Star-shaped Risk Measures

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In this paper monetary risk measures that are positively superhomogeneous, called *star-shaped risk measures*, are characterized and their properties are studied. The measures in this class, which arise when the subadditivity property of coherent risk measures is dispensed with and positive homogeneity is weakened, include all practically used risk measures, in particular, both convex risk measures and Value-at-Risk. From a financial viewpoint, our relaxation of convexity is necessary to quantify the capital requirements for risk exposure in the presence of liquidity risk, competitive delegation, or robust aggregation mechanisms. From a decision theoretical perspective, star-shaped risk measures emerge from variational preferences when risk mitigation strategies can be adopted by a rational decision maker.

Key words: Convexity, capital charge, liquidity risk, competitive pricing, monotonicity along rays

1. Introduction

The calculation of regulatory capital requirements is critical for the well-functioning of the financial sector. On this important issue, Artzner et al. [2] redefined the way of thinking about risk measurement. In their language, a risk measure ρ associates to every uncertain future loss X a capital requirement $\rho(X)$. This quantity is interpreted as the minimum amount of money a financial institution, which holds a position described by X, must have in order to absorb the realized losses and keep solvent in good times and bad. For instance, Value-at-Risk (VaR) at level 99% is the minimum amount of money that shields against insolvency (losses greater than reserves) in at least the 99% of cases.

A *coherent risk measure* of Artzner et al. [2] is a risk measure which satisfies monotonicity, translation invariance, normalization, positive homogeneity and subadditivity (to be defined formally in

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Section 3). All of these properties are motivated by the interpretation of $\rho(X)$ as a buffer against future losses, possibly imposed on a financial institution by an (internal or external) regulator. The *monotonicity* property, just says that higher losses require higher reserves. In the presence of a market of riskless bonds, *translation invariance* establishes that the acquisition of a claim with sure payoff reduces by the same amount the risk of the position previously held. The meaning of *normalization*, i.e., $\rho(0)=0$, is self-evident (and it is implied by the other properties). These first three properties are broadly accepted, and we assume them throughout when we speak of *risk measures.*[1]

On the other hand, since the very beginning, *positive homogeneity* has been object of concerns, and it actually motivated the introduction of convex risk measures by Föllmer and Schied [21, p. 430] and Frittelli and Rosazza Gianin [25, p. 1475]. To justify positive homogeneity, Artzner et al. [2, p. 209] argue as follows: for each risk X, it must be the case that

$$\rho(\lambda X) \geq \lambda \rho(X) \qquad \forall \lambda > 1 \tag{$\bigstar$}$$

because additional liquidity risk may arise if a position is multiplied by a large factor. The "star shape" property ($\bigstar$) has a simple and straight economic motivation, but under subadditivity, it is equivalent to positive homogeneity. As such, subadditivity tends to ignore concentration effects by forcing the inequality in ($\bigstar$) to be an equality, which may be an unintended consequence. This emphasizes the problematic nature of subadditivity in some specific contexts. Section 2 presents several examples where a strict inequality in ($\bigstar$) is compelling.

The present paper studies the theory of risk measures when subadditivity is dispensed with, and positive homogeneity replaced with the genuinely weaker star shapedness. Throughout the paper, a map ρ is *star-shaped* if it satisfies condition ($\bigstar$).

As noted by both Frittelli and Rosazza Gianin [25] and Föllmer and Schied [21], convex risk measures satisfy this property, which might be called "convexity at 0" and has also been called "positive superhomogeneity" for obvious mathematical reasons. Like convexity, star shapedness penalizes concentration of risk and the ensuing liquidity problems (the risk-to-exposure ratio of any position is weakly increasing; see Proposition 1), but differently from convexity, star shapedness does not take stance on the effects of "merge-and-downsize" strategies. More precisely, star shapedness reflects the idea that downsizing reduces risk more than proportionally, and on top of that, convexity further requires that the risk-diminishing effects of downsizing more than compensate any possible additional risk entailed by a merger.

Our choice of the name "star-shaped" over its competitors, is motivated by the fact that a risk measure is star-shaped if, and only if, the set of acceptable risky positions it determines —that is, the set of positions that do not require any accumulation of capital— is star-shaped. The meaning is transparent: reducing the exposure in an acceptable position cannot make it unacceptable. The implications are relevant: since both cones and convex sets that contain the origin are star-shaped, both positively homogeneous risk measures, such as Value-at-Risk, and convex risk measures, such as Expected Shortfall, are star-shaped. Star-shaped sets and functions are well studied in mathematics; see, e.g., Rubinov and Yagubov [46].

It is safe to say that star-shaped risk measures encompass virtually all monetary (i.e., monotone and translation invariant) risk measures used in the literature and in the financial practice. Perhaps more importantly, star-shaped risk measures, which are in general not convex, emerge in a number of relevant situations in risk management. In the next Section 2, we present several important examples illustrating the natural appearance of these risk measures when computing margin charges in the presence of liquidity risk and competitive allocation, when different capital requirements are robustly aggregated, when risk mitigation is possible for a decision maker, and when a non-concave utility function is involved in risk assessment.

The rest of the paper is dedicated to a comprehensive study of star-shaped risk measures. Our main contributions are briefly summarized below.

In a style similar to the axiomatic foundation of risk measures due to Artzner et al. [2] and Föllmer and Schied [22], the main theory of star-shaped risk measures is developed in Sections 3 (basic definitions), 4 (aggregative operations), and 5 (representation theorems). On the mathematical side, we obtain an insightful minimax representation of star-shaped risk measures as minima of convex risk measures (Theorem 2). *This fact extends to any star-shaped risk measure the elegant representation of Value-at-Risk as the minimum of all convex risk measures that dominate it*, and actually characterizes star-shaped risk measures through this envelope property. We also show that the class of star-shaped risk measures is closed under the fundamental aggregative operations of taking infima, suprema, mixtures, and inf-convolutions (Theorem 1), and that the corresponding minimax representations admit very tractable formulas (Theorem 3). Section 6 studies the optimization properties of star-shaped risk measures, while law-invariant star-shaped risk measures are analyzed in Section 7. All proofs are relegated to the Appendix.

With this, we provide the general framework for star-shaped risk measures, which, for the first time, unifies many risk measurement approaches that up to the present day were considered irreconcilable: convex risk measures, Value-at-Risk, and robust aggregation of risk measures; moreover, we connect them to competitive risk sharing and to risk mitigation. Given the considerable flexibility of this framework, many future research directions are possible, and some of them are described in the concluding Section 8. In this paper, we use the convention that a positive value of the realized random risk X means a loss (i.e., a sign change from Artzner et al. [2]).

2. Four motivating examples

2.1. Margin requirements with liquidity risk and multiple CCPs

In order to stabilize markets after the 2007-2008 financial crisis, Dodd-Frank reform act in US and EMIR regulation in Europe impose that in many derivative markets transactions are cleared by specialized intermediaries, the so-called Central Counterparties (CCP herafter). In order to reduce their exposure to insolvency risk, CCPs set a capital charge, the margin requirement, to each market participant. Glassermann, Moallemi, and Yuan [29] discuss margin settlement and explain that, to capture liquidity costs at default, margin requirements at the level of individual CCP need to be superlinear in position size. In the language of risk measures, this is equivalent to requiring that the capital charge is star-shaped with respect to the position size (see Proposition 1), i.e., the individual CCP will set a risk capital charge using a star-shaped risk measure.

As observed by [29], usually dealers have access simultaneously to multiple competing CCPs and may split the execution of a single deal across different CCPs. Next, we illustrate that in this setting, the effective margin charge is star-shaped, but it may be non-convex even if each risk measure used by CCPs is convex. Suppose that there are several CCPs that may clear the position X of a dealer, and assume that these CCPs are associated with convex (hence star-shaped) risk measures $\rho_1, \dots, \rho_n$. If all of the CCPs are simultaneously accessible, as in [29, (1)], the dealer would need to solve

$$\min_{X_1,\dots,X_n \in \mathcal{X}} \left\{ \sum_{i=1}^{n} \rho_i(X_i) \;\middle|\; \sum_{i=1}^{n} X_i = X \right\}$$

which is known as the operation of inf-convolution, discussed in Section 4. In practice, it may be infeasible or operationally costly to trade with all CCPs at once; the dealer may face constraints on the configuration of CCPs that can be involved for a specific deal.[2] Let $\mathcal{A}$ be the set of non-empty subsets of $\{1, \dots, n\}$ representing compositions of CCPs the dealer can clear her position with. This leads to the dealer's minimization problem

$$\min_{A \in \mathcal{A}} \min_{\substack{X_i \in \mathcal{X} \\ \text{s.t. } i \in A}} \left\{ \sum_{i \in A} \rho_i(X_i) \;\middle|\; \sum_{i \in A} X_i = X \right\},$$

and its minimum value, denoted by $\rho(X)$, is the effective margin charge for the position X. We can check that ρ is a star-shaped risk measure but not convex in general, because the minimum and the inf-convolution of convex risk measures are star-shaped (see Theorem 1) but not necessarily convex due to the minimum operation over $\mathcal{A}$. In particular, if $\mathcal{A}$ contains only all sets with one element, then ρ is the minimum of $\rho_1, \dots, \rho_n$.

2.2. Aggregation of convex risk measures and the ECB asset purchase program

Assume that a supervising agency consists of a board I of experts, and assume that each expert $i \in I$, based on her information and uncertainty attitudes, proposes a convex risk measure ρ_i. Now the agency has to aggregate these opinions, and of course any weighted average

$$\rho_\mu(X) = \sum_{i \in I} \mu_i \rho_i(X)$$

is again a convex risk measure, where by a weighted average we mean that the μ_is are non-negative and adding up to 1. At the same time, it is well-known that averaging is sensitive to outliers, and taking for instance a median is a popular robust alternative.

Simple calculations show that a median, any order statistics, and any L-estimator of convex risk measures is a star-shaped risk measure, which is typically not convex.

To make the example applied to an economic context, we consider the capital charge design in the European Central Bank (ECB) asset purchase program, an unconventional monetary policy with which the ECB fosters the purchase of assets (issued both by private and public institutions) to achieve the target inflation rate.

Let I be the set of all national central banks of the Euro-zone that participate in the program. The program is designed according to a skin-in-the-game mechanism: a fraction of the losses deriving from a default of the issuer of a purchased bond falls under a full European risk-sharing regime, and the relevant national central banks bear the remaining losses.

How should the ECB define the capital charge required to face the resulting risk? Assume that each national regulator $i \in I$ adopts a convex risk measure ρ_i to assess the potential losses arising from a portfolio X of purchased assets. We consider the selection a proper capital charge by the ECB for each purchase. In the extreme case of full loss sharing, the precautionary amount

$$\rho_\vee(X) = \max_{i \in I} \rho_i(X),$$

would be considered as a safe capital charge by all the involved national banks. In the opposite extreme case of no loss sharing, the potential loss deriving from a default loss is bore in full by the national central bank j that will manage the execution of the purchase. In this case, the ECB acts like a "dealer" in Section 2.1 by delegating risk management of the bond to the nation with minimal capital charge

$$\rho_\wedge(X) = \min_{i \in I} \rho_i(X).$$

Arguably, both rules above are too extreme: the first immobilizes a great amount of resources, and the second is quite fragile and does not exploit risk sharing. As a natural compromise, one may use the convex combination

$$\rho(X) = \mu \max_{i \in I} \rho_i(X) + (1-\mu) \min_{i \in I} \rho_i(X)$$

as the effective capital charge, where the weight $\mu \in [0,1]$ has a clear interpretation as an index of caution of the ECB. Of course, the resulting risk measure ρ is star-shaped and not convex in general.

2.3. Risk mitigation

From a decision theoretic perspective, a convex risk measure ρ can be seen as the certain equivalent loss of an ambiguity averse (and risk neutral, as a financial institution is typically assumed to be) decision maker who satisfies the rationality axioms of Maccheroni, Marinacci, and Rustichini [36]. Since such a decision maker (DM) favors prospects X with lower certain equivalent losses, his preferences are represented by the certainty equivalent

$$\nu(X) = -\rho(X) = \min_{P \in \mathcal{M}} \{\mathbb{E}_P[u(X)] + \alpha(P)\}$$

where $u(x) = -x$ because of risk neutrality and $\alpha(P)$ is an ambiguity index that penalizes probabilistic scenarios according to their plausibility. Formally, α is a convex and weak*-lower semicontinuous function on the set $\mathcal{M}$ of probabilistic scenarios (finitely additive probabilities) which is *grounded*, that is, such that $\min_{P \in \mathcal{M}} \alpha(P) = 0$. Implicit in the uncertainty aversion axiom of [36] (which corresponds to convexity of ρ) is the idea that a DM facing prospect X cannot affect the plausibility of probabilistic scenarios.

