Prudence and higher-order risk attitudes in the rank-dependent utility model

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Abstract

We obtain a full characterization of consistency with respect to higher-order stochastic dominance within the rank-dependent utility model. Different from the results in the literature, we do not assume any condition on the utility functions and the probability weighting functions, such as differentiability or continuity. It turns out that the level of generality that we offer leads to models that do not have a continuous probability weighting function and yet they satisfy prudence. In particular, the corresponding probability weighting function can only have a jump at 1, and must be linear on $[0, 1)$.

Keywords: Stochastic dominance; expected utility model; completely monotone functions; probability weighting; discontinuity.

1 Introduction

Stochastic dominance is a widely used concept in economics, finance, and engineering for comparing different distributions of uncertain outcomes, particularly in the context of risk preferences in decision theory. Stochastic dominance is considered as a robust way of risk comparison as it allows for analysis without a specific utility function or preference model; see Levy (2015) and Shaked and Shanthikumar (2007).

The most popular stochastic dominance rules are the first-order stochastic dominance (FSD) and the second-order stochastic dominance (SSD). More recently, higher-order risk attitudes,[1] captured by higher-order stochastic dominance, become popular concepts in decision theory; see e.g., Eeckhoudt and Schlesinger (2006), Eeckhoudt et al. (2009), Crainich et al. (2013) and Deck and

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[1] By "higher-order" we meant an order that is larger than 2.

Schlesinger (2014). In this paper, we refer to higher-order risk attitudes as *consistency with higher-order stochastic dominance*. Among these attitudes, prudence (described by third-order stochastic dominance, TSD) is particularly significant as it relates to precautionary behavior, highlighting how prudence influences savings behavior when future income is uncertain, as shown by Kimball (1990). Through the concept of risk apportionment, Eeckhoudt and Schlesinger (2006) established elegant descriptions of consistency with respect to higher-order risk attitudes on preference relations. These studies emphasize the importance of prudence in modeling and analyzing economic behavior, making it a crucial component in the broader analysis of risk attitudes. Accurately characterizing higher-order risk attitudes, especially prudence, within different decision-making frameworks is thus important for a deeper understanding of behavior and decision under risk.

In this paper, our goal is to fully characterize higher-order risk attitudes in the rank-dependent utility (RDU) model, introduced by Quiggin (1982). The RDU model is one of the most popular models for decision under risk, and it serves as the building block for the cumulative prospect theory of Tversky and Kahneman (1992). The RDU models include both the expected utility (EU) model and the dual utility model of Yaari (1987) as special cases. Characterization of other notions of risk attitudes for RDU can be found in Chew et al. (1987), Wakker (1994), Ryan (2006), and Wang and Wu (2025a). We refer to Wakker (2010) for a general background on RDU and related decision models.

All EU models that are consistent with nth-order stochastic dominance are precisely those with an n-monotone utility function. This follows from a result of Müller (1997) and is reported in Proposition 1. Such functions have derivatives up to degree $n-2$, but not necessarily differentiable at degree $n-1$ or n.

In the RDU framework, it is straightforward to verify that, for a risk-averse decision maker (i.e., SSD consistent), the utility function must be concave, and the probability weighting function must be convex; see Chew et al. (1987), where some differentiability is assumed. The most relevant result on RDU with higher-order consistency is Theorem 2.1 of Eeckhoudt et al. (2020), which characterizes the expectation through consistency with TSD under the dual utility model, where the utility function is assumed to be the identity, and the probability weighting function is assumed differentiable up to arbitrary degree. This restriction reduces the class of potential probability weighting functions and offers technical convenience. The result of Eeckhoudt et al. (2020) is that among all probability weighting functions in the dual utility model, only the identity is consistent with TSD. In contrast, our main result, Theorem 1, shows that when differentiability is not assumed, there are more probability weighting functions that yield dual utility models consistent with TSD, and other higher-order risk attitudes. In particular, in the dual utility model, such probability weighting functions, except for the identity, are not continuous, but they are linear on $[0,1)$ and indexed by one parameter. The corresponding preference model is a mixture of an EU model and a worst-case, most pessimistic, RDU model. Our results unify existing theories and offers a clear way for evaluating preference relations that align with higher-order risk attitudes.

