6–7, which are much more sophisticated. We will elaborate on the proof of Result 6, as the proof of Result 7 is logically similar. We first consider the case of $N = 1$, which requires the next lemma.

Lemma A2. *Let the cdfs of $\tilde{x}$ and $\tilde{y}$ be F and G, with F crossing G once from below.*

i) Suppose additive invariance holds for both $\succeq_{1,\phi}$ and $\not\preceq_{1,\phi}$. If there exist $\beta > 0$ and $a \le \bar{x} < \bar{y} \le b$ such that $a \le \tilde{x}, \tilde{y} \le b - (\bar{y} - \bar{x})$, $\phi(\bar{x}, \bar{y}) = e^{\beta(\bar{x} - \bar{y})}$ and

$$\int_a^b [G(x) - F(x)] e^{\beta x} dx \ge 0, \tag{A4}$$

then $\tilde{x} \succeq_{1,\phi} \tilde{y}$.

ii) Suppose multiplicative invariance holds for both $\succeq_{1,\phi}$ and $\not\preceq_{1,\phi}$. If there exist $\beta > 0$ and $a \le \bar{x} < \bar{y} \le b$ such that $a \le \tilde{x}, \tilde{y} \le b\bar{x}/\bar{y}$, $\phi(\bar{x}, \bar{y}) = (\bar{x}/\bar{y})^\beta$ and

$$\int_a^b [G(x) - F(x)] x^\beta dx \ge 0, \tag{A5}$$

then $\tilde{x} \succeq_{1,\phi} \tilde{y}$.

Proof. (i) Let $\tilde{z}$ follow the density function $\frac{\beta e^{-\beta z}}{1 - e^{-\beta\theta}}$, where $\theta = \bar{y} - \bar{x}$ and $z \in [0, \theta]$. Denote by H_1 and H_2 the cdfs of $\tilde{x} + \tilde{z}$ and $\tilde{y} + \tilde{z}$, respectively. Our condition $a \le \tilde{x}, \tilde{y} \le b - \theta$ guarantees that both $\tilde{x} + \tilde{z}$ and $\tilde{y} + \tilde{z}$ are supported on $[a, b]$. Direct computation shows that

$$H_1(w) = \int_0^\theta \frac{\beta e^{-\beta z}}{1 - e^{-\beta\theta}} F(w-z) dz = \frac{\beta}{1 - e^{-\beta\theta}} \int_{w-\theta}^w e^{-\beta(w-x)} F(x) dx$$

and $H_2(w) = \frac{\beta}{1-e^{-\beta\theta}} \int_{w-\theta}^w e^{-\beta(w-x)} G(x) dx$ for $w \in \mathbb{R}$, resulting in $H_2(w) - H_1(w) = \frac{\beta e^{-\beta w}}{1 - e^{-\beta\theta}} [H(w) - H(w-\theta)]$, where $H(x) = \int_a^x [G(t) - F(t)] e^{\beta t} dt$. Noting that $H(x) = 0$ for $x \le a$, condition (A4) and the single-crossing property imply $H(x) \ge 0$ for all $x \in \mathbb{R}$. Using integration by parts, we can verify

$$\mathbb{E}u(\tilde{x} + \tilde{z}) - \mathbb{E}u(\tilde{y} + \tilde{z}) = \int_a^b [H_2(w) - H_1(w)] u'(w) dw$$

$$
\begin{aligned}
&= \int_a^b \frac{\beta e^{-\beta w}}{1-e^{-\beta\theta}} H(w)u'(w)dw - \int_a^b \frac{\beta e^{-\beta w}}{1-e^{-\beta\theta}} H(w-\theta)u'(w)dw \\
&= \int_a^b \frac{\beta e^{-\beta w}}{1-e^{-\beta\theta}} H(w)u'(w)dw - \int_{a-\theta}^{b-\theta} \frac{\beta e^{-\beta(w+\theta)}}{1-e^{-\beta\theta}} H(w)u'(w+\theta)dw \\
&\geq \int_a^{b-\theta} \frac{\beta e^{-\beta w}}{1-e^{-\beta\theta}} H(w)u'(w)dw - \int_a^{b-\theta} \frac{\beta e^{-\beta(w+\theta)}}{1-e^{-\beta\theta}} H(w)u'(w+\theta)dw \\
&= \frac{\beta}{1-e^{-\beta\theta}} \int_a^{b-\theta} H(x)e^{-\beta x}[u'(x) - e^{-\beta\theta}u'(x+\theta)]dx,
\end{aligned}
$$