As defended in a series of infuential papers by Jaques Drèze and Edi Karni (see, e.g., Drèze [17] and Karni [32]), there are some relevant cases in which a DM, by adopting a risk-mitigation strategy a, may indeed affect such plausibility. In these cases, depending on a, the certain equivalent loss is given by

$$\rho(a, X) = \max_{P \in \mathcal{M}} \{\mathbb{E}_P[X] - \alpha_a(P)\}$$

and the DM facing X will choose a to minimize it, so that his certainty equivalent becomes

$$\nu(X) = -\min_{a \in A} \rho(a, X) = \max_{a \in A} \min_{P \in \mathcal{M}} \{\mathbb{E}_P[u(X)] + \alpha_a(P)\} \tag{1}$$

where A is the set of risk-mitigation strategies (commonly called actions). For this interpretation to be accurate it is necessary that the function $\rho(a, \cdot)$ be a *bona fide* certainty equivalent loss for each a, and in particular $\rho(a, c) = \rho(b, c) = c$ for all actions $a, b \in A$ and every certain loss $c \in \mathbb{R}$. In other words, the requirement $\min_{P \in \mathcal{M}} \alpha_a(P) = 0$ for all $a \in A$, is what allows to interpret the "penalty functions" $\{\alpha_a : a \in A\}$ as ambiguity indexes rather than generic costs.

As observed in the previous example, $\min_{a \in A} \rho(a, \cdot)$ is a star-shaped risk measure, which is not necessarily convex. Thus the preferences

$$X \succsim Y \iff \nu(X) \geq \nu(Y)$$

are not necessarily uncertainty averse. Yet, these preferences satisfy

$$X \succsim c \Longrightarrow \beta X + (1-\beta)\, c \succsim c \tag{2}$$

for every uncertain prospect $X \in \mathcal{X}$, every certain prospect $c \in \mathbb{R}$, and every weight $\beta \in (0,1)$, thus they are *increasingly relative ambiguity averse* in the sense of Xue [51] and Cerreia-Vioglio, Maccheroni, and Marinacci [12].[3] This simple observation is important, in that it clarifies the decision-theoretic appeal of star-shaped risk measures. They describe institutional decision makers who, in the face of uncertainty, will take any available measure to mitigate its adverse effects, and who, as the total capital of the financial institution increases, will decrease the fraction of it that is exposed to those effects (see the seminal Arrow [1]).

Drèze meets Hansen and Sargent The first axiomatic treatment of risk-mitigation dates back to Drèze (1987). In his work, for every uncertain prospect $X \in \mathcal{X}$,

$$\nu(X) = \max_{a \in A} \mathbb{E}_{Q_a}[u(X)] \tag{3}$$

this means that Drèze's decision makers are confident that by choosing $a \in A$ they can induce Q_a without error. Hansen and Sargent [30] would say "without fear of misspecification." They also show that misspecification concerns on part of DMs can be addressed through robustification. Formally, this is achieved by replacing $\mathbb{E}_{Q_a}[u(X)]$ in (3) with

$$-\lambda \log \mathbb{E}_{Q_a}\left[\exp\left(-\frac{u(X)}{\lambda}\right)\right] = \min_{P \in \mathcal{M}} \{\mathbb{E}_P[u(X)] + \lambda \mathcal{R}(P \mid Q_a)\}$$

where $\lambda > 0$ captures the level of trust in the relation between a and Q_a, and $\mathcal{R}$ is the relative entropy. This adjustment leads to

$$\nu_\lambda(X) = \max_{a \in A} \min_{P \in \mathcal{M}} \{\mathbb{E}_P[u(X)] + \lambda \mathcal{R}(P \mid Q_a)\}$$

which for $\lambda = +\infty$ corresponds to Drèze's initial proposal, while lower values of λ correspond to lower confidence on part of DMs in their ability to affect probabilistic scenarios. Clearly, this is a special case of (1) with $\alpha_a(\cdot) = \lambda \mathcal{R}(\cdot \mid Q_a)$.

2.4. Non-concave utilities

Föllmer and Schied [21, 22] proposed the class of utility-based shortfall risk measures

$$\rho_u(X) = \inf\{m \in \mathbb{R} \mid \mathbb{E}_P[u(m-X)] \geq u(0)\} \qquad X \in \mathcal{X} \tag{4}$$

where P is a given probability measure and u is an increasing and non-constant utility function on $\mathbb{R}$ such that (w.l.o.g.) $u(0) = 0$. The interpretation of ρ_u in (4) is that the acceptable risk positions are the ones which have non-negative reservation price. Note that (4) always defines a risk measure (in the formal sense of Definition 1 below) which is convex if and only if u is concave (see, e.g., [22, Section 4.9]).

Concavity of u corresponds to a strong form of risk aversion. Utility functions with local convexities have been studied and normatively justified since Friedman and Savage [24]. Motivated by this, Landsberger and Meilijson [33] and [34] consider the broader class of utility functions such that

$$\lambda \mapsto \frac{u(\lambda)}{\lambda} \text{ is decreasing on } (0,\infty) \text{ and } (-\infty,0) \tag{5}$$

which allow for convex kinks (see also Müller [39] and Müller et al. [40]), and they show that these functions capture "aversion to fatter profit/loss tails." Note that (5) is equivalent to

$$\lambda \mapsto \frac{u(\lambda x)}{\lambda} \text{ is decreasing on } (0,\infty) \text{ for all } x \in \mathbb{R}. \tag{6}$$

If u satisfies (5), then the corresponding utility-based shortfall risk measure ρ_u in (4) is star-shaped. This claim can be verified by, for $\lambda > 1$,

$$\begin{aligned}\rho_u(\lambda X) &= \inf\{\lambda m \in \mathbb{R} \mid \mathbb{E}_P[u(\lambda m - \lambda X)] \geq 0\}\\ &= \lambda \inf\left\{m \in \mathbb{R} \mid \mathbb{E}_P\left[\frac{u(\lambda m - \lambda X)}{\lambda}\right] \geq 0\right\}\\ &\geq \lambda \inf\{m \in \mathbb{R} \mid \mathbb{E}_P[u(m - X)] \geq 0\} = \lambda \rho_u(X)\end{aligned}$$

where the inequality is due to (6). Therefore, star-shaped risk measures arise naturally if non-concave utility functions are involved in utility-based shortfall risk measurement *à la* Föllmer and Schied.

3. Star-shaped risk measures: basic definitions

The points of departure of our analysis are the standard definitions that shaped the theory of risk measurement as originally introduced in the landmark papers by Artzner et al. [2], Delbaen [15], Föllmer and Schied [21], and Frittelli and Rosazza Gianin [25]. As mentioned, we use the convention of McNeil, Frey, and Embrechts [38] that random variables represent future *losses* of positions held by financial institutions over some time horizon T. In particular, for a position with loss given by X, a negative value of $X(\omega)$ corresponds to a gain (if ω occurs). The time horizon T is left unspecified and, in order to simplify the presentation, we set interest rates equal to zero so that there is no discounting.

Specifically, the possible losses of financial positions are represented by a linear space $\mathcal{X}$ of bounded random variables containing all constants. We do not assume that a probability measure is *a priori* given on the underlying measurable space Ω of states of the environment. The space $\mathcal{X}$ is endowed with the pointwise order, so that $X \geqq Y$ if, and only if, $X(\omega) \geq Y(\omega)$ for all states ω.

Definition 1. A *risk measure* is a function $\rho : \mathcal{X} \to \mathbb{R}$ that satisfies

1. *Monotonicity:* If $X \geqq Y$, then $\rho(X) \geq \rho(Y)$;
2. *Translation invariance:* $\rho(X - m) = \rho(X) - m$ for all $X \in \mathcal{X}$ and all $m \in \mathbb{R}$;
3. *Normalization:* $\rho(0) = 0$.

A risk measure may satisfy the following further properties

4. *Star-shapedness:* $\rho(\lambda X) \geq \lambda \rho(X)$ for all $X \in \mathcal{X}$ and all $\lambda > 1$;
5. *Convexity:* $\rho(\lambda X + (1-\lambda)Y) \leq \lambda \rho(X) + (1-\lambda)\rho(Y)$ for all $X, Y \in \mathcal{X}$ and all $\lambda \in (0,1)$;
6. *Positive homogeneity:* $\rho(\lambda X) = \lambda \rho(X)$ for all $X \in \mathcal{X}$ and all $\lambda > 0$;
7. *Subadditivity:* $\rho(X + Y) \leq \rho(X) + \rho(Y)$ for all $X, Y \in \mathcal{X}$.

A risk measure ρ is *coherent* if it satisfies positive homogeneity and subadditivity.

Introduced by [21] and [25], convex risk measures are a very popular class of risk measures that capture preference for diversification; see also Cerreia-Vioglio et al. [11] for quasi-convex risk measures. As anticipated, convex risk measures are known to be star-shaped. In fact, for all $X \in \mathcal{X}$ and all $\lambda > 1$,

$$\rho(X) = \rho\left(\frac{1}{\lambda}(\lambda X) + \left(1 - \frac{1}{\lambda}\right)0\right) \leq \frac{1}{\lambda}\rho(\lambda X) + \left(1 - \frac{1}{\lambda}\right)\rho(0) = \frac{1}{\lambda}\rho(\lambda X).$$

However, the converse is generally not true, because all positively homogeneous risk measures are star-shaped too. For instance, Value-at-Risk (VaR) is positively homogeneous without being convex. Specifically, given a probability measure Q, VaR at level $\beta \in (0,1]$ is defined by

$$\mathrm{VaR}^Q_\beta(X) = \inf\{x \in \mathbb{R} : Q(X > x) \leq 1 - \beta\}$$

and it is the minimum capital reserve that brings default probability below $1 - \beta$. Non-convexity of VaR led to the introduction of Expected Shortfall (ES)

$$\mathrm{ES}^Q_\beta(X) = \frac{1}{1-\beta}\int_\beta^1 \mathrm{VaR}^Q_t(X)\mathrm{d}t$$

which of course is star-shaped too. ES is the standard risk measure in Basel IV and has been recently axiomized by Wang and Zitikis [50].

A main concern regarding both $\mathrm{VaR}^Q_\beta(X)$ and $\mathrm{ES}^Q_\beta(X)$ is the fact that their value depends crucially on Q, thus the possible misspecification of Q makes these measures very fragile. This, in turn, led recent research on risk measures to investigate robustifications like the ones described in the next two examples. Let us anticipate that these robustifications generate monetary risk measures which are always star-shaped and typically non-convex.

Example 1 (Scenario-based risk measures). Wang and Ziegel [49] consider a collection $\mathcal{Q}$ of probability measures and define

$$\mathrm{MaxVaR}_\beta^{\mathcal{Q}}(X) = \sup_{Q \in \mathcal{Q}} \mathrm{VaR}_\beta^Q(X).$$

This risk measure has a natural interpretation in terms of robustness, it is widely used in applications (see, e.g., Natarajan et al. [42]), and like VaR it is generally not convex, but being positively homogeneous it is star-shaped.

Alternatively, for the finite $\mathcal{Q}$'s that typically appear in applications, one might consider the less extreme robustification

$$\mathrm{MedVaR}_\beta^{\mathcal{Q}}(X) = \mathrm{Median}\{\mathrm{VaR}_\beta^Q(X) \mid Q \in \mathcal{Q}\}.$$

Like $\mathrm{MaxVaR}_\beta^{\mathcal{Q}}$ also $\mathrm{MedVaR}_\beta^{\mathcal{Q}}(X)$ is positively homogeneous and generally not convex.

When VaR is replaced by ES, it is easy to show that $\mathrm{MaxES}_\beta^{\mathcal{Q}}(X)$ is a coherent risk measure (like ES) while $\mathrm{MedES}_\beta^{\mathcal{Q}}(X)$ is star-shaped without being coherent or even convex.

The next example goes one step further by yielding a non-convex and non-positively homogeneous, yet very much tractable risk measure.

Example 2 (Benchmark loss VaR). The benchmark loss VaR in Bignozzi et al. [9] is defined as

$$\mathrm{LVaR}_\alpha^Q(X) = \sup_{t \geq 0}\{\mathrm{VaR}_{\alpha(t)}^Q(X) - t\}$$

where $\alpha : [0, \infty) \to (0, 1]$ is an increasing and right-continuous function. It is easy to check that LVaR_α^Q is neither positively homogeneous nor convex, and yet it is a star-shaped risk measure.