The rest of paper is organized as follows. In Section 2, we introduces the necessary notations and definitions. Section 3 presents the main results and discusses the characterization of other notions of risk attitudes for the RDU model found in the literature. All proofs are presented in Section 4.

2 Preliminaries

Let $a, b \in \mathbb{R}$ with $a < b$. We assume that all random variables take values in the interval $[a, b]$, and we denote this space of random variables as $\mathcal{X}_{[a,b]}$. Capital letters, such as X and Y, are used to represent random variables, and F and G for distribution functions.

For $X \in \mathcal{X}_{[a,b]}$, we write $\mathbb{E}[X]$ for the expectation of X. Denote by F_X and F_X^{-1} the distribution function and the left-quantile function of X, respectively, where we have the relation that $F_X^{-1}(s) = \inf\{x : F_X(x) \geq s\}$ for $s \in (0, 1]$. We use $\max(X)$ and $\min(X)$ to represent the essential supremum and the essential infimum of X, i.e., $\max(X) := F_X^{-1}(1)$ and $\min(X) := \inf\{x : F_X(x) > 0\}$. Denote by δ_η the point-mass at $\eta \in \mathbb{R}$. For a real-valued function f, let f'_- and f'_+ be the left and right derivative of f, respectively, and denote by $f^{(n)}$ the nth derivative for $n \in \mathbb{N}$. Whenever we use the notation f'_-, f'_+ and $f^{(n)}$, it is understood that they exist. We recall that the left derivative of a convex or concave function always exists (see e.g., Proposition A.4 of Föllmer and Schied (2016)). Denote by $[n] := \{1, \dots, n\}$ for $n \in \mathbb{N}$. In this paper, all terms like "increasing", "decreasing", "convex" and "concave" are in the weak sense.

A decision maker's preference relation $\succsim$ is a weak order[2] on $\mathcal{X}_{[a,b]}$, with asymmetric part $\succ$ and symmetric part $\sim$. For $X, Y \in \mathcal{X}_{[a,b]}$, $X \succsim Y$ means that X is at least as good as Y for the decision maker.

For a distribution function F, denote by $F^{[1]} = F$ and define

$$F^{[n]}(\eta) = \int_{-\infty}^{\eta} F^{[n-1]}(\xi)\mathrm{d}\xi, \quad \eta \in \mathbb{R} \text{ and } n \geq 2.$$

It is well-known that $F_X^{[n]}(\eta)$ is connected to the expectation of $(\eta - X)_+^n$ (see e.g., Proposition 1 of

[2]That is, for $X, Y, Z \in \mathcal{X}$, (a) either $X \succsim Y$ or $Y \succsim X$; (b) $X \succsim Y$ and $Y \succsim Z$ imply $X \succsim Z$.

Ogryczak and Ruszczyński (2001)):

$$F_X^{[n+1]}(\eta) = \frac{1}{n!}\mathbb{E}[(\eta - X)_+^n], \quad X \in \mathcal{X}_{[a,b]}, \ \eta \in \mathbb{R}, \ n \geq 1, \tag{1}$$

where $x_+ = \max\{0, x\}$ for $x \in \mathbb{R}$.

The following outlines the definitions of nth-order stochastic dominance.

Definition 1. For $n \in \mathbb{N}$, we say that X dominates Y in the sense of nth-order stochastic dominance (nSD), denoted by $X \geq_n Y$ or $F_X \geq_n F_Y$ if

$$F_X^{[n]}(\eta) \leq F_Y^{[n]}(\eta), \ \forall \eta \in [a,b] \text{ and } F_X^{[k]}(b) \leq F_Y^{[k]}(b) \text{ for } k \in [n]$$

or equivalently,

$$\mathbb{E}[(\eta - X)_+^{n-1}] \leq \mathbb{E}[(\eta - Y)_+^{n-1}], \ \forall \eta \in [a,b] \text{ and } \mathbb{E}[(b-X)^k] \leq \mathbb{E}[(b-Y)^k] \text{ for } k \in [n-1].$$

For $n \in \{1, 2, 3\}$, nSD corresponds to the well-known FSD, SSD, and TSD. The partial order $\geq_n$ for these cases is commonly written as $\geq_{\text{FSD}}$, $\geq_{\text{SSD}}$ or $\geq_{\text{TSD}}$. A direct conclusion is that nSD is stronger than $(n+1)$SD for $n \geq 1$, i.e., $X \geq_n Y$ implies $X \geq_{n+1} Y$.