where the inequality follows from that $H(w) = 0$ for $w \leq a$, and $H(w) \geq 0$ and $u'(w) \geq 0$ for $w \in [b-\theta, b]$. Since $u'(x) \geq \phi(x, x+\theta)u'(x+\theta)$ for all $u \in U_{1,\phi}$, and additive invariance for $\succeq_{1,\phi}$ implies by Result 3 that $\phi(x, x+\theta)$ depends only on θ, the condition $\phi(\bar{x}, \bar{x}+\theta) = e^{-\beta\theta}$ implies $u'(x) \geq e^{-\beta\theta}u'(x+\theta)$. Hence, $\mathbb{E}u(\tilde{x}+\tilde{z}) \geq \mathbb{E}u(\tilde{y}+\tilde{z})$ for all $u \in U_{1,\phi}$, yielding $\tilde{x}+\tilde{z} \succeq_{1,\phi} \tilde{y}+\tilde{z}$, which implies $\tilde{x} \succeq_{1,\phi} \tilde{y}$ by additive invariance for non-dominance.

(ii) Let $\tilde{z}$ follow the density function $\frac{(\beta+1)z^{-(\beta+2)}}{1-\theta^{-(\beta+1)}}$, where $\theta = \bar{y}/\bar{x}$ and $z \in [1, \theta]$. Denote by H_1 and H_2 the cdfs of $\tilde{x}\tilde{z}$ and $\tilde{y}\tilde{z}$, respectively. We can obtain

$$
H_1(w) = \int_1^\theta \frac{(\beta+1)z^{-\beta-2}}{1-\theta^{-(\beta+1)}} F(w/z)dz = \frac{\beta}{1-\theta^{-(\beta+1)}} \int_{w/\theta}^{w} \frac{x^\beta}{w^{\beta+1}} F(x)dx
$$

and $H_2(w) = \frac{\beta}{1-\theta^{-(\beta+1)}} \int_{w/\theta}^{w} \frac{x^\beta}{w^{\beta+1}} G(x)dx$, leading to $H_2(w) - H_1(w) = \frac{\beta w^{-(\beta+1)}}{1-\theta^{-(\beta+1)}}[H(w) - H(w/\theta)]$, where $H(x) = \int_a^x [G(t) - F(t)]t^\beta dt$. Due to (A5) and the single-crossing property, we have $H(x) \geq 0$ for all $x \in \mathbb{R}$. Using integration by parts, we can verify

$$
\begin{aligned}
\mathbb{E}u(\tilde{x}\tilde{z}) - \mathbb{E}u(\tilde{y}\tilde{z}) &= \int_a^b [H_2(w) - H_1(w)]u'(w)dw \\
&\geq \frac{\beta}{1-\theta^{-(\beta+1)}} \int_a^{b/\theta} H(x)x^{-(\beta+1)}[u'(x) - \theta^{-\beta}u'(x\theta)]dx.
\end{aligned}
$$

By arguments similar to those in (i), we obtain $u'(x) \geq \theta^{-\beta}u'(x\theta)$ for all $u \in U_{1,\phi}$ and $\tilde{x}\tilde{z} \succeq_{1,\phi} \tilde{y}\tilde{z}$, which implies $\tilde{x} \succeq_{1,\phi} \tilde{y}$ by multiplicative invariance for non-dominance.