Next we formalize some properties of star-shaped risk measures that we informally anticipated in the introduction. We begin with a simple, but conceptually important characterization in terms of risk-to-exposure ratios.

Proposition 1. *For a risk measure $\rho : \mathcal{X} \to \mathbb{R}$, the following are equivalent:*

(i) ρ is star-shaped;

(ii) $\rho(\alpha X) \leq \alpha\rho(X)$ for all $X \in \mathcal{X}$ and all $\alpha \in (0, 1)$;

(iii) for each $X \in \mathcal{X}$, the risk-to-exposure ratio $r_X : \beta \mapsto \rho(\beta X)/\beta$ is an increasing function of β on $(0, \infty)$.

These properties of star-shaped functions explain why they have also been called "increasing along rays" or "radiant" in the context of abstract convex analysis, where this property was first discovered; see, e.g., Zaffaroni [52] and Penot [43].

An acceptance set is naturally associated with each risk measure ρ. It is the set of positions that do not require any additional capital. Formally, it coincides with the lower level set of ρ at 0, that is,

$$\mathcal{A}_\rho = \{X \in \mathcal{X} \mid \rho(X) \leq 0\}.$$

As well known, $\mathcal{A}_\rho$ completely determines ρ; in fact, the translation invariance property guarantees that

$$\rho(X) = \min\{m \in \mathbb{R} \mid X - m \in \mathcal{A}_\rho\} \qquad X \in \mathcal{X}. \tag{7}$$

In other words, the risk of X is measured as the minimum amount by which the loss it represents must be uniformly reduced to make the adjusted position acceptable.

In general, a subset $\mathcal{A}$ of $\mathcal{X}$ such that:

$$\sup\{m \in \mathbb{R} \mid m \in \mathcal{A}\} = 0 \text{ and}$$
$$X \in \mathcal{A},\ Y \in \mathcal{X},\ Y \leqq X \implies Y \in \mathcal{A}$$

is called an *acceptance set*, and it generates a risk measure

$$\rho_{\mathcal{A}}(X) = \inf\{m \in \mathbb{R} \mid X - m \in \mathcal{A}\} \qquad X \in \mathcal{X}$$

which is convex (resp., positively homogeneous) if $\mathcal{A}$ is convex (resp., a cone); see Föllmer and Schied [22] for details. For obvious reasons, it is convenient to call *coherent* an acceptance set which is a convex cone.

The fact that similar relations hold between star shapedness of a risk measure and star shapedness of its acceptance sets, shows that a risk measure is star-shaped if and only if it is based on the following principle: deleveraging an acceptable position cannot make it unacceptable. Equivalently, increasing the exposure to an unacceptable position cannot make it acceptable.

Below, recall that a subset S of a vector space is *star-shaped* if, and only if, $\lambda s \in S$ for all $\lambda \in [0,1]$ and all $s \in S$.

Proposition 2. *For a risk measure $\rho : \mathcal{X} \to \mathbb{R}$, the following are equivalent:*

(i) ρ is star-shaped;

(ii) the set $\mathcal{A}_\rho$ is star-shaped in $\mathcal{X}$;

(iii) there exists a star-shaped acceptance set $\mathcal{A}$ such that $\rho = \rho_{\mathcal{A}}$.

Finally, coherent risk measures coincide with subadditive and star-shaped risk measures; thus here positive homogeneity can be replaced by the weaker property of star shapedness. This fact sheds some light on the strengths and weaknesses of the subadditivity assumption.

Proposition 3. *For a subadditive risk measure $\rho : \mathcal{X} \to \mathbb{R}$, the following are equivalent:*

(i) ρ *is star-shaped;*

(ii) ρ *is positively homogeneous (thus coherent);*

(iii) ρ *is convex.*

4. Aggregation operations

As explained in Section 2, star-shaped risk measures naturally emerge when a collection of risk measures $\{\rho_i\}_{i\in I}$ have to be aggregated in single risk measure ρ. To analyze these situations, we introduce some operations that cover a broad range of aggregators.

- For each probability μ on the parts of I, the average ρ_μ, is defined as

$$\rho_\mu(X) = \int_I \rho_i(X)\,\mathrm{d}\mu(i) \qquad X \in \mathcal{X}. \tag{8}$$

In particular, if I is finite or μ is supported on a finite subset of I, then ρ_μ is a convex combination of $\{\rho_i\}_{i\in I}$, i.e., $\rho_\mu = \sum_{i\in I} c_i\rho_i$ where $c_i = \mu(\{i\})$, $i \in I$.

- The supremum $\rho_\vee$, is defined as

$$\rho_\vee(X) = \sup_{i\in I} \rho_i(X) \qquad X \in \mathcal{X}.$$

- The infimum $\rho_\wedge$, is defined as

$$\rho_\wedge(X) = \inf_{i\in I} \rho_i(X) \qquad X \in \mathcal{X}.$$

- The inf-convolution $\rho_\diamond$, is defined as

$$\rho_\diamond(X) = \inf\left\{\sum_{i\in I}\rho_i(Y_i) \;\middle|\; Y_i \in \mathcal{X} \text{ for all } i \in I \text{ and } \sum_{i\in I} Y_i = X\right\} \qquad X \in \mathcal{X} \tag{9}$$

provided $I = \{1, 2, ..., n\}$ is finite and

$$\sum_{i\in I}\rho_i(Z_i) \geq 0 \tag{10}$$

for all $Z_1, Z_2, ..., Z_n \in \mathcal{X}$ such that $\sum_{i\in I} Z_i = 0$.[4]

The following theorem shows that the class of star-shaped risk measures is closed under these operations.

Theorem 1. *For a collection of star-shaped risk measures, their average, supremum, infimum, and inf-convolution (when defined) are star-shaped risk measures.*

Theorem 1 implies that star-shaped risk measures form a complete lattice, i.e., a set that is closed with respect to the pointwise supremum and infimum operations. Moreover, the inf-convolution risk measure is important because it can be considered as the one used by a representative agent in a risk-sharing or order-splitting problem (see, e.g., Embrechts et al. [18] and the references therein), and star shapedness is preserved in such situations.

In the proof of Theorem 1 we show that (8) defines a star-shaped risk measure even if μ is a capacity on I and the integral is in the sense of Choquet (a capacity on I is a set function on a σ-field of I such that $0=\mu(\varnothing)\leq\mu(J)\leq\mu(K)\leq\mu(I)=1$ for all $J\subseteq K\subseteq I$). This is not done for mathematical elegance, but because: first, the supremum case $\rho_\vee$ and the infimum case $\rho_\wedge$ are special cases of Choquet averages; second, when I is finite, any order statistic (such as the median) of $\{\rho_i(X)\}_{i\in I}$ has form (8) for a suitable capacity μ (see Murofushi and Sugeno [41]).

Remark 1. Theorem 1 and what we have just observed about Choquet averages show that the class of star-shaped risk measures is closed under many commonly used economic aggregation mechanisms that disrupt convexity (see also Cerreia-Vioglio, Corrao, and Lanzani [13]). Later we will see in Section 7 that some subclasses of star-shaped risk measures, in particular, law-invariant ones and $\mathcal{SSD}$-consistent ones, are also closed under the same operations.

Finally, the inf-convolution $\rho_\diamond$ is denoted by $\Box_{i\in I}\rho_i$ when it is necessary to make explicit the collection $\{\rho_i\}_{i\in I}$ of risk measures that it aggregates.

5. Representation of star-shaped risk measures

Proposition 4.47 of Föllmer and Schied [22] shows that VaR is the minimum of the collection of all convex risk measures that dominate it. Our main representation theorem shows that, not only this is true for all star-shaped risk measures, but this property actually characterizes them.

Theorem 2. *For a risk measure $\rho:\mathcal{X}\to\mathbb{R}$, the following are equivalent:*

(i) ρ is star-shaped (resp. positively homogeneous);

(ii) there exists a collection Γ of convex (resp. coherent) risk measures such that

$$\rho(X)=\min_{\gamma\in\Gamma}\gamma(X)\qquad X\in\mathcal{X};\tag{11}$$

(iii) there exists a family $\{\mathcal{A}_\beta\}_{\beta\in B}$ of convex (resp. coherent) acceptance sets such that

$$\rho(X)=\min\{m\in\mathbb{R}\mid X-m\in\mathcal{A}_\beta \text{ for some } \beta\in B\}\qquad X\in\mathcal{X}.$$

Moreover, in (ii) Γ can be chosen as the collection of all convex (resp. coherent) risk measures dominating ρ and in (iii) $\{\mathcal{A}_\beta\}_{\beta\in B}$ can be chosen as the family of their acceptance sets.

The intuition behind Theorem 2, in particular the implication (i)⇒(ii), is best explained with Figure 1 in the simple case of $|\Omega|=2$ (without positive homogeneity). The main idea is to write the acceptance set $\mathcal{A}_\rho$ of a star-shaped risk measure ρ as the union of convex sets $\mathcal{A}_Y$ for $Y\in\mathcal{X}$, where $\mathcal{A}_Y$ contains all random variables dominated by $\alpha(Y-\rho(Y))$ for some $\alpha\in[0,1]$. As such, ρ can be written as the minimum of convex risk measures ρ_Y, each associated with the acceptance set $\mathcal{A}_Y$.

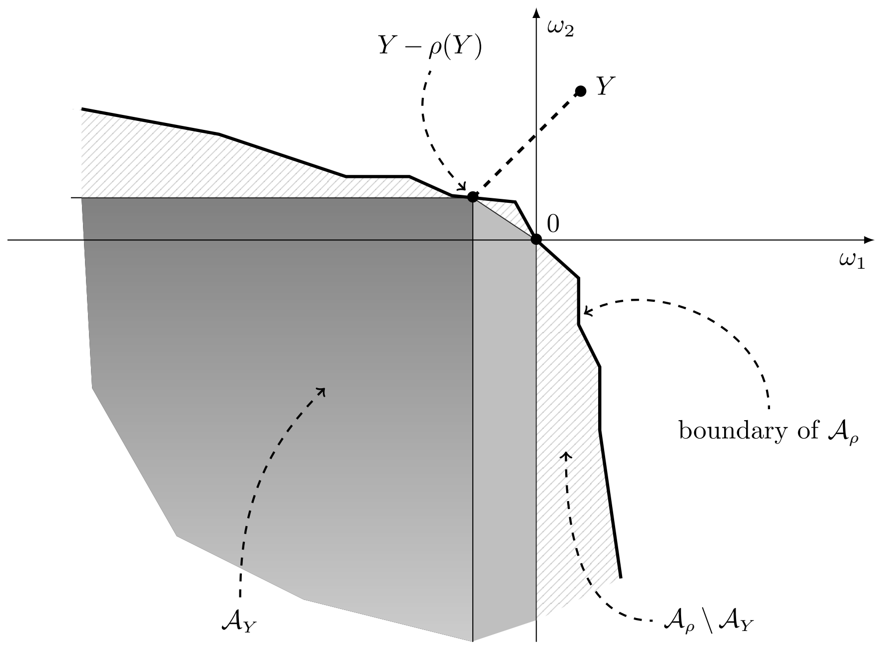

Figure 1 **The intuition behind Theorem 2 in case $\Omega = \{\omega_1, \omega_2\}$, where $\mathcal{A}_Y$ is the set of all random variables dominated by $\alpha(Y - \rho(Y))$ for some $\alpha \in [0,1]$.**

Theorem 2 yields a tractable representation of star-shaped risk measures, and its first use in applications dates back to Castagnoli et al. [10] (where it is stated without proof as Proposition 2). The special case of coherent risk measures with $\mathcal{X} = \mathbb{R}^n$ also appears in the independent Chandrasekher et al. [14].

As it happens for convex (and coherent) risk measures this envelope representation is not unique, unless a suitable relaxation is considered. For each set of convex (resp. coherent) risk measures Γ define its relaxation $\tilde{\Gamma}$ by

$$\tilde{\Gamma} = \left\{ \tilde{\gamma} : \mathcal{X} \to \mathbb{R} \;\middle|\; \begin{array}{l} \tilde{\gamma} \text{ is a convex (resp. coherent) risk measure and} \\ \text{for each } X \in \mathcal{X} \text{ there is } \gamma \in \Gamma \text{ such that } \tilde{\gamma}(X) \geq \gamma(X) \end{array} \right\}.$$

Proposition 4. *Let ρ be a star-shaped risk measure and Γ_1 and Γ_2 be two sets of convex risk measures such that*

$$\rho(X) = \min_{\gamma_1 \in \Gamma_1} \gamma_1(X) = \min_{\gamma_2 \in \Gamma_2} \gamma_2(X) \qquad X \in \mathcal{X}$$

then $\tilde{\Gamma}_1 = \tilde{\Gamma}_2 = \{\gamma : \mathcal{X} \to \mathbb{R} \mid \gamma$ is a convex risk measure and $\gamma \geqq \rho\}$.