We say that a preference relation $\succsim$ is *consistent with nSD* if $X \succsim Y$ for all $X, Y \in \mathcal{X}_{[a,b]}$ with $X \geq_n Y$. Intuitively, nSD compares two uncertain outcome, and consistency with nSD implies that the decision maker prefers the less risky outcome according to nSD. Specifically, consistency with FSD states that higher outcomes are always preferred. Consistency with SSD is related to risk aversion, defined as an aversion to mean-preserving spreads (see Rothschild and Stiglitz (1970)). Consistency with higher-order stochastic dominance accommodates decision makers who exhibit more refined preference relations, such as prudence when $n = 3$ (Kimball (1990)) and temperance when $n = 4$ (Kimball (1992)). Their preference descriptions are obtained by Eeckhoudt and Schlesinger (2006) via risk apportionment, which generalizes the idea of mean-preserving spreads.

In some literature, higher-order stochastic dominance is applicable to unbounded random variables; see Rolski (1976); Fishburn (1980); Shaked and Shanthikumar (2007). Contrary to Definition 1, this version is referred to as "unrestricted" stochastic dominance because it does not impose boundary conditions at point b, requiring instead that $F_X^{[n]}(\eta) \leq F_Y^{[n]}(\eta)$ for all $\eta \in \mathbb{R}$. The nSD in Definition 1 is a more stringent rule than its unrestricted counterpart, and while they provide the same comparisons of random variables within $\mathcal{X}_{[a,b]}$ for $n \leq 3$, distinction emerge for $n \geq 4$; see Wang and Wu (2025b). Note that the more stringent a stochastic dominance rule, the weaker its consistency property tends to be, leading to stronger characterization results derived from this consistency. Therefore, we opt for the restricted stochastic dominance in Definition 1 over the more lenient unrestricted version.

3 Characterization

3.1 Expected utility

Before understanding the consistency properties in rank-dependent utility models, one needs to understand the more basic expected utility (EU) model. First, we introduce some definitions below. A preference relation $\succsim$ admits an EU representation with $u : [a, b] \to \mathbb{R}$ if

$$X \succsim Y \iff \mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)].$$

To avoid trivial cases, we assume that u is nonconstant and the relevant set of u is defined as

$$\mathcal{U} = \{u : \mathbb{R} \to \mathbb{R} : \ u \text{ is nonconstant}\}.$$

Definition 2 (n-monotone functions)**.** Let $f : [a, b] \to \mathbb{R}$. For $n \geq 2$, we say that f is an n-monotone function if $(-1)^{k-1} f^{(k)} \geq 0$ for $k \in [n-2]$ and $(-1)^{n-1} f^{(n-2)}$ is decreasing and convex, where $f^{(k)}$ is the kth derivative of f and we assume that $f^{(0)} = f$. In particular, f is a 1-monotone function if it is increasing.

The class of n-monotone functions are useful in many fields. For instance, they fully describe all Archimedean copulas in statistics; see McNeil and Nešlehová (2009). For a mathematical treatment, see Williamson (1956). If a function f is n-monotone for all $n \in \mathbb{N}$, it is called completely monotone; this property is well studied in the mathematics literature and it is closely linked to Laplace–Stieltjes transforms; see e.g., Schoenberg (1938). Furthermore, Whitmore (1989) characterized the preference relations that admit an EU representation with all completely monotone utility functions.

Intuitively, an n-monotone function generalizes the notion of monotonicity beyond first-order (increasing functions) and second-order (concave functions) behavior. The next result shows that consistency with nSD in the EU framework can be characterized by all n-monotone functions.

Proposition 1. *Let $n \in \mathbb{N}$, and suppose that a preference relation $\succsim$ admits an EU representation with $u \in \mathcal{U}$. Then, $\succsim$ is consistent with nSD if and only if u is an n-monotone function.*

In Fishburn (1976), the congruent set U of utility functions for a stochastic dominance $\geq_{\mathrm{SD}}$

is defined as follows: $X \geq_{\mathrm{SD}} Y$ if and only if $\mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)]$ for all $u \in U$. Note that the congruent set for a stochastic dominance is not unique. There are several congruent sets for nSD that have been extensively studied (see e.g., Denuit and Eeckhoudt (2013) and Section 4.A.7 of Shaked and Shanthikumar (2007)). Proposition 1 identifies the largest congruent set of nSD within the class of all utility functions.