Proof of Results 6–7 for $N = 1$. We only prove the "only if" statement, as the "if" statement is straightforward. We first prove Result 6. Let $d = b - a > 0$. We know $\phi(x, y)$ has the form $\psi(x-y)$ from Result 3, and need to show $\psi(z) = e^{\beta z}$ for some constant β and all $z \in (-d, 0)$. Let $L(z) = \log\psi(z)/z$. Suppose for contradiction that $L(z)$ is not a constant on $(-d, 0)$, we claim that there exist $s,t \in (-d,0)$ satisfying $L(s) > L(t)$ and $|s+t| < d$. Note that there exist $r,t \in (-d,0)$ such that $L(r) > L(t)$, and maximality of ϕ implies for any $-d < s < 0$, $\phi(x,x-s) \geq \phi(x,x-s/2)\phi(x-s/2,x-s)$ (see Lemma 2), leading to $L(s) \leq L(s/2)$. If $|r+t| > d$, taking an integer k large enough and using $L(x/2) \geq L(x)$ repeatedly yields $L(r/2^k) \geq L(r) > L(t)$ and $|r/2^k + t| < d$. It then suffices to choose $s = r/2^k$. The above argument establishes the claim and guarantees that we can take $\hat{x} \in (a,b)$ such that the point $\hat{x} - s - t$ is also in (a,b). Let $\beta_1 = L(s)$ and $\beta_2 = L(t)$. Since $L(s) > L(t)$, there exists $\alpha \in (e^{\beta_1 s}, e^{\beta_2 s})$. Because of $\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[e^{\beta_2 \hat{x}} - e^{\beta_2(\hat{x}-\varepsilon)}] = \beta_2 e^{\beta_2 \hat{x}}$ and $\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[e^{\beta_2(\hat{x}-s+\alpha\varepsilon)} - e^{\beta_2(\hat{x}-s)}] = \alpha\beta_2 e^{\beta_2(\hat{x}-s)}$, it holds $e^{\beta_2\hat{x}} + e^{\beta_2(\hat{x}-s)} \geq e^{\beta_2(\hat{x}-\varepsilon)} + e^{\beta_2(\hat{x}-s+\alpha\varepsilon)}$ for $\varepsilon > 0$ small enough. That is, (A4) holds for $\tilde{x} \overset{d}{=} [\hat{x}; \hat{x}-s]$ and $\tilde{y} \overset{d}{=} [\hat{x}-\varepsilon; \hat{x}-s+\alpha\varepsilon]$ with $\beta = \beta_2$. Applying (i) of Lemma A2 with $\beta = \beta_2$ and $\bar{x} = \hat{x} - s$ and $\bar{y} = \hat{x} - s - t$, which uses the condition $\hat{x} - s - t \in (a,b)$, we have $[\hat{x}; \hat{x}-s] \succeq_{1,\phi} [\hat{x}-\varepsilon; \hat{x}-s+\alpha\varepsilon]$ and thus $\alpha \leq \phi(\hat{x}, \hat{x}-s) = \psi(s)$ by (ii) of Lemma A1, a contradiction to $\alpha > e^{\beta_1 s} = e^{L(s)s} = \psi(s)$. Result 7 follows by applying the same arguments as above, replacing exponential functions with power functions.

We now turn to proving Results 6–7 for general N, which require two additional lemmas.

Lemma A3. *Fix $x < y$. Let F_n and G_n be the cdfs of $\tilde{z}_n(x,y)$ and $\tilde{w}_n(x,y)$ in* (A3).

i) F_n and G_n are $(n-1)$th-degree single-crossing.

ii) $F_n^{(k)}(b) = G_n^{(k)}(b)$ for $k = 1, \dots, n-1$.

Proof. (i) Note that F_1 and G_1 are 0th-degree single-crossing. By induction, assume that F_n and G_n are $(n-1)$th-degree single-crossing. By definition, $2F_{n+1}(x) = F_n(x) + G_n(x-\varepsilon_n)$ and $2G_{n+1}(x) = F_n(x-\varepsilon_n) + G_n(x)$. By using $F_n^{(n-1)}(x) = G_n^{(n-1)}(x) = 0$ for $x \leq a$, we have

$$2[F_{n+1}^{(n)}(x) - G_{n+1}^{(n)}(x)] = \int_a^x \left[F_n^{(n-1)}(y) + G_n^{(n-1)}(y-\varepsilon_n) - F_n^{(n-1)}(y-\varepsilon_n) - G_n^{(n-1)}(y)\right] dy$$
$$= \int_{x-\varepsilon_n}^x \left[F_n^{(n-1)}(y) - G_n^{(n-1)}(y)\right] dy.$$