In particular, the set $\tilde{\Gamma} = \tilde{\Gamma}_1 = \tilde{\Gamma}_2$ does not depend on the selected representation, but only on ρ.

The next result shows that the representation obtained in Theorem 2, is well behaved with respect to the aggregative operations described in Section 4.

Theorem 3. *Let $\{\rho_i\}_{i\in I}$ be a collection of star-shaped risk measures and $\{\Gamma_i\}_{i\in I}$ a collection of sets of convex risk measures such that, for each $i \in I$,*

$$\rho_i(X) = \min_{\gamma_i\in\Gamma_i} \gamma_i(X) \qquad X \in \mathcal{X}. \tag{12}$$

The average, supremum, infimum, and inf-convolution (when defined) of the collection $\{\rho_i\}_{i\in I}$ are given by

$$\int_I \rho_i(X)\,\mathrm{d}\mu(i) = \min_{\gamma\in\int\Gamma_i\,\mathrm{d}\mu(i)} \gamma(X) \qquad \text{where } \int \Gamma_i\,\mathrm{d}\mu(i) = \left\{ \int_I \gamma_i\,\mathrm{d}\mu(i) \;\middle|\; \gamma_i \in \Gamma_i \text{ for all } i \in I \right\} \tag{13}$$

$$\sup_{i\in I} \rho_i(X) = \min_{\gamma\in\cap_i\tilde{\Gamma}_i} \gamma(X) \tag{14}$$

$$\inf_{i\in I} \rho_i(X) = \inf_{\gamma\in\cup_i\Gamma_i} \gamma(X) \tag{15}$$

$$\rho_\diamond(X) = \inf_{\gamma\in\Box_{i\in I}\Gamma_i} \gamma(X) \qquad \text{where } \Box_{i\in I}\Gamma_i = \{\Box_{i\in I}\gamma_i \mid \gamma_i \in \Gamma_i \text{ for all } i \in I\}. \tag{16}$$

for all $X \in \mathcal{X}$. Moreover, the infimum on the r.h.s. of (15) is attained when $\inf_{i\in I}\rho_i(X)$ is attained on I, and the infimum on the r.h.s. of (16) is attained when

$$\rho_\diamond(X) = \min\left\{ \sum_{i\in I} \rho_i(Y_i) \;\middle|\; Y_i \in \mathcal{X} \text{ for all } i \in I \text{ and } \sum_{i\in I} Y_i = X \right\}$$

that is, if the infimum in the definition of $\rho_\diamond$ is attained.

Remark 2. In formula (14), it is in general necessary to use the relaxations $\tilde{\Gamma}_i$ of the original sets Γ_i. A counterexample to the use of the Γ_i's can be found by choosing $\Gamma_1 = \{\gamma_1\}$ and $\Gamma_2 = \{\gamma_2\}$ with $\gamma_1 \neq \gamma_2$ so that $\Gamma_1 \cap \Gamma_2 = \varnothing$.

The counterparts of the latter two statements for positively homogeneous risk measures are straightforward. Finally, assume that $\mathcal{X} = L^\infty(\Omega, \mathcal{F}, P)$ where P is any given probability measure. Let $\mathcal{P}$ be the set of all probability measures on $(\Omega, \mathcal{F})$ which are absolutely continuous with respect to P. Any convex risk measure γ on $\mathcal{X}$ with the Fatou property can be written as ([22, Theorem 4.33])

$$\gamma(X) = \sup_{Q\in\mathcal{P}} \{\mathbb{E}_Q[X] - \alpha_\gamma(Q)\}, \qquad X \in \mathcal{X},$$

where $\alpha_\gamma : \mathcal{P} \to [0,\infty]$ is such that $\inf_{Q\in\mathcal{P}} \alpha_\gamma(Q) = 0$. If the Fatou property is not assumed, then the set $\mathcal{P}$ needs to be replaced by the set of all finitely additive probabilities. In the next result, we obtain a robust representation of star-shaped risk measures based on Theorem 2, in which each convex risk measure γ satisfies the Fatou property and thus can be represented via $\alpha_\gamma : \mathcal{P} \to [0,\infty]$.

Proposition 5. *A risk measure $\rho : \mathcal{X} \to \mathbb{R}$ is star-shaped if and only if there exists a collection $\{\alpha_\gamma\}_{\gamma\in\Gamma}$ of functions $\alpha_\gamma : \mathcal{P} \to [0,\infty]$, with $\inf_{Q\in\mathcal{P}} \alpha_\gamma(Q) = 0$ for all $\gamma \in \Gamma$, such that*

$$\rho(X) = \min_{\gamma\in\Gamma} \sup_{Q\in\mathcal{P}} \{\mathbb{E}_Q[X] - \alpha_\gamma(Q)\}, \qquad X \in \mathcal{X}. \tag{17}$$

6. Optimization of star-shaped risk measures

Theorem 2 allows us to use convex optimization techniques to optimize star-shaped risk measures. Let $\{\rho_i\}_{i\in I}$ be a collection of star-shaped risk measures with representations

$$\rho_i(X) = \min_{\gamma\in\Gamma_i} \gamma(X) \qquad X \in \mathcal{X}$$

where Γ_i is a set of convex risk measures for all $i \in I$. Like in Example 1, $I = \mathcal{Q}$ may be a set of probability measures on Ω and each ρ_Q may be a risk measure which is law-invariant under Q, say VaR_β^Q.

Let $\boldsymbol{S}$ be a vector of risk factors, that is, random variables on Ω, A be a set of available actions, and $\ell : A \times \mathbb{R}^n \to \mathbb{R}$ be a loss function. The natural interpretation of $\ell(a, \boldsymbol{S})$ is as the random loss corresponding to action a and risk factor $\boldsymbol{S}$. Consider the standard risk minimization problem

$$\text{to minimize} \quad \rho_i(\ell(a, \boldsymbol{S})) \quad \text{over } a \in A \tag{18}$$

and its robust version

$$\text{to minimize} \quad \sup_{i\in I} \rho_i(\ell(a, \boldsymbol{S})) \quad \text{over } a \in A. \tag{19}$$

Again note that the objective ρ_i in (18) and its robust version $\rho_\vee = \sup_{i\in I} \rho_i$ in (19) are star-shaped, but not necessarily convex risk measures. Nevertheless, using Theorems 2 and 3, the two optimization problems above can be converted into standard optimization problems for convex risk measures.

Proposition 6. *If $\ell(a, \boldsymbol{S}) \in \mathcal{X}$ for all $a \in A$, then*

$$\inf_{a\in A} \rho_i(\ell(a, \boldsymbol{S})) = \inf_{\gamma\in\Gamma_i} \inf_{a\in A} \gamma(\ell(a, \boldsymbol{S})) \tag{20}$$

and

$$\inf_{a\in A} \sup_{i\in I} \rho_i(\ell(a, \boldsymbol{S})) = \inf_{\gamma\in\tilde{\Gamma}_I} \inf_{a\in A} \gamma(\ell(a, \boldsymbol{S})). \tag{21}$$

where $\tilde{\Gamma}_I = \bigcap_{i\in I} \tilde{\Gamma}_i$. Moreover,

1. a^* is a minimizer of problem (18) *if, and only if, there exists $\gamma^* \in \Gamma_i$ such that (a^*, γ^*) minimizes $\gamma(\ell(a, \boldsymbol{S}))$ over $(a, \gamma) \in A \times \Gamma_i$;*

2. a^* is a minimizer of problem (19) *if, and only if, there exists $\gamma^* \in \tilde{\Gamma}_I$ such that (a^*, γ^*) minimizes $\gamma(\ell(a, \boldsymbol{S}))$ over $(a, \gamma) \in A \times \tilde{\Gamma}_I$.*

As a direct consequence of Proposition 6, the optimal financial position for a star-shaped risk measure ρ in a given subset is the minimum among the optimal financial positions of the convex risk measures which represent it in the sense of Theorem 2.

Corollary 1. *If $\rho : \mathcal{X} \to \mathbb{R}$ is a star-shaped risk measure with representation* (11) *and $\mathcal{Y} \subseteq \mathcal{X}$, then*

$$\inf_{X \in \mathcal{Y}} \rho(X) = \inf_{\gamma \in \Gamma} \inf_{X \in \mathcal{Y}} \gamma(X).$$

Example 3 (Portfolio selection with risk measures). An important problem in finance is portfolio selection with various objectives and constraints. If the objective is to minimize a risk measure ρ, a classic problem can be formulated as

$$\text{to minimize: } \rho(-Y_T) \quad \text{over portfolios } Y \quad \text{subject to } Y_0 \leq x_0 \text{ and } Y_T \in \mathcal{Y}, \tag{22}$$

where x_0 is a constant representing the budget, Y_T is the terminal payoff at time T of a dynamically traded portfolio process Y, Y_0 is its initial price, and $\mathcal{Y}$ is a set of acceptable positions on the terminal payoff; e.g., there may be a bound on the maximum loss from Y_T.

For the purpose of illustration, we assume a complete market in which the risk-free interest rate is zero. In a complete market, it is well known that problem (22) can be solved in two steps (see e.g., Föllmer, Schied, and Weber [23]). First, solve the static problem

$$\text{to minimize: } \rho(-X) \quad \text{over } X \in \mathcal{X} \quad \text{subject to } \mathbb{E}^Q[X] \leq x_0 \text{ and } X \in \mathcal{Y}, \tag{23}$$

where Q is the unique martingale measure in the financial market, and $\mathcal{X}$ is the set of random variables representing time-T payoffs. Second, replicate a portfolio with $Y_T = X$ and $Y_0 = \mathbb{E}^Q[X]$ using a standard method of martingale representation.

For various choices of $\mathcal{Y}$ and a convex risk measure ρ, problem (23) has been well studied and often admits explicit solutions; see Chapter 8 of Föllmer and Schied [22]. On the other hand, analytical solutions to problem (22) for non-convex risk measures are rarely available as it involves non-convex optimization. Using Corollary 1, we can solve (23) for any star-shaped risk measure ρ. First, we write $\rho = \min_{\gamma \in \Gamma} \gamma$ for some set Γ of convex risk measures. Second, we solve for

$$X^*_\gamma \in \operatorname{argmin}\{\gamma(-X) : X \in \mathcal{X} \cap \mathcal{Y} \text{ and } \mathbb{E}^Q[X] \leq x_0\}$$

for each $\gamma \in \Gamma$, using results à la [22]. Third, we take a minimum of $\gamma(X^*_\gamma)$ over $\gamma \in \Gamma$, to obtain the minimizer γ^*. Finally, the position $X^*_{\gamma^*}$ is a minimizer to the original problem (23).

The main point of this example is to show that optimization for star-shaped risk measures is convenient if corresponding results on convex risk measures are available.

Remark 3. Recall that the convex risk measures $\gamma \in \Gamma$ in Theorem 2 satisfy normalization, that is, $\gamma(0) = 0$ for all $\gamma \in \Gamma$. This simple property has a few implications. Consider the problem of minimizing $\rho(\ell(a, \boldsymbol{S}))$ over $a \in A$ as in (18), where $\rho = \min_{\gamma \in \Gamma} \gamma$. Suppose that there exists $a_0 \in A$ representing a risk-free action (e.g., perfect hedge, full insurance, non-defaultable bond purchase, or no participation, in different contexts), giving rise to $\ell(a_0, \boldsymbol{S}) = c$ which is a constant. Our first observation is that, since each γ is normalized, the problem can be rewritten as

$$\min_{\gamma\in\Gamma}\min_{a\in A}\Big(\gamma(\ell(a,\boldsymbol{S})) - \gamma(\ell(a_0,\boldsymbol{S}))\Big) + c.$$

That is, we can first optimize the relative improvement of our decision a over the benchmark a_0 for each γ and then look for the best improvement over all $\gamma \in \Gamma$. Second, if $a \mapsto \ell(a, \boldsymbol{S})$ is bounded, then $a \mapsto \gamma(\ell(a, \boldsymbol{S}))$ is uniformly bounded for $\gamma \in \Gamma$. Both the above observations may fail to hold if γ is not normalized. In Section 8, we will briefly discuss issues on normalization of γ.