3.2 Rank-dependent utility

Next, we present the definition of the *rank-dependent utility* (RDU) model (Quiggin (1982)). An RDU function incorporates an increasing utility function $u \in \mathcal{U}$ and a probability weighting function h that is an element of the following set:

$$\mathcal{H} = \{h : [0,1] \to [0,1] : \ h \text{ is increasing}, \ h(0) = 0, \ \text{and } h(1) = 1\},$$

and it has the form

$$R_{u,h}(X) = \int_0^\infty h \circ \overline{F}_{u(X)}(\eta)\mathrm{d}\eta + \int_{-\infty}^0 (h \circ \overline{F}_{u(X)}(\eta) - 1)\mathrm{d}\eta,$$

where $\overline{F} = 1 - F$ is the survival function. If h is the identity function, then the RDU model reduces to the expected utility. On the other hand, if u is the identity, then the RDU model is the *dual utility* (Yaari (1987)) that has the following definition:

$$I_h(X) = \int_0^\infty h \circ \overline{F}_X(\eta)\mathrm{d}\eta + \int_{-\infty}^0 (h \circ \overline{F}_X(\eta) - 1)\mathrm{d}\eta,$$

For simplicity, we consider $R_{u,h}(F_X)$ (resp. $I_h(F_X)$) and $R_{u,h}(X)$ (resp. $I_h(X)$) to be identical.

A preference relation $\succsim$ admits an RDU representation with $u \in \mathcal{U}$ and $h \in \mathcal{H}$ if

$$X \succsim Y \iff R_{u,h}(X) \geq R_{u,h}(Y).$$

It is straightforward to see that a preference relation that admits an RDU representation satisfies consistency with FSD. Moreover, if u and h are both differentiable, then consistency with SSD holds if and only if u is concave and h is convex; see Chew et al. (1987). The following theorem is our main result, which establishes the characterization of consistency with higher-order stochastic dominance in the RDU model.

Theorem 1. *Let $n \geq 3$, and suppose that a preference relation admits an RDU representation with $u \in \mathcal{U}$ and $h \in \mathcal{H}$. Then, $\succsim$ is consistent with nSD if and only if the one of the two following cases* *hold:*

(i) $u \in \mathcal{U}$ *is increasing and* $h(s) = \mathbb{1}_{\{s=1\}}$ *for all* $s \in [0,1]$*. In this case,* $R_{u,h}(X) = \min u(X)$ *for all* $X \in \mathcal{X}_{[a,b]}$*.*

(ii) u *is an* n*-monotone function and* $h(s) = \lambda s \mathbb{1}_{\{s<1\}} + \mathbb{1}_{\{s=1\}}$ *for all* $s \in [0,1]$ *with some* $\lambda \in (0,1]$*. In this case,* $R_{u,h}(X) = \lambda \mathbb{E}[u(X)] + (1-\lambda) \min u(X)$ *for all* $X \in \mathcal{X}_{[a,b]}$*.*

Remark 1. In Theorem 1, the two cases can be combined under a strict monotonicity condition. Suppose that a preference relation $\succsim$ that admits an RDU model is monotone for constant, that is, $c > d$ implies $c \succ d$. Then $\succsim$ is consistent with nSD if and only if it can be represented by some strictly increasing n-monotone function u and $h(s) = \lambda s \mathbb{1}_{\{s<1\}} + \mathbb{1}_{\{s=1\}}$ for all $s \in [0,1]$ with some $\lambda \in [0,1]$. This is because in case (i), choosing different strictly increasing functions u leads to the same preference relation.

Prudence, which reflects a precautionary saving behavior (Kimball (1990)), corresponds to consistency with TSD. Theorem 1 includes as a special case the characterization of prudent decision makers under the RDU framework with general utility functions and probability weighting functions. In contrast to Eeckhoudt et al. (2020), who showed that only the identity weighting function is consistent with TSD under some smoothness assumptions, our result characterizes a class of preference relations that can be represented as convex combinations of an EU model and a worst-case pessimistic RDU model, offering a nontrivial extension. This is an instance where smoothness assumptions exclude some economically meaningful models.