Since $F_n^{(n-1)}$ and $G_n^{(n-1)}$ are single-crossing, we may suppose that c is their crossing point. Then $F_{n+1}^{(n)}(x) - G_{n+1}^{(n)}(x)$ is monotonic on $[c, c+\varepsilon_n]$, and its sign remains unchanged for $x < c$ and for $x > c + \varepsilon_n$. It follows that $F_{n+1}^{(n)}$ and $G_{n+1}^{(n)}$ must also be single-crossing. (ii) Write $\tilde{z}_n = \tilde{z}_n(x,y)$ and

$\tilde{w}_n = \tilde{w}_n(x,y)$. First, $2\mathbb{E}\tilde{z}_2 = \mathbb{E}\tilde{z}_1 + \mathbb{E}[\tilde{w}_1 + \varepsilon_2] = \mathbb{E}[\tilde{z}_1 + \varepsilon_2] + \mathbb{E}\tilde{w}_1 = 2\mathbb{E}\tilde{w}_2$. Assuming that $\mathbb{E}\tilde{z}_n^j = \mathbb{E}\tilde{w}_n^j$ for $j = 1,\ldots,n-1$, we have for $k = 1,\ldots,n$ that

$$\begin{aligned}\mathbb{E}\tilde{z}_{n+1}^k - \mathbb{E}\tilde{w}_{n+1}^k &= \frac{1}{2}\mathbb{E}\tilde{z}_n^k + \frac{1}{2}\mathbb{E}(\tilde{w}_n + \varepsilon_{n+1})^k - \frac{1}{2}\mathbb{E}(\tilde{z}_n + \varepsilon_{n+1})^k - \frac{1}{2}\mathbb{E}\tilde{w}_n^k \\ &= \frac{1}{2}\sum_{i=1}^{k}\binom{k}{i}\varepsilon_{n+1}^i\left(\mathbb{E}\tilde{w}_n^{k-i} - \mathbb{E}\tilde{z}_n^{k-i}\right) = 0,\end{aligned}$$

where the last equality follows by induction. The equal moments lead to $F_n^{(k)}(b) = G_n^{(k)}(b)$.

The following lemma extends Lemma A2 to higher degrees. To avoid repetition, we only present the extension of part (i), since part (ii) is similar.

Lemma A4. *Let the cdfs of $\tilde{x}$ and $\tilde{y}$ be F and G, satisfying $G^{(n)}(b) = F^{(n)}(b)$ for $n = 1,\ldots,N-1$ and $F^{(N-1)}$ crossing $G^{(N-1)}$ once from below. Suppose additive invariance holds for both $\succeq_{N,\phi}$ and $\not\preceq_{N,\phi}$. If there exist $\beta > 0$ and $a \le \bar{x} < \bar{y} \le b$ such that $a \le \tilde{x},\tilde{y} \le b - (\bar{y} - \bar{x})$, $\phi(\bar{x},\bar{y}) = e^{\beta(\bar{x}-\bar{y})}$ and*

$$\int_a^b [G^{(N-1)}(x) - F^{(N-1)}(x)]e^{\beta x}dx \ge 0, \tag{A6}$$

then $\tilde{x} \succeq_{N,\phi} \tilde{y}$.