7. Law-invariant star-shaped risk measures

In this section, we discuss the important class of law-invariant risk measures. Assume now that $\mathcal{X} = L^\infty(\Omega, \mathcal{F}, P)$ where P is an atomless probability measure. A risk measure ρ is *law-invariant* if two losses $X, Y \in \mathcal{X}$ that share the same law under P are equally risky:

$$X \overset{\mathrm{d}}{=} Y \implies \rho(X) = \rho(Y).$$

Most of risk measures that are used in the financial practice, like VaR and ES, are law-invariant; we omit P in ES^P_α and VaR^P_α in this section. Law-invariance is the key property that facilitates statistical inference for risk measures, thus converting distributional information into risk assessment. For this reason, law-invariant risk measures are also called *statistical functionals*.

Since, according to Theorem 2, star-shaped risk measures can be represented as minima of convex risk measures, one may naturally wonder whether law-invariant and star-shaped risk measures are minima of law-invariant and convex risk measures. Quite unexpectedly, the answer to this question is no. Indeed, as shown by Mao and Wang [37], minima of law-invariant and convex risk measures are consistent with respect to second-order stochastic dominance ($\mathcal{SSD}$), and VaR —while law-invariant and star-shaped— is not consistent with $\mathcal{SSD}$.

More specifically, we denote by $X \succcurlyeq_{\mathcal{SSD}} Y$ the fact that $X \in \mathcal{X}$ second-order stochastically dominates $Y \in \mathcal{X}$, which means

$$\mathbb{E}_P[u(-X)] \geq \mathbb{E}_P[u(-Y)]$$

for all increasing and concave $u : \mathbb{R} \to \mathbb{R}$ such that the two expectations exist. Recall that our random variables describe losses, hence their negatives describe gains. A risk measure ρ is said to be *$\mathcal{SSD}$-consistent* if

$$X \succcurlyeq_{\mathcal{SSD}} Y \Longrightarrow \rho(X) \leq \rho(Y).$$

In words, if less risky losses demand smaller capital reserves. The following characterization is derived from the aforementioned [37].

THEOREM 4. *For a function $\rho : \mathcal{X} \to \mathbb{R}$, the following are equivalent:*

(i) ρ is a star-shaped and $\mathcal{SSD}$-consistent risk measure;

(ii) there exists a star-shaped set $\mathcal{G}$ of increasing functions $g : (0,1) \to \mathbb{R}$ with $g(0+) \leq 0$ such that

$$\rho(X) = \inf_{g \in \mathcal{G}} \sup_{\alpha \in (0,1)} \{\mathrm{ES}_\alpha(X) - g(\alpha)\} \qquad X \in \mathcal{X}. \tag{24}$$

Moreover, the class of star-shaped and $\mathcal{SSD}$-consistent risk measures is closed under the operations considered in Theorem 1.

The previous discussion on VaR, shows that the class of star-shaped and $\mathcal{SSD}$-consistent risk measures is strictly smaller than the class of star-shaped and law-invariant ones. The characterization of this class of risk measures concludes our analysis, by showing that star-shaped and law-invariant risk measures are robustifications of VaR in the same sense in which star-shaped and $\mathcal{SSD}$-consistent ones are robustifications of ES.

THEOREM 5. *For a function $\rho : \mathcal{X} \to \mathbb{R}$, the following are equivalent:*

(i) ρ is a star-shaped and law-invariant risk measure;

(ii) there exists a star-shaped set $\mathcal{G}$ of increasing functions $g : (0,1) \to \mathbb{R}$ with $g(0+) \leq 0$ such that

$$\rho(X) = \inf_{g \in \mathcal{G}} \sup_{\alpha \in (0,1)} \{\mathrm{VaR}_\alpha(X) - g(\alpha)\} \qquad X \in \mathcal{X}. \tag{25}$$

Moreover, the class of star-shaped and law-invariant risk measures is closed under the operations considered in Theorem 1.

According to Theorem 4, there is a minimal element in the set of all star-shaped and $\mathcal{SSD}$-consistent risk measures dominating VaR_α. One may wonder whether this minimal element coincides with ES_α, since ES_α is known as the smallest law-invariant convex risk measure dominating VaR_α (Proposition 5.4 of [2] and Theorem 4.67 of [22]).[5] The answer is affirmative, as stated in the following proposition, generalizing the aforementioned minimality of ES_α.

PROPOSITION 7. *For $\alpha \in (0,1)$, the smallest $\mathcal{SSD}$-consistent risk measure dominating* VaR_α *is* ES_α.

In Proposition 7, star shapedness is not mentioned, but since ES_α is star-shaped, it is also the smallest star-shaped $\mathcal{SSD}$-consistent risk measure dominating VaR_α.

8. Concluding remarks

This paper presents the first systematic investigation of the minimalistic assumption of *star shapedness* of risk measures, highlighting the following features:

- star shapedness captures the equivalent rationality assumptions of *increasing risk-to-exposure ratio* and *increasing relative ambiguity aversion*;
- star shapedness of risk measures is intimately linked to utility functions that have *local convexities* and possibly convex kinks;
- star shapedness is the shared property of most risk measures adopted in the financial practice and proposed in the literature, including *Value-at-Risk* and its robustifications, on the one hand, and *convex risk measures* on the other;
- star shapedness is preserved by *competitive risk sharing* and *robust aggregation* of risk measures, a property which is not enjoyed by convex risk measures;
- a risk measure is star-shaped if, and only if, it is a *minimum of (normalized) convex risk measures*, this makes its *optimization tractable*.

The risk measures studied in this paper are static and real-valued (unconditional). A promising direction of investigation is towards dynamic and conditional risk measures. The natural extension of the definition of star shapedness to L^0-valued risk measures presents itself as an economically compelling and mathematically powerful one, sharing the gist of subscale invariance of Bielecki, Cialenco, and Chen [8].

Aouani and Chateauneuf [4] characterized the class of distorted probabilities that are exact capacities by means of probability distortions $f:[0,1]\to\mathbb{R}$ that are star-shaped at both 0 and 1. Their Lemma 3.1 shows that these distortions are minima of convex distortions, thus, it has a "flavor" which is similar to our Theorem 2. But in this regard, it is worthwhile observing that the risk measures obtained by Choquet integration of these distorted probabilities are positively homogeneous (because they are comonotonic), and not only star-shaped. A gem, their Figure 1 shows that the iconic Friedman-Savage utility is star-shaped without being convex (see Section 2.4 of this paper).

In this paper, all risk measures by definition satisfy normalization, a basic requirement for computing capital charges. If normalization is dropped from the definition of a risk measure, then every monotone and translation-invariant mapping is the minimum of (non-normalized) convex risk measures.[6] This result was formalized in a recent study of Jia, Xia, and Zhao [31]. The key difference between a star-shaped ρ and a generic ρ is normalization of the convex risk measures $\gamma\in\Gamma$ representing ρ via $\rho=\min_{\gamma\in\Gamma}\gamma$. We mention a few important differences made by normalization of $\gamma\in\Gamma$ (thus, star-shapedness of ρ). First, normalization is essential for the economic interpretation of the minimum representation, where each member γ is interpreted a standalone risk measure (see Section 2.2) or certainty equivalent (see Section 2.3). Second, normalization of γ leads to a few non-trivial implications in optimization problems, as discussed in Remark 3. Third, there are available tools for optimizing star-shaped functions in the literature; see e.g., Rubinov and Andramonov [45].

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Appendix. Proofs

Proof of Proposition 1. (i) implies (ii). If $\alpha \in (0,1)$, then $1/\alpha \in (1,\infty)$, and, for all $X \in \mathcal{X}$, star shapedness implies

$$\rho(X) = \rho\left(\frac{1}{\alpha}(\alpha X)\right) \geq \frac{1}{\alpha}\rho(\alpha X).$$

(ii) implies (iii). Let $\beta > \gamma > 0$, then, for all $X \in \mathcal{X}$, $\gamma/\beta \in (0,1)$ implies

$$\rho(\gamma X) = \rho\left(\frac{\gamma}{\beta}(\beta X)\right) \leq \frac{\gamma}{\beta}\rho(\beta X)$$

that is, $\rho(\beta X)/\beta \geq \rho(\gamma X)/\gamma$.

(iii) implies (i). For all $X \in \mathcal{X}$ and all $\lambda > 1$, since r_X is increasing, one has

$$\frac{\rho(\lambda X)}{\lambda} \geq \frac{\rho(1X)}{1} = \rho(X)$$

as wanted. ■

Proof of Proposition 2. (i) implies (ii). Let $X \in \mathcal{A}_\rho$ and $\alpha \in (0,1)$. By Proposition 1, $\rho(\alpha X) \leq \alpha\rho(X) \leq \alpha 0 = 0$, that is, $\alpha X \in \mathcal{A}_\rho$. Normalization implies that the same happens for $\alpha = 0$, while the fact that $1X = X$ covers the case $\alpha = 1$.

(ii) implies (iii). In fact, $\mathcal{A}_\rho$ is an acceptance set, it is star-shaped, and by (7) we have $\rho = \rho_{\mathcal{A}_\rho}$.

(iii) implies (i). Let $X \in \mathcal{X}$ and $\alpha \in (0,1)$. If $m \in \mathbb{R}$ and $X - m \in \mathcal{A}$, then star shapedness of $\mathcal{A}$ implies $\alpha X - \alpha m \in \mathcal{A}$, thus

$$\rho(\alpha X) = \inf\{k \in \mathbb{R} \mid \alpha X - k \in \mathcal{A}\} \leq \alpha m.$$

Since m was arbitrarily chosen, one has

$$\frac{\rho(\alpha X)}{\alpha} \leq \inf\{m \in \mathbb{R} \mid X - m \in \mathcal{A}\} = \rho(X).$$

By Proposition 1, ρ is star-shaped. ■

Proof of Proposition 3. (i) implies (ii). Let $X \in \mathcal{X}$. We know from Proposition 1 that $r_X : \beta \mapsto \rho(\beta X)/\beta$ is increasing on $(0,\infty)$. Subadditivity of ρ implies that

$$\rho(2^n X) = \rho(2^{n-1}X + 2^{n-1}X) \leq 2\rho(2^{n-1}X)$$

for all $n \in \mathbb{Z}$, and hence

$$r_X(2^n) = \frac{\rho(2^n X)}{2^n} \leq \frac{\rho(2^{n-1}X)}{2^{n-1}} = r_X(2^{n-1}) \leq r_X(2^n).$$

Therefore, the increasing function r_X is constant on the set $\{2^n : n \in \mathbb{Z}\}$, and this forces it to be constant on $(0, \infty)$, showing positive homogeneity.

The rest is straightforward. ∎

Proof of Theorem 1. It is sufficient to prove the theorem for Choquet averages and inf-convolutions, since the supremum and infimum operations are special cases of the former.

Choquet averages. Let μ be a capacity on the set of all parts of I. Note that, for each $X \in \mathcal{X}$,

$$\begin{array}{l} \varrho^X : I \to \mathbb{R} \\ \qquad i \mapsto \varrho^X(i) = \rho_i(X) \end{array}$$

is such that $\inf_{\omega\in\Omega} X(\omega) \leq \rho_i(X) \leq \sup_{\omega\in\Omega} X(\omega)$ for all $i \in I$, thus ϱ^X is a bounded function from I to $\mathbb{R}$. Denote by $B(I)$ the set of all bounded functions from I to $\mathbb{R}$ and observe that

$$\begin{array}{lll} \mu : & B(I) & \to \mathbb{R} \\ & f & \mapsto \mu(f) = \int_I f(i)\, \mathrm{d}\mu(i) \end{array}$$

is a positively homogeneous risk measure on $B(I)$.

Next we show that if $\{\rho_i\}_{i\in I}$ is a collection of star-shaped risk measures, then

$$\rho_\mu(X) = \int_I \rho_i(X)\, \mathrm{d}\mu(i) = \mu\left(\varrho^X\right) \qquad X \in \mathcal{X}$$

is a star-shaped risk measure too.