As an immediate consequence of Theorem 1, if a preference relation $\succsim$ is represented by

$$X \succsim Y \iff \lambda \mathbb{E}[u(X)] + (1-\lambda) \min v(X) \geq \lambda \mathbb{E}[u(Y)] + (1-\lambda) \min v(Y),$$

where u is n-monotone and $v \in \mathcal{U}$ is increasing, then $\succsim$ is consistent with nSD. Note that this preference relation cannot be represented by the RDU model unless $u = v$. More generally, a preference relation represented by a mixture of several RDU functions $R_{u,h}$ that are consistent with nSD is again consistent with nSD, although it does not necessarily admit an RDU model representation.

3.3 Discussion

Next, we discuss our characterization results and other notions of risk attitudes for the RDU model in the literature. Consistency with respect to each risk attitude corresponds to a set $\mathcal{M} \subseteq$

$\mathcal{U} \times \mathcal{H}$ of pairs of utility functions and probability weighting functions. For many risk attitudes, the set $\mathcal{M}$, has a *separable form*; that is, it imposes conditions on $u \in \mathcal{U}$ and $h \in \mathcal{H}$ separately, except for the trivial case that $h_{\mathbb{1}}(s) = \mathbb{1}_{\{s=1\}}$ (in this case, u does not matter). We write this separable form as $\mathcal{U}_* \times \mathcal{H}_*$, which means

$$\mathcal{M} = (\mathcal{U}_* \times \mathcal{H}_*) \cup (\mathcal{U} \times \{h_{\mathbb{1}}\}).$$

Remarkably, there are some notions of attitude that impose a joint condition on the interplay of u and h. In what follows, RA stands for risk aversion.

1. Our Theorem 1 demonstrates that the set $\mathcal{M}$ corresponding to consistency with higher-order stochastic dominance has a separable form, with $\mathcal{U}_*$ being the set of n-monotone functions, and the $\mathcal{H}_*$ being linear on $[0, 1)$.

2. The set $\mathcal{M}$ corresponding to consistency with SSD has a separable form; see Chew et al. (1987) and Ryan (2006). In this case, $\mathcal{U}_*$ is the set of increasing and concave elements of $\mathcal{U}$, and $\mathcal{H}_*$ is the set of all convex elements of $\mathcal{H}$. This consistency is also known as strong RA.

3. The case of FSD is trivial as all RDU models are consistent with FSD.

4. Probabilistic risk aversion (P-RA) in the RDU model means quasi-convexity of the RDU functional with respect to distribution functions; see Wakker (1994). As shown by Wang and Wu (2025a), the set $\mathcal{M}$ corresponding to P-RA has a separable form, in which $\mathcal{U}_* = \mathcal{U}$ and $\mathcal{H}_*$ is slightly larger than the set of convex probability weighting functions.

5. Next, we discuss some notions of risk attitudes whose characterization in RDU leads to joint conditions on u and h. Chateauneuf et al. (2005) studied the characterization of monotone RA in the RDU model.[3] They showed that, under the assumption that u is continuous and strictly increasing, and h is strictly increasing, the characterization of this consistency property is $G_u \leq P_h$, where G_u and P_h are the *index of greediness* for u and *the index of pessimism* for h, defined as

$$G_u = \sup_{a \leq x_1 < x_2 \leq x_3 < x_4 \leq b} \frac{u(x_4) - u(x_3)}{x_4 - x_3} \Big/ \frac{u(x_2) - u(x_1)}{x_2 - x_1}; \quad P_h = \inf_{0<s<1} \frac{1 - h(s)}{1 - s} \Big/ \frac{h(s)}{s}. \quad (2)$$

[3] For two random variables $X, Y : \Omega \to \mathbb{R}$, we say that Y is a monotone mean-preserving increase in risk of X if there exists a random variable $Z : \Omega \to \mathbb{R}$ with $\mathbb{E}[Z] = 0$ such that X and Z are comonotone and Y has the same distribution function as $X + Z$, where the statement that X and Z are comonotone means that $(X(\omega) - X(\omega'))(Z(\omega) - Z(\omega')) \geq 0$ for all $\omega, \omega' \in \Omega$. A preference relation $\succsim$ exhibits monotone RA if $X \succsim Y$ whenever Y is a monotone mean-preserving increase in risk of X.