Proof. Let $\bar{b} = b - (\bar{y} - \bar{x})$. For $\tilde{x},\tilde{y}$ supported on $[a,\bar{b}]$ and $x \ge \bar{b}$, it follows from the identity $G^{(n)}(x) - F^{(n)}(x) = G^{(n)}(\bar{b}) - F^{(n)}(\bar{b}) + \sum_{k=1}^{n-1}[G^{(k)}(\bar{b}) - F^{(k)}(\bar{b})](x-\bar{b})^{n-k}/(n-k)!$ that conditions $G^{(n)}(b) = F^{(n)}(b)$ for $n = 1,\ldots,N-1$ and $a \le \tilde{x},\tilde{y} \le \bar{b}$ imply

$$G^{(n)}(x) - F^{(n)}(x) = 0 \quad \text{for all } x \in [\bar{b},b] \text{ and } n = 1,\ldots,N-1.$$

For a random variable $\tilde{w}$ with cdf W and $t > a$, we use $\tilde{w}_t^{(N-1)}$ to represent the random variable supported on $[a,t]$ whose cdf is $x \mapsto W^{(N-1)}(x)/W^{(N-1)}(t)$ for $x \in [a,t]$ (Denuit and Eeckhoudt 2010). Let $\tilde{z}$ follow the density function $\frac{\beta e^{-\beta z}}{1-e^{-\beta\theta}}$ for $z \in [0,\theta]$, where $\theta = \bar{y} - \bar{x}$. Similar reasoning to that proving (i) of Lemma A2 yields

$$\tilde{x}_{\bar{b}}^{(N-1)} + \tilde{z} \succeq_{1,\phi} \tilde{y}_{\bar{b}}^{(N-1)} + \tilde{z}. \tag{A7}$$

Let $M = F^{(N-1)}(\bar{b}) = G^{(N-1)}(\bar{b}) > 0$, and $\hat{F}$ and $\hat{G}$ be the cdfs of $\tilde{x}_{\bar{b}}^{(N-1)}$ and $\tilde{y}_{\bar{b}}^{(N-1)}$, respectively. Keep in mind that $\hat{F}(x) = F^{(N-1)}(x)/M$ and $\hat{G}(x) = G^{(N-1)}(x)/M$ for $x \le \bar{b}$, and $\hat{F}(x) = \hat{G}(x) = 1$ for $x > \bar{b}$. For all $x \in \mathbb{R}$, we have

$$\begin{aligned}
&\hat{F} * H(x) - \hat{G} * H(x) \\
=&\int_0^\theta [\hat{F}(x-y) - \hat{G}(x-y)]dH(y) = \frac{1}{M}\int_0^\theta \left[F^{(N-1)}(x-y) - G^{(N-1)}(x-y)\right] dH(y) \\
=&\frac{1}{M}[(F^{(N-1)} * H)(x) - (G^{(N-1)} * H)(x)] = \frac{1}{M}[(F*H)^{(N-1)}(x) - (G*H)^{(N-1)}(x)], \qquad \text{(A8)}
\end{aligned}$$

where the first and third equalities follow by definition, the second one uses $F^{(N-1)}(x-y) = G^{(N-1)}(x-y)$ whenever $x - y > \bar{b}$, and the final equality follows from (A2). Since $\tilde{x} + \tilde{z}$ and $\tilde{y} + \tilde{z}$ have the same first $(N-1)$ moments, we obtain

$$\begin{aligned}
\mathbb{E}u(\tilde{x}+\tilde{z}) - \mathbb{E}u(\tilde{y}+\tilde{z}) &= (F*H)^{(N-1)}(b)(-1)^{N-1}\left[\mathbb{E}u^{(N-1)}((\tilde{x}+\tilde{z})_b^{(N-1)}) - \mathbb{E}u^{(N-1)}((\tilde{y}+\tilde{z})_b^{(N-1)})\right] \\
&= F^{(N-1)}(b)(-1)^{N-1}\left[\mathbb{E}u^{(N-1)}(\tilde{x}_{\bar{b}}^{(N-1)} + \tilde{z}) - \mathbb{E}u^{(N-1)}(\tilde{y}_{\bar{b}}^{(N-1)} + \tilde{z})\right] \ge 0
\end{aligned}$$

for all $u \in U_{N,\phi}$, where the first equality follows from integration by parts and the cancellation of lower-order terms, the second one follows from (A8), and the inequality in the final step follows from $(-1)^{N-1}u^{(N-1)} \in U_{1,\phi}$ and (A7). This establishes $\tilde{x} + \tilde{z} \succeq_{N,\phi} \tilde{y} + \tilde{z}$, which further implies $\tilde{x} \succeq_{N,\phi} \tilde{y}$ by additive invariance for non-dominance.