1. *Monotonicity:* if $X \geqq Y$, then $\rho_\mu(X) \geq \rho_\mu(Y)$.
   In fact, $X \geqq Y$ implies $\rho_i(X) \geq \rho_i(Y)$ for all $i \in I$, that is, $\varrho^X \geqq \varrho^Y$, whence

$$\rho_\mu(X) = \mu\left(\varrho^X\right) \geq \mu\left(\varrho^Y\right) = \rho_\mu(Y).$$

2. *Translation invariance:* $\rho_\mu(X - m) = \rho_\mu(X) - m$ for all $X \in \mathcal{X}$ and all $m \in \mathbb{R}$.
   Given any $X \in \mathcal{X}$ and any $m \in \mathbb{R}$,

$$\varrho^{X-m}(i) = \rho_i(X - m) = \rho_i(X) - m = \varrho^X(i) - m \qquad i \in I$$

that is, $\varrho^{X-m} = \varrho^X - m$, whence

$$\rho_\mu(X - m) = \mu\left(\varrho^{X-m}\right) = \mu\left(\varrho^X - m\right) = \mu\left(\varrho^X\right) - m = \rho_\mu(X) - m.$$

3. *Normalization:* $\rho_\mu(0) = 0$.
   Clearly $\varrho^0(i) = \rho_i(0) = 0$ for all $i \in I$, that is, $\varrho^0 \equiv 0$, whence

$$\rho_\mu(0) = \mu\left(\varrho^0\right) = \mu(0) = 0.$$

4. *Star shapedness:* $\rho_\mu(\lambda X) \geq \lambda\rho_\mu(X)$ for all $X \in \mathcal{X}$ and all $\lambda > 1$.

Given any $X \in \mathcal{X}$ and any $\lambda > 1$, $\rho_i(\lambda X) \geq \lambda \rho_i(X)$ for all $i \in I$, that is, $\varrho^{\lambda X} \geqq \lambda \varrho^X$, whence

$$\rho_\mu(\lambda X) = \mu\left(\varrho^{\lambda X}\right) \geq \mu\left(\lambda \varrho^X\right) = \lambda \mu\left(\varrho^X\right) = \lambda \rho_\mu(X).$$

Infimal convolutions. Set $I = \{1, 2, ..., n\}$. For each $X \in \mathcal{X}$ define the sets

$$\mathcal{D}(X) = \left\{ \boldsymbol{Y} \in \mathcal{X}^I \mid \sum_{i \in I} Y_i = X \right\}$$
$$\mathcal{S}(X) = \left\{ \boldsymbol{Y} \in \mathcal{X}^I \mid \sum_{i \in I} Y_i \geqq X \right\}$$

and note that $\mathcal{D}(X) \subseteq \mathcal{S}(X) \subseteq \mathcal{S}(Y)$ if Y in $\mathcal{X}$ is such that $Y \leqq X$. Consider the function

$$\begin{aligned} \boldsymbol{\rho} : \mathcal{X}^I &\to \mathbb{R} \\ \boldsymbol{Y} &\mapsto \boldsymbol{\rho}(\boldsymbol{Y}) = \sum_{i \in I} \rho_i(Y_i). \end{aligned}$$

Note that

$$\rho_\diamond(X) = \inf\left\{ \sum_{i \in I} \rho_i(Y_i) \;\middle|\; Y_i \in \mathcal{X} \text{ for all } i \in I \text{ and } \sum_{i \in I} Y_i = X \right\} = \inf_{\boldsymbol{Y} \in \mathcal{D}(X)} \boldsymbol{\rho}(\boldsymbol{Y})$$

for all $X \in \mathcal{X}$. Next we show that

$$\rho_\diamond(X) = \inf_{\boldsymbol{Y} \in \mathcal{S}(X)} \boldsymbol{\rho}(\boldsymbol{Y}).$$

Since $\mathcal{D}(X) \subseteq \mathcal{S}(X)$, one has $\rho_\diamond(X) = \inf_{\boldsymbol{Y} \in \mathcal{D}(X)} \boldsymbol{\rho}(\boldsymbol{Y}) \geq \inf_{\boldsymbol{Y} \in \mathcal{S}(X)} \boldsymbol{\rho}(\boldsymbol{Y})$. As to the converse inequality, observe that, for each $\boldsymbol{Y} \in \mathcal{S}(X)$, we have that $Z = X - \sum_{i \in I} Y_i \leqq 0$ and $Z \in \mathcal{X}$, and we can define $\bar{Y}_1 = Y_1 + Z \leqq Y_1$. With this $\left(\bar{Y}_1, Y_2, \ldots, Y_n\right) \in \mathcal{X}^I$ and $\bar{Y}_1 + Y_2 + \cdots + Y_n = Y_1 + Z + Y_2 + \cdots + Y_n = X$, thus $\left(\bar{Y}_1, Y_2, \ldots, Y_n\right) \in \mathcal{D}(X)$ and

$$\rho_\diamond(X) = \inf_{\boldsymbol{V} \in \mathcal{D}(X)} \boldsymbol{\rho}(\boldsymbol{V}) \leq \boldsymbol{\rho}\left(\bar{Y}_1, Y_2, \ldots, Y_n\right) = \rho_1\left(\bar{Y}_1\right) + \cdots + \rho_n(Y_n) \leq \sum_{i \in I} \rho_i(Y_i) = \boldsymbol{\rho}(\boldsymbol{Y}).$$

Then $\rho_\diamond(X)$ is a lower bound for $\{\boldsymbol{\rho}(\boldsymbol{Y}) \mid \boldsymbol{Y} \in \mathcal{S}(X)\}$ hence $\rho_\diamond(X) \leq \inf_{\boldsymbol{Y} \in \mathcal{S}(X)} \boldsymbol{\rho}(\boldsymbol{Y})$.

1. *Monotonicity:* if $X \geqq Y$, then $\rho_\diamond(X) \geq \rho_\diamond(Y)$.

   As observed, $X \geqq Y$ implies $\mathcal{S}(X) \subseteq \mathcal{S}(Y)$, whence

$$\rho_\diamond(X) = \inf_{\boldsymbol{V} \in \mathcal{S}(X)} \boldsymbol{\rho}(\boldsymbol{V}) \geq \inf_{\boldsymbol{V} \in \mathcal{S}(Y)} \boldsymbol{\rho}(\boldsymbol{V}) = \rho_\diamond(Y).$$

2. *Translation invariance:* $\rho_\diamond(X - m) = \rho_\diamond(X) - m$ for all $X \in \mathcal{X}$ and all $m \in \mathbb{R}$.

   Given any $X \in \mathcal{X}$ and any $m \in \mathbb{R}$, if $\boldsymbol{Y} \in \mathcal{D}(X - m)$, then $(Y_1 + m, Y_2, ..., Y_n) \in \mathcal{D}(X)$, thus

$$\rho_\diamond(X) \leq \boldsymbol{\rho}(Y_1 + m, Y_2, ..., Y_n) = \rho_1(Y_1 + m) + ... + \rho_n(Y_n)$$

and

$$\rho_\diamond(X) - m \leq \rho_1(Y_1) + ... + \rho_n(Y_n) = \boldsymbol{\rho}(\boldsymbol{Y}).$$

Hence

$$\rho_\diamond(X) - m \leq \inf_{\boldsymbol{Y} \in \mathcal{D}(X-m)} \boldsymbol{\rho}(\boldsymbol{Y}) = \rho_\diamond(X - m)$$

that is, $\rho_\diamond(X) - m \le \rho_\diamond(X - m)$ for all $(X, m) \in \mathcal{X} \times \mathbb{R}$. But then, replacing m with $-m$, and X with $X - m$,

$$\rho_\diamond(X - m) + m = \rho_\diamond(X - m) - (-m) \le \rho_\diamond([X - m] - (-m)) = \rho_\diamond(X)$$

that is, $\rho_\diamond(X) - m \ge \rho_\diamond(X - m)$ for all $(X, m) \in \mathcal{X} \times \mathbb{R}$.

3. *Normalization:* $\rho_\diamond(0) = 0$.

Clearly,

$$\rho_\diamond(0) = \inf\left\{\sum_{i\in I} \rho_i(Y_i) \;\middle|\; Y_i \in \mathcal{X} \text{ for all } i \in I \text{ and } \sum_{i\in I} Y_i = 0\right\} \le \sum_{i\in I} \rho_i(0) = 0.$$

Then $\rho_\diamond(0) = 0$, if, and only if,

$$\inf\left\{\sum_{i\in I} \rho_i(Y_i) \;\middle|\; Y_i \in \mathcal{X} \text{ for all } i \in I \text{ and } \sum_{i\in I} Y_i = 0\right\} \ge 0$$

that is, if, and only if,

$$\sum_{i\in I} \rho_i(Y_i) \ge 0$$

for all $Y_1, Y_2, ..., Y_n \in \mathcal{X}$ such that $\sum_{i\in I} Y_i = 0$, which is precisely condition (10).

Remark. Observe that $\rho_\diamond(X) = \inf_{\boldsymbol{Y}\in\mathcal{D}(X)} \boldsymbol{\rho}(\boldsymbol{Y})$ guarantees that $\rho_\diamond(X) \in [-\infty, \infty)$ for all $X \in \mathcal{X}$. But together with translation invariance normalization implies that $\rho_\diamond(m) = m$ for all $m \in \mathbb{R}$, then monotonicity of $\rho_\diamond$ and boundedness of the elements of $\mathcal{X}$ guarantee that $\rho_\diamond(X) \in \mathbb{R}$ for all $X \in \mathcal{X}$.

4. *Star shapedness:* $\rho_\diamond(\lambda X) \ge \lambda\rho_\diamond(X)$ for all $X \in \mathcal{X}$ and all $\lambda > 1$.

Given any $X \in \mathcal{X}$ and any $\lambda > 0$, note that

$$\mathcal{D}(\lambda X) = \lambda\mathcal{D}(X).$$

In fact, $\boldsymbol{Y} \in \mathcal{D}(\lambda X)$ implies $\boldsymbol{Y} \in \mathcal{X}^I$ and $\sum_{i\in I} Y_i = \lambda X$, then $\boldsymbol{Z} = \lambda^{-1}\boldsymbol{Y} \in \mathcal{X}^I$ and $\sum_{i\in I} Z_i = \sum_{i\in I} \lambda^{-1} Y_i = \lambda^{-1}\sum_{i\in I} Y_i = X$, thus $\boldsymbol{Z} \in \mathcal{D}(X)$ and $\boldsymbol{Y} = \lambda\boldsymbol{Z}$. This shows that $\mathcal{D}(\lambda X) \subseteq \lambda\mathcal{D}(X)$ for all $(X, \lambda) \in \mathcal{X} \times (0, \infty)$, but then

$$\lambda\mathcal{D}(X) = \lambda\mathcal{D}\left(\frac{1}{\lambda}(\lambda X)\right) \subseteq \lambda\left(\frac{1}{\lambda}\mathcal{D}(\lambda X)\right) = \mathcal{D}(\lambda X)$$

and equality holds. Therefore, if $\lambda > 1$,

$$\begin{aligned}\rho_\diamond(\lambda X) &= \inf_{\boldsymbol{Y}\in\mathcal{D}(\lambda X)} \sum_{i\in I} \rho_i(Y_i) = \inf_{\boldsymbol{V}\in\mathcal{D}(X)} \sum_{i\in I} \rho_i(\lambda V_i) \\ &\ge \inf_{\boldsymbol{V}\in\mathcal{D}(X)} \sum_{i\in I} \lambda\rho_i(V_i) = \lambda \inf_{\boldsymbol{V}\in\mathcal{D}(X)} \sum_{i\in I} \rho_i(V_i) = \lambda\rho_\diamond(X)\end{aligned}$$

as wanted. ■

Proof of Theorem 2. Since positively homogeneous risk measures are obviously star-shaped, the statement says that they admit both representations:

$$\rho(X) = \min\{\gamma(X) \mid \gamma \geqq \rho \text{ is a convex risk measure}\}$$

and

$$\rho(X) = \min\{\delta(X) \mid \delta \geqq \rho \text{ is a coherent risk measure}\}.$$

In this proof, when we write "star-shaped (resp. positively homogeneous)," we mean "to obtain the general result that holds for all star-shaped risk measures (resp. to obtain the special result that holds for the positively homogeneous ones)."

(i) implies (ii). For each $Y \in \mathcal{X}$ set

$$\mathcal{A}_Y = \mathrm{co}\{Y - \rho(Y), 0\} - \mathcal{X}^+$$

where $\mathcal{X}^+$ is the cone of non-negative elements in $\mathcal{X}$, and $\mathrm{co}\{Y - \rho(Y), 0\}$ is the convex set (resp. the convex cone) generated by $\{Y - \rho(Y), 0\}$. Recall that the *convex cone* of a subset S of a vector space is the set of all elements of the form $as + bt$ such that $a, b \in \mathbb{R}_+$ and $s, t \in S$. We note that $\mathcal{A}_Y$ is a convex set (resp. a convex cone), because it is a sum of convex sets (resp. convex cones).