The condition $G_u \leq P_h$ clearly illustrates the interplay between u and h.

6. Two notions of fractional SD were introduced by Müller et al. (2017) and Huang et al. (2020). Let us focus on the most relevant cases of fractional SD between first-order and second-order SD. For the notion of Müller et al. (2017) with parameter $\gamma \in (0,1)$, denoted f-γ-SD,[4] Mao and Wang (2022) showed that (under continuity) the consistency in RDU is equivalent to $Q_h \geq \gamma G_u$, where G_u is given in (2) and Q_h is given by

$$Q_h = \inf_{0 \leq s_1 < s_2 \leq s_3 < s_4 \leq 1} \frac{h(s_4) - h(s_3)}{s_4 - s_3} \Big/ \frac{h(s_2) - h(s_1)}{s_2 - s_1}.$$

Hence, the set $\mathcal{M}$ does not have a separable form. However, for the notion of Huang et al. (2020) with parameter $c \in (0,1)$, denoted f-c-SD,[5] Mao and Wang (2022) showed that the set $\mathcal{M}$ has a separable form, where $\mathcal{U}_*$ contains u with $x \mapsto u(c\log(x)/(1-c))$ being concave and $\mathcal{H}_*$ contains all convex elements of $\mathcal{H}$.

7. To the best of our knowledge, a full characterization of weak RA[6] in the RDU model has not been established in the literature; see Cohen (1995) for some sufficient conditions. The existing results imply that $\mathcal{M}$ does not have a separable form.

We summarize in Table 1 the above cases. To systemically understand which notions of risk attitude leads to a separable form of $\mathcal{M}$ seems not clear at this point.

4 Proofs

4.1 Proposition 1

Define

$$\mathcal{U}_n = \{u \mid u(x) = (\eta - x)_+^{n-1}, \eta \in [a,b]\} \cup \{u \mid u(x) = (b-x)^k, k \in [n-1]\}.$$

By Definition 1, we know that $X \geq_n Y$ if and only if $\mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)]$ for all $u \in \mathcal{U}_n$. Note that each $u \in \mathcal{U}_n$ is an n-monotone function. Moreover, the set of all n-monotone functions is a convex

[4] We say that X dominates Y under f-γ-SD if $\mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)]$ for all functions u satisfying $0 \leq \gamma u'(y) \leq u'(x)$ for all $x \leq y$.

[5] We say that X dominates Y under f-c-SD if $\mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)]$ for all functions u satisfying $u'(x) > 0$ and $u''(x)/u'(x) \leq 1/c - 1$ for all x.

[6] We say that a decision maker with a preference relation $\succsim$ exhibits weak RA if $\mathbb{E}[X] \succsim X$ for all random variables X.

Risk attitude	$\mathcal{M}$ separable?	Source
FSD	YES	definition
SSD	YES	Chew et al. (1987); Ryan (2006)
nSD	YES	Theorem 1
P-RA	YES	Wakker (1994); Wang and Wu (2025a)
M-RA	NO	Chateauneuf et al. (2005)
f-γ-SD	NO	Mao and Wang (2022)
f-c-SD	YES	Mao and Wang (2022)
weak RA	NO	Cohen (1995) (not fully characterized)

Table 1: Summary of whether $\mathcal{M}$ has a separable form; some results imposed regularity conditions.

cone and closed with respect to pointwise convergence. Hence, the result follows immediately from Corollary 3.8 of Müller (1997). □

4.2 Theorem 1

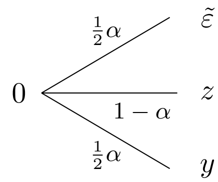
In this proof, we will encounter simple random variables. An explicit representation of $R_{h,u}$ with simple random variables is given below. For $X \in \mathcal{X}_{[a,b]}$ with the distribution $F_X = \sum_{i=1}^n p_i \delta_{x_i}$ where $x_1 \geq \cdots \geq x_n$, $p_1, \ldots, p_n \geq 0$ and $\sum_{i=1}^n p_i = 1$, it holds that

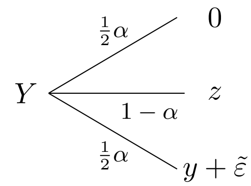
$$R_{u,h}(X) = \sum_{i=1}^{n} (h(p_1 + \cdots + p_i) - h(p_1 + \cdots + p_{i-1}))u(x_i).$$

The necessity statement of Theorem 1 is the most challenging. To establish it, we introduce a useful lemma that outlines the constraints on h.