Proof of Results 6–7 for general N. To prove Result 6, we first prove the "if" statement. Suppose $\tilde{z} + \tilde{x} \succeq_{N,\phi} \tilde{z} + \tilde{y}$. Denote by H the cdf of $\tilde{z}$. We have

$$\begin{aligned}
&\int_a^x \left[\int_a^b G^{(N-1)}(t-s)dH(s) - \int_a^b F^{(N-1)}(t-s)dH(s)\right] e^{(1/c-1)t}dt \ge 0 \quad \text{for all } x \in [a,b] \\
\Leftrightarrow &\int_a^b \left[\int_a^x \left[G^{(N-1)}(t-s) - F^{(N-1)}(t-s)\right] e^{(1/c-1)(t-s)}dt\right] e^{(1/c-1)s}dH(s) \ge 0 \quad \text{for all } x \in [a,b].
\end{aligned}$$

Since $G^{(N-1)}$ and $F^{(N-1)}$ are single-crossing, $\int_a^x G^{(N-1)}(t)e^{(1/c-1)t}dt$ and $\int_a^x F^{(N-1)}(t)e^{(1/c-1)t}dt$ are single-crossing. Similar arguments to those proving Theorem 1 yield (7), which together with (1) implies $\tilde{x} \succeq_{N,\phi} \tilde{y}$. Below we prove the "only if" statement. Similar to the proof of Result 6 for $N = 1$, let $d = b - a > 0$ and $L(z) = \log\psi(z)/z$, where $\phi(x,y) = \psi(x-y)$. Suppose for contradiction that $L(z)$ is not a constant. By similar arguments, there exist $s, t \in (-d, 0)$ and $\hat{x} \in (a, b)$

satisfying $L(s) > L(t)$ and $|s+t| < d$, and the point $\hat{x} - s - t$ is also in (a,b). Let $\beta_1 = L(s)$ and $\beta_2 = L(t)$. Since $L(s) > L(t)$, there exists $\alpha \in (e^{\beta_1 s}, e^{\beta_2 s})$. Letting $x_1 = \hat{x}$ and $y_1 = \hat{x} - s$, define $\tilde{z}_N(x_1, y_1)$ and $\tilde{w}_N(x_1, y_1)$ as in (A3), whose cdfs are denoted by F_N and G_N. Because of $\lim_{\varepsilon_1,\ldots,\varepsilon_N \to 0} \frac{1}{\varepsilon_1 \cdots \varepsilon_N} \left(\mathbb{E}e^{\beta_2 \tilde{z}_N} - \mathbb{E}e^{\beta_2 \tilde{w}_N}\right) = \beta_2^N \left(e^{\beta_2 x_1} - \alpha e^{\beta_2 y_1}\right) > 0$, and using integration by parts, $\int_a^b [G_N^{(N-1)}(x) - F_N^{(N-1)}(x)] e^{\beta_2 x} dx = \frac{1}{\beta_2} \left(\mathbb{E}e^{\beta_2 \tilde{z}_N} - \mathbb{E}e^{\beta_2 \tilde{w}_N}\right) \geq 0$ for $\varepsilon_1, \ldots, \varepsilon_N > 0$ small enough. That is, (A6) holds for $\tilde{x} \overset{d}{=} \tilde{z}_N(x_1, y_1)$ and $\tilde{y} \overset{d}{=} \tilde{w}_N(x_1, y_1)$. By Lemma A4, we have $\tilde{z}_N(x_1, y_1) \succeq_{N,\phi} \tilde{w}_N(x_1, y_1)$, and thus $\alpha \leq \phi(\hat{x}, \hat{x} - s) = \psi(s)$ follows from (ii) of Lemma A1, a contradiction to $\alpha > e^{\beta_1 s} = e^{L(s)s} = \psi(s)$. Result 7 can be proved in a similar way.