Since ρ is translation invariant, $\rho(Y - \rho(Y)) = 0$ and $Y - \rho(Y) \in \mathcal{A}_\rho$.

- In the star-shaped case, by Proposition 2, $\mathcal{A}_\rho$ is star-shaped. For any $\alpha \in [0,1]$ we have $\alpha(Y - \rho(Y)) \in \mathcal{A}_\rho$, implying $\alpha(Y - \rho(Y)) - \mathcal{X}^+ \subseteq \mathcal{A}_\rho$. Hence, $\mathcal{A}_Y \subseteq \mathcal{A}_\rho$.
- In the positively homogeneous case, $\mathcal{A}_\rho$ is conic. For any $\alpha \in [0,\infty)$ we have $\alpha(Y - \rho(Y)) \in \mathcal{A}_\rho$, implying $\alpha(Y - \rho(Y)) - \mathcal{X}^+ \subseteq \mathcal{A}_\rho$. Hence, $\mathcal{A}_Y \subseteq \mathcal{A}_\rho$.

Since $0 \in \mathrm{co}\{Y - \rho(Y), 0\}$ then $0 \in \mathcal{A}_Y$, it follows that

$$0 \leq \sup\{m \in \mathbb{R} \mid m \in \mathcal{A}_Y\} \leq \max\{m \in \mathbb{R} \mid m \in \mathcal{A}_\rho\} = 0,$$

and so $0 = \max\{m \in \mathbb{R} \mid m \in \mathcal{A}_Y\}$. Moreover, it is straightforward to check that $Z \leqq X$ implies $Z \in \mathcal{A}_Y$ if $X \in \mathcal{A}_Y$.

So far, we know that $\mathcal{A}_Y$ is a convex (resp. coherent) acceptance set included in $\mathcal{A}_\rho$.

Now, notice that since $Y - \rho(Y) \in \mathcal{A}_Y$, we have that

$$\rho_{\mathcal{A}_Y}(Y) = \inf\{m \in \mathbb{R} \mid Y - m \in \mathcal{A}_Y\} \leq \rho(Y).$$

But since $\mathcal{A}_Y \subseteq \mathcal{A}_\rho$, then, for all $X \in \mathcal{X}$,

$$\begin{aligned}\rho(X) &= \inf\{m \in \mathbb{R} \mid X - m \in \mathcal{A}_\rho\} \\ &\leq \inf\{m \in \mathbb{R} \mid X - m \in \mathcal{A}_Y\} = \rho_{\mathcal{A}_Y}(X),\end{aligned}$$

and in particular, $\rho(Y) = \rho_{\mathcal{A}_Y}(Y)$.

Summing up, for all $X \in \mathcal{X}$,

$$\rho(X) \leq \inf_{Y \in \mathcal{X}} \rho_{\mathcal{A}_Y}(X) \quad \text{and} \quad \rho(X) = \rho_{\mathcal{A}_X}(X)$$

that is

$$\rho(X) = \min_{Y \in \mathcal{X}} \rho_{\mathcal{A}_Y}(X).$$

A fortiori, letting Γ be the set of all convex (resp. coherent) risk measures dominating ρ, since $\rho_{\mathcal{A}_X}$ belongs to Γ,

$$\rho(X) \leq \inf_{\gamma \in \Gamma} \gamma(X) \quad \text{ and } \quad \rho(X) = \rho_{\mathcal{A}_X}(X)$$

that is

$$\rho(X) = \min_{\gamma \in \Gamma} \gamma(X).$$

This completes the main implication of the theorem and shows that Γ can be chosen as the set of all convex (resp. coherent) risk measures dominating ρ.

(ii) implies (iii). Let Γ be a collection of convex (resp. coherent) risk measures such that

$$\rho(X) = \min_{\gamma \in \Gamma} \gamma(X) \qquad X \in \mathcal{X}$$

and denote by $\mathcal{A}_\gamma$ the acceptance set of each $\gamma \in \Gamma$. Theorem 1 guarantees that ρ is a star-shaped risk measure, moreover,

$$\begin{aligned} \mathcal{A}_\rho &= \{X \in \mathcal{X} \mid \rho(X) \leq 0\} = \{X \in \mathcal{X} \mid \gamma(X) \leq 0 \text{ for some } \gamma \in \Gamma\} \\ &= \{X \in \mathcal{X} \mid X \in \mathcal{A}_\gamma \text{ for some } \gamma \in \Gamma\} = \bigcup_{\gamma \in \Gamma} \mathcal{A}_\gamma. \end{aligned}$$

Then $\{\mathcal{A}_\gamma\}_{\gamma \in \Gamma}$ is a family of convex (resp. coherent) acceptance sets such that

$$\begin{aligned} \rho(X) &= \min\{m \in \mathbb{R} \mid X - m \in \mathcal{A}_\rho\} = \min\left\{m \in \mathbb{R} \;\middle|\; X - m \in \bigcup_{\gamma \in \Gamma} \mathcal{A}_\gamma\right\} \\ &= \min\{m \in \mathbb{R} \mid X - m \in \mathcal{A}_\gamma \text{ for some } \gamma \in \Gamma\} \end{aligned}$$

for all $X \in \mathcal{X}$. Clearly, when Γ is the set of all convex (resp. coherent) risk measures dominating ρ, $\{\mathcal{A}_\gamma\}_{\gamma \in \Gamma}$ is the family of their acceptance sets.

(iii) implies (i). Let $\{\mathcal{A}_\beta\}_{\beta \in B}$ be a family of convex (resp. coherent) acceptance sets such that

$$\begin{aligned} \rho(X) &= \min\{m \in \mathbb{R} \mid X - m \in \mathcal{A}_\beta \text{ for some } \beta \in B\} \\ &= \min\left\{m \in \mathbb{R} \;\middle|\; X - m \in \bigcup_{\beta \in B} \mathcal{A}_\beta\right\}. \end{aligned}$$

For each $X \in \mathcal{X}$ define

$$\begin{aligned} f_X : \mathbb{R} &\to \mathcal{X} \\ m &\mapsto X - m \end{aligned}$$

and note that

$$X - m \in \bigcup_{\beta \in B} \mathcal{A}_\beta \iff f_X(m) \in \bigcup_{\beta \in B} \mathcal{A}_\beta \iff m \in f_X^{-1}\left(\bigcup_{\beta \in B} \mathcal{A}_\beta\right) \iff m \in \bigcup_{\beta \in B} f_X^{-1}(\mathcal{A}_\beta).$$

With this

$$\begin{aligned} \rho(X) &= \inf\left\{m \in \mathbb{R} \;\middle|\; m \in \bigcup_{\beta \in B} f_X^{-1}(\mathcal{A}_\beta)\right\} = \inf\left(\bigcup_{\beta \in B} f_X^{-1}(\mathcal{A}_\beta)\right) \\ &= \inf_{\beta \in B}\left\{\inf f_X^{-1}(\mathcal{A}_\beta)\right\} = \inf_{\beta \in B}\left\{\inf\left\{m \in \mathbb{R} \mid m \in f_X^{-1}(\mathcal{A}_\beta)\right\}\right\} \\ &= \inf_{\beta \in B} \underbrace{\inf\{m \in \mathbb{R} \mid X - m \in \mathcal{A}_\beta\}}_{\rho_\beta(X)} \end{aligned}$$

where ρ_β is the risk measure generated by $\mathcal{A}_\beta$, which is convex (resp. coherent) since the $\mathcal{A}_\beta$'s are convex (resp. coherent) acceptance sets. But then Theorem 1 guarantees that ρ is a star-shaped risk measure (if, in addition, the ρ_β's are coherent, immediate verification yields positive homogeneity of ρ). ■

Proof of Theorem 3. By Theorem 1, we know that the Choquet average, supremum, infimum, and inf-convolution are star-shaped risk measures. We proceed to show that representations (13)-(16) hold for an arbitrary $X \in \mathcal{X}$. For ease of notation, set $\boldsymbol{\Gamma} = \prod_{i\in I} \Gamma_i$ and denote by $\boldsymbol{\gamma} = (\gamma_i)_{i\in I}$ its generic element, and note that

$$\int \Gamma_i \,\mathrm{d}\mu\,(i) = \left\{ \int_I \gamma_i \,\mathrm{d}\mu\,(i) \ \mid \boldsymbol{\gamma} \in \boldsymbol{\Gamma} \right\} \quad \text{and} \quad \Box_{i\in I}\Gamma_i = \{\Box_{i\in I}\gamma_i \mid \boldsymbol{\gamma} \in \boldsymbol{\Gamma}\}\,.$$

Choquet averages. For each $i \in I$, choose $\gamma_i^* \in \Gamma_i$ such that $\gamma_i^*(X) = \rho_i(X)$, so that

$$\inf_{\boldsymbol{\gamma}\in\boldsymbol{\Gamma}} \int_I \gamma_i(X)\,\mathrm{d}\mu\,(i) \le \int_I \gamma_i^*(X)\,\mathrm{d}\mu\,(i) = \int_I \rho_i(X)\,\mathrm{d}\mu\,(i) = \int_I \min_{\gamma_i\in\Gamma_i} \gamma_i(X)\,\mathrm{d}\mu\,(i) \le \inf_{\boldsymbol{\gamma}\in\boldsymbol{\Gamma}} \int_I \gamma_i(X)\,\mathrm{d}\mu\,(i)\,.$$

Since $\boldsymbol{\gamma}^* \in \boldsymbol{\Gamma}$, one has

$$\int_I \rho_i(X)\,\mathrm{d}\mu\,(i) = \int_I \gamma_i^*(X)\ \ \mathrm{d}\mu\,(i) = \min_{\boldsymbol{\gamma}\in\boldsymbol{\Gamma}} \int_I \gamma_i(X)\,\mathrm{d}\mu\,(i) = \min_{\delta\in\int\Gamma_i\,\mathrm{d}\mu(i)} \delta\,(X)\,.$$

Suprema. Since

$$\sup_{i\in I} \rho_i(X)$$

is a star-shaped risk measure, by Theorem 2,

$$\begin{aligned}
\sup_{i\in I} \rho_i(X) &= \min\left\{ \gamma\,(X) \mid \gamma \text{ is a convex risk measure and, } \forall Y \in \mathcal{X},\ \gamma\,(Y) \ge \sup_{i\in I} \rho_i\,(Y) \right\} \\
&= \min\left\{ \gamma\,(X) \mid \gamma \text{ is a convex risk measure and } \forall Y \in \mathcal{X},\ \forall i \in I,\ \gamma\,(Y) \ge \rho_i\,(Y) \right\} \\
&= \min\left\{ \gamma\,(X) \mid \gamma \text{ is a convex risk measure and } \forall i \in I,\ \forall Y \in \mathcal{X},\ \gamma\,(Y) \ge \rho_i\,(Y) \right\} \\
&= \min\left\{ \gamma\,(X) \mid \gamma \text{ is a convex risk measure and } \forall i \in I,\ \gamma \in \tilde{\Gamma}_i \right\}.
\end{aligned}$$

Moreover, since each $\tilde{\Gamma}_i$ consists of convex risk measures, one has

$$\sup_{i\in I} \rho_i(X) = \min\left\{ \gamma\,(X) \mid \forall i \in I,\ \gamma \in \tilde{\Gamma}_i \right\} = \min\left\{ \gamma\,(X) \;\middle|\; \gamma \in \bigcap_{i\in I} \tilde{\Gamma}_i \right\}$$

also note that —since the set over which minimization was performed never changed— the set

$$\bigcap_{i\in I} \tilde{\Gamma}_i$$

consists of all convex risk measures that dominate $\sup_{i\in I} \rho_i$.

Infima. As well known,

$$\inf_{i\in I} \min_{\gamma_i\in\Gamma_i} \gamma_i(X) = \inf_{\gamma\in\cup_{i\in I}\Gamma_i} \gamma(X).$$

Moreover, if the infimum on the left-hand side is a minimum, then the infimum on the right-hand side is also a minimum.