Lemma 1. *Let $n \geq 3$, and let $u \in \mathcal{U}$ and $h \in \mathcal{H}$. If the RDU function $R_{u,h} : \mathcal{X}_{[a,b]} \to \mathbb{R}$ is consistent with nSD, then $h(s) = \lambda s \mathbb{1}_{\{s<1\}} + \mathbb{1}_{\{s=1\}}$ for all $s \in [0, 1]$ with some $\lambda \in [0, 1]$.*

Proof of Lemma 1. Note that a larger n corresponds to a weaker form of nSD, which in turn corresponds to a stronger consistency condition. Therefore, it suffices to verify the result for the case of TSD, i.e., $n = 3$. Suppose that $R_{u,h} : \mathcal{X}_{[a,b]} \to \mathbb{R}$ is consistent with TSD. Note that consistency with TSD is stronger than consistency with SSD. By Corollary 12 of Ryan (2006), we know that one of the following cases holds: (a) u is increasing and concave, and h is continuous and convex; (b) $u \in \mathcal{U}$ and $h(s) = \lambda s \mathbb{1}_{\{s<1\}} + \mathbb{1}_{\{s=1\}}$ for all $s \in [0, 1]$ with some $\lambda \in [0, 1)$. Case (b) is included in this lemma. Suppose now that Case (a) holds. We aim to show that h is an identity function on $[0,1]$. Note that $u \in \mathcal{U}$ is increasing and concave. We assume without loss of generality that $a < 0 < b$ and $u'(0) > 0$ where u' is the derivative of u. For $n \in \mathbb{N}$, $\alpha \in (0,1)$ and $y, z, \epsilon \in \mathbb{R}$ with $0 < y \le \epsilon$ and $a \le z < -n(n-1)\epsilon$ and $y + n\epsilon \le b$, we construct two simple random variables as follows:





where $\widetilde{\epsilon}$ is a two-point random variable with a zero mean and the distribution of the form $F_{\widetilde{\epsilon}} = (1-1/n)\delta_{n\epsilon} + (1/n)\delta_{-n(n-1)\epsilon}$. Specifically, we have

$$F_X = \left(\frac{1}{2} - \frac{1}{2n}\right)\alpha\delta_{n\epsilon} + \frac{1}{2n}\alpha\delta_{-n(n-1)\epsilon} + \frac{1}{2}\alpha\delta_y + (1-\alpha)\delta_z$$

and

$$F_Y = \left(\frac{1}{2} - \frac{1}{2n}\right)\alpha\delta_{y+n\epsilon} + \frac{1}{2n}\alpha\delta_{y-n(n-1)\epsilon} + \frac{1}{2}\alpha\delta_0 + (1-\alpha)\delta_z.$$

Note that $y - \epsilon \le 0 < y$ and $z < -n(n-1)\epsilon$, and we have

$$b \ge y + n\epsilon > n\epsilon > y > 0 > y - n(n-1)\epsilon > -n(n-1)\epsilon > z \ge a,$$

which implies $X, Y \in \mathcal{X}_{[a,b]}$. It is straightforward to check that $X \le_{\mathrm{TSD}} Y$ (see e.g., Crainich et al. (2013)). Denote by $a_n = (1-1/n)/2$ and $b_n = 1 - 1/(2n)$. It holds that

$$\begin{aligned} R_{u,h}(X) = {} & u(n\epsilon)h(\alpha a_n) + u(y)(h(\alpha b_n) - h(\alpha a_n)) \\ & + u(-n(n-1)\epsilon)(h(\alpha) - h(\alpha b_n)) + u(z)(1-h(\alpha)) \end{aligned}$$

and

$$\begin{aligned} R_{u,h}(Y) = {} & u(y+n\epsilon)h(\alpha a_n) + u(0)(h(\alpha b_n) - h(\alpha a_n)) \\ & + u(y - n(n-1)\epsilon)(h(\alpha) - h(\alpha b_n)) + u(z)(1-h(\alpha)). \end{aligned}$$