By the above steps, the proof of Theorem 2 is now completed. □

Proof of Lemma 4. First consider the "if" part. Let $\tilde{x}, \tilde{y}$ satisfy (11) and $u \in U_N$ with (10). Let $H^{(0)} = H = F - G$, and $H^{(n)}(x) = \int_{-\infty}^x H^{(n-1)}(t) dt$, $n \geq 1$. Based on (10) and that $\tilde{x}$ and $\tilde{y}$ have finite first N moments, we can verify that $\lim_{a \to -\infty} u^{(n)}(a) H^{(n)}(a) = 0$ for $n = 0, \ldots, N-1$, $\lim_{b \to +\infty} |u(b) H(b)| \leq \lim_{b \to +\infty} |\int_b^\infty u(x) dF(x)| + |\int_b^\infty u(x) dG(x)| = 0$, and $\lim_{b \to +\infty} u'(b) H^{(1)}(b) = -(\mathbb{E}\tilde{x} - \mathbb{E}\tilde{y}) \lim_{b \to +\infty} u'(b) \leq 0$. For $2 \leq n \leq N-1$, we have

$$\lim_{b \to +\infty} u^{(n)}(b) H^{(n)}(b) = \lim_{b \to +\infty} b^{n-1} u^{(n)}(b) \frac{G^{(n)}(b) - F^{(n)}(b)}{b^{n-1}} = \lim_{b \to +\infty} b^{n-1} u^{(n)}(b) \frac{\mathbb{E}\tilde{x} - \mathbb{E}\tilde{y}}{(n-1)!} = 0,$$

where the second equality follows from the L'Hospital principle. Using integration by parts N times and by the above four limits, we have

$$\mathbb{E}u(\tilde{x}) - \mathbb{E}u(\tilde{y}) = \lim_{\substack{a \to -\infty \\ b \to +\infty}} \int_a^b u(x) \mathrm{d}H(x) \geq \lim_{\substack{a \to -\infty \\ b \to +\infty}} \int_a^b (-1)^N u^{(N)}(x) H^{(N-1)}(x) \mathrm{d}x \geq 0.$$

This completes the proof of the "if" part. For the "only if" part, $\mathbb{E}\tilde{x} \geq \mathbb{E}\tilde{y}$ follows from that $u(x) = x$ belongs to U_N. For $x \in \mathbb{R}$, define $u_\varepsilon(t) = (2\varepsilon)^{-1} \int_{-\varepsilon}^{\varepsilon} (x + c - t)_+^{N-1} dc$ with $\varepsilon > 0$. We have $u_\varepsilon \in U_N$, and thus, $\mathbb{E}u_\varepsilon(\tilde{x}) \geq \mathbb{E}u_\varepsilon(\tilde{y})$ for any $\varepsilon > 0$. By $u_\varepsilon \to u_0$ uniformly as $\varepsilon \to 0$ with $u_0(t) := (x - t)_+^{N-1}$, we have $\mathbb{E}u_0(\tilde{x}) = \mathbb{E}(x - \tilde{x})_+^{N-1} \geq \mathbb{E}(x - \tilde{y})_+^{N-1} = \mathbb{E}u_0(\tilde{y})$, that is, (11) holds. □

Proof of Theorem 3. There are two directions to show. The first is to verify these invariance rules when the support of $\tilde{x}$ and $\tilde{y}$ is $\mathbb{R}$ or $\mathbb{R}_+$. This can be done by noting that the proof of Theorem 1 carries through verbatim when $\tilde{z}$ has a bounded support, and similarly for the fractional degree cases. The second is the construction of these special forms of maximal interpolators. In our proofs of Results 1 through 7, extending $[a,b]$ to $\mathbb{R}$ or $\mathbb{R}_+$ allows for the same arguments in all proofs to work because the random variables that we constructed on $[a,b]$ also live on $\mathbb{R}$ or $\mathbb{R}_+$. □

Proof of Proposition 1. Note that stochastic dominance with degrees up to three means considering $\succeq_{N,\phi}$ for $N \leq 2$. The independence of the integral conditions (6) to (8) of our rules from the choice of $[a,b]$ follows directly from the facts shown in Lemma 1. □

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