Inf-convolutions. First observe that for each $\boldsymbol{\gamma}=(\gamma_i)_{i\in I}\in\prod_{i\in I}\Gamma_i=\boldsymbol{\Gamma}$, and all $Y_1,Y_2,...,Y_n\in\mathcal{X}$ such that $\sum_{i\in I}Y_i=0$, we have

$$\sum_{i\in I}\gamma_i(Y_i)\geq\sum_{i\in I}\rho_i(Y_i)\geq 0$$

because $\gamma_i\geqq\rho_i$, thus each $\square_{i\in I}\gamma_i\in\square_{i\in I}\Gamma_i$ is a (well defined) convex risk measure.

We want to show that:

- each $\square_{i\in I}\gamma_i\in\square_{i\in I}\Gamma_i$ dominates $\rho_\diamond=\square_{i\in I}\rho_i$;
- $\rho_\diamond(X)=\inf\{\square_{i\in I}\gamma_i(X)\mid\boldsymbol{\gamma}\in\boldsymbol{\Gamma}\}=\inf\{\delta(X)\mid\delta\in\square_{i\in I}\Gamma_i\}$.

As to the first point, note that, for all $\boldsymbol{\gamma}\in\boldsymbol{\Gamma}$ and all $Y_1,...,Y_n\in\mathcal{X}$ such that $\sum_{i\in I}Y_i=X$, we have

$$\sum_{i\in I}\gamma_i(Y_i)\geq\sum_{i\in I}\rho_i(Y_i)$$

but then

$$\begin{aligned}\square_{i\in I}\gamma_i(X)&=\inf\left\{\sum_{i\in I}\gamma_i(Y_i)\;\middle|\;Y_i\in\mathcal{X}\text{ for all }i\in I\text{ and }\sum_{i\in I}Y_i=X\right\}\\&\geq\inf\left\{\sum_{i\in I}\rho_i(Y_i)\;\middle|\;Y_i\in\mathcal{X}\text{ for all }i\in I\text{ and }\sum_{i\in I}Y_i=X\right\}\\&=\rho_\diamond(X)\end{aligned}$$

as wanted.

As to the second, for each $X\in\mathcal{X}$ and all $\varepsilon>0$, there exist $Y_1,...,Y_n\in\mathcal{X}$ such that $\sum_{i\in I}Y_i=X$ and

$$\rho_\diamond(X)+\varepsilon\geq\sum_{i\in I}\rho_i(Y_i)=\sum_{i\in I}\bar{\gamma}_i(Y_i)$$

for some $(\bar{\gamma}_1,...,\bar{\gamma}_n)\in\prod_{i\in I}\Gamma_i$ because minima are attained in (12); otherwise one could use the fact that $\sum_{i\in I}\rho_i(Y_i)\geq\sum_{i\in I}(\bar{\gamma}_i(Y_i)-\varepsilon/2n)$. It follows that

$$\rho_\diamond(X)+\varepsilon\geq\square_{i\in I}\bar{\gamma}_i(X)\geq\inf\left\{\square_{i\in I}\gamma_i(X)\mid(\gamma_i)_{i\in I}\in\prod\nolimits_{i\in I}\Gamma_i\right\}=\inf\{\delta(X)\mid\delta\in\square_{i\in I}\Gamma_i\}.$$

Letting $\varepsilon\to 0$, we conclude that

$$\rho_\diamond(X)\geq\inf\{\delta(X)\mid\delta\in\square_{i\in I}\Gamma_i\}$$

and the converse inequality descends from the previous point.

Finally, if

$$\rho_\diamond(X)=\min\left\{\sum_{i\in I}\rho_i(Y_i)\;\middle|\;Y_i\in\mathcal{X}\text{ for all }i\in I\text{ and }\sum_{i\in I}Y_i=X\right\}.$$

Then, for each $X\in\mathcal{X}$, take $Y_i\in\mathcal{X}$ $(i\in I)$ such that $\sum_{i\in I}Y_i=X$ and

$$\rho_\diamond(X)=\sum_{i\in I}\rho_i(Y_i).$$

Since minima are attained in (12), then there exists some $(\bar{\gamma}_1,...,\bar{\gamma}_n)\in\prod_{i\in I}\Gamma_i$ such that

$$\rho_\diamond(X)=\sum_{i\in I}\rho_i(Y_i)=\sum_{i\in I}\bar{\gamma}_i(Y_i)$$

but then

$$\inf\left\{\Box_{i\in I}\gamma_i(X) \mid (\gamma_i)_{i\in I}\in\prod\nolimits_{i\in I}\Gamma_i\right\}=\rho_\diamond(X)=\sum_{i\in I}\bar{\gamma}_i(Y_i)\geq\Box_{i\in I}\bar{\gamma}_i(X)$$

thus the infimum is attained, which concludes the proof. ■

Proof of Proposition 5. We have shown in the proof of Theorem 2 that the star-shaped risk measure ρ has representation

$$\rho(X)=\min_{Y\in\mathcal{X}}\rho_{\mathcal{A}_Y}(X)\qquad X\in\mathcal{X},$$

where $\rho_{\mathcal{A}_Y}$ has acceptance set

$$\mathcal{A}_Y=\mathrm{co}\{Y-\rho(Y),0\}-\mathcal{X}^+=\{Z\in\mathcal{X}:Z\leqq\alpha(Y-\rho(Y))\text{ for some }\alpha\in[0,1]\}.$$

To get the representation (17), it suffices to show that each $\rho_{\mathcal{A}_Y}$ has a representation $\rho_{\mathcal{A}_Y}=\sup_{Q\in\mathcal{P}}\{\mathbb{E}_Q-\alpha_\gamma(Q)\}$. This is equivalent to the Fatou property of $\rho_{\mathcal{A}_Y}$ as in Theorem 4.33 of [22]. The Fatou property of $\rho_{\mathcal{A}_Y}$ can be stated via the Fatou closure of $\mathcal{A}_Y$, that is,

$$X_n\xrightarrow{\text{a.s.}}X,\ (X_n)_{n\in\mathbb{N}}\text{ is uniformly bounded, and }(X_n)_{n\in\mathbb{N}}\subseteq\mathcal{A}_Y\implies X\in\mathcal{A}_Y,$$

where $\xrightarrow{\text{a.s.}}$ represents almost sure convergence. We next show this Fatou closure. For each $n\in\mathbb{N}$, since $X_n\in\mathcal{A}_Y$, we have $X_n\leqq\alpha_n(Y-\rho(Y))$ for some $\alpha_n\in[0,1]$. Noting that $[0,1]$ is compact, take a subsequence $(\alpha_{n_k})_{k\in\mathbb{N}}$ of $(\alpha_n)_{n\in\mathbb{N}}$ which converges to a limit, which is denoted by $\alpha_0\in[0,1]$. We have $X_{n_k}\xrightarrow{\text{a.s.}}X$, and therefore

$$\alpha_0(Y-\rho(Y))=\lim_{k\to\infty}\alpha_{n_k}(Y-\rho(Y))\geqq\lim_{k\to\infty}X_{n_k}=X.$$

Thus, $X\in\mathcal{A}_Y$, showing that $\mathcal{A}_Y$ is Fatou closed. This leads to (17). ■

Proof of Proposition 6. The proof is a routine verification once the reader recalls that $\rho_i=\min_{\gamma\in\Gamma_i}\gamma$ for each $i\in I$ (Theorem 2) and $\sup_{i\in I}\rho_i=\rho_\vee=\min_{\gamma\in\tilde{\Gamma}_I}\gamma$ (Theorem 3). ■

Proof of Theorem 4. By Theorem 3.1 of [37], for an $\mathcal{SSD}$-consistent risk measure ρ, the representation (24) holds with

$$\mathcal{G}=\{g_Y:(0,1)\to\mathbb{R},\ \alpha\mapsto\mathrm{ES}_\alpha(Y)\mid Y\in\mathcal{A}_\rho\}.$$

The closed interval $[0,1]$ is used in [37], but it is easy to check that one can also use the open interval $(0,1)$, since $\alpha\mapsto\mathrm{ES}_\alpha(Y)$ is continuous.

(i) implies (ii). It suffices to check that the set $\mathcal{G}$ is star-shaped when ρ is star-shaped. This follows immediately from the fact that $Y\in\mathcal{A}_\rho$ implies $\lambda Y\in\mathcal{A}_\rho$ for $\lambda\in[0,1]$, and positive homogeneity of ES gives $\lambda g_Y\in\mathcal{G}$.

(ii) implies (i). It suffices to check that ρ is star-shaped when $\mathcal{G}$ is star-shaped. For $\lambda\in[0,1]$, noting that $\lambda\mathcal{G}\subseteq\mathcal{G}$, we have, for $X\in\mathcal{X}$,

$$\begin{aligned}\rho(\lambda X)&=\inf_{g\in\mathcal{G}}\sup_{\alpha\in(0,1)}\{\mathrm{ES}_\alpha(\lambda X)-g(\alpha)\}\\&\leq\inf_{g\in\lambda\mathcal{G}}\sup_{\alpha\in(0,1)}\{\lambda\mathrm{ES}_\alpha(X)-g(\alpha)\}\\&=\lambda\inf_{g\in\mathcal{G}}\sup_{\alpha\in(0,1)}\{\mathrm{ES}_\alpha(X)-g(\alpha)\}=\lambda\rho(X).\end{aligned}$$

Hence, ρ is star-shaped.

The final part of the statement regarding inf-convolutions follows from Theorem 4.1 of [37]. The rest is straightforward. ■

Proof of Theorem 5. (i) implies (ii). We denote by $X \succcurlyeq_{\mathcal{FSD}} Y$ the fact that $X \in \mathcal{X}$ first-order stochastically dominates $Y \in \mathcal{X}$, which means

$$F_X \geq F_Y, \text{ or equivalently, } \mathrm{VaR}_\alpha(X) \leq \mathrm{VaR}_\alpha(Y) \text{ for all } \alpha \in (0,1), \tag{26}$$

where F_X and F_Y are the distribution functions of X and Y under P. Since ρ is law-invariant and monotonic,

$$X \succcurlyeq_{\mathcal{FSD}} Y \implies \rho(X) \leq \rho(Y).$$

Let $\mathcal{A}_\rho = \{Y \in \mathcal{X} : \rho(Y) \leq 0\}$ be the acceptance set of ρ. For each $Y \in \mathcal{A}_\rho$, let $g_Y : (0,1) \to \mathbb{R}$ be given by $g_Y(\alpha) = \mathrm{VaR}_\alpha(Y)$. Note that if $Y \in \mathcal{A}_\rho$, then:

- The same is true for βY for any $\beta \in [0,1]$, hence, by positive homogeneity of VaR,

$$\mathcal{G} = \{g_Z : Z \in \mathcal{A}_\rho\}$$

is a star-shaped set.

- We have $g_Y(0+) \leq 0$. To see this, as well known (see, e.g., [22, Remark 4.50]), $g_Y(0+)$ is the essential infimum of Y, if it were strictly positive, it would follow that $Y \geqq \varepsilon$ for some $\varepsilon > 0$, and so $\varepsilon = \rho(\varepsilon) \leq \rho(Y) \leq 0$ because $Y \in \mathcal{A}_\rho$, a contradiction.
- The same is true for any $Z \in \mathcal{X}$ such that $Z \succcurlyeq_{\mathcal{FSD}} Y$, and hence

$$\mathcal{A}_\rho = \bigcup_{Y \in \mathcal{A}_\rho} \{Z \in \mathcal{X} \mid Z \succcurlyeq_{\mathcal{FSD}} Y\}.$$

It follows that

$$\begin{aligned}
\rho(X) &= \inf\{m \in \mathbb{R} \mid X - m \in \mathcal{A}_\rho\} \\
&= \inf\left\{ m \in \mathbb{R} \mid X - m \in \bigcup_{Y \in \mathcal{A}_\rho} \{Z \in \mathcal{X} \mid Z \succcurlyeq_{\mathcal{FSD}} Y\} \right\} \\
&= \inf \bigcup_{Y \in \mathcal{A}_\rho} \{m \in \mathbb{R} \mid X - m \in \{Z \in \mathcal{X} \mid Z \succcurlyeq_{\mathcal{FSD}} Y\}\} \\
&= \inf \bigcup_{Y \in \mathcal{A}_\rho} \{m \in \mathbb{R} \mid X - m \succcurlyeq_{\mathcal{FSD}} Y\} \\
&= \inf_{Y \in \mathcal{A}_\rho} \inf\{m \in \mathbb{R} \mid X - m \succcurlyeq_{\mathcal{FSD}} Y\}.
\end{aligned}$$

By (26), we have

$$\begin{aligned}
\inf\{m \in \mathbb{R} \mid X - m \succcurlyeq_{\mathcal{