Since $R_{u,h}$ is consistent with TSD, we have $R_{u,h}(X) \le R_{u,h}(Y)$, and hence,

$$\begin{aligned}&(u(y+n\epsilon)-u(n\epsilon))h(\alpha a_n)+(u(y-n(n-1)\epsilon)-u(-n(n-1)\epsilon))(h(\alpha)-h(\alpha b_n))\\&\ge (u(y)-u(0))(h(\alpha b_n)-h(\alpha a_n)).\end{aligned}$$

Letting $y \downarrow 0$, it holds that for all $\alpha \in (0,1)$, $n \ge 1$ and sufficiently small $\epsilon > 0$,

$$u'_+(n\epsilon)h(\alpha a_n)+u'_+(-n(n-1)\epsilon)(h(\alpha)-h(\alpha b_n)) \ge u'(0)(h(\alpha b_n)-h(\alpha a_n)),$$

where u'_+ is the right-derivative of u. Letting $\epsilon \downarrow 0$ in the above equation and noting that $u'(0) > 0$, we have

$$h(\alpha a_n)+h(\alpha)-h(\alpha b_n) \ge h(\alpha b_n)-h(\alpha a_n).$$

With the relation that $b_n - a_n = 1/2$ for all $n \ge 1$, it follows that

$$\frac{h(\alpha)}{\alpha} \ge \frac{h(\alpha b_n)-h(\alpha a_n)}{\alpha(b_n-a_n)}, \quad \forall \alpha \in (0,1),\ n \ge 1.$$

Next, letting $n \to \infty$ and noting that h is continuous in Case (a) yields

$$\frac{h(\alpha)}{\alpha} \ge \frac{h(\alpha)-h(\alpha/2)}{\alpha/2}, \quad \forall \alpha \in (0,1).$$

On the other hand, using the convexity of h and $h(0) = 0$ yields

$$\frac{h(\alpha)}{\alpha} \le \frac{h(\alpha)-h(\alpha/2)}{\alpha/2}, \quad \forall \alpha \in (0,1).$$

Therefore, we have concluded that $h(\alpha) = 2h(\alpha/2)$ for all $\alpha \in (0,1)$. Since h is continuous and convex on $[0,1]$, it is straightforward to verify that $h(s) = s$ for $s \in [0,1]$. This completes the proof. □

Proof of Theorem 1. **Sufficiency.** Case (i) is trivial because u is increasing and the mapping $X \mapsto \min X$ is consistent with nSD. Suppose now Case (ii) holds. Proposition 1 implies that the mapping $X \mapsto \mathbb{E}[u(X)]$ is consistent with nSD. Combining with the result in Case (i), we have that $R_{u,h}$ is consistent with nSD.

Necessity. By Lemma 1, we know that $h(s) = \lambda s \mathbb{1}_{\{s<1\}} + \mathbb{1}_{\{s=1\}}$ for all $s \in [0,1]$ with some

$\lambda \in [0,1]$. Hence,

$$R_{u,h}(X) = \lambda \mathbb{E}[u(X)] + (1-\lambda) \min u(X), \quad X \in \mathcal{X}_{[a,b]}.$$

If $\lambda = 0$, then $R_{u,h}$ has the form in Case (i). If $\lambda > 0$, we will verify that $X \mapsto \mathbb{E}[u(X)]$ is consistent with nSD. For $X, Y \in \mathcal{X}_{[a,b]}$ with $X \le_n Y$, define $X', Y' \in \mathcal{X}_{[a,b]}$ with their distributions as follows:

$$F_{X'} = \frac{1}{2}\delta_a + \frac{1}{2}F_X \quad \text{and} \quad F_{Y'} = \frac{1}{2}\delta_a + \frac{1}{2}F_Y.$$

It is straightforward to check that $X' \le_n Y'$ and $\min X' = \min Y' = a$. Since $R_{u,h}$ is consistent with nSD, we have

$$0 \le R_{u,h}(Y') - R_{u,h}(X') = \lambda(\mathbb{E}[u(Y')] - \mathbb{E}[u(X')]) = \frac{\lambda}{2}(\mathbb{E}[u(Y)] - \mathbb{E}[u(X)]).$$

Hence, we have verified that $X \mapsto \mathbb{E}[u(X)]$ is consistent with nSD, and Proposition 1 shows that u is an n-monotone function. This completes the proof of necessity. □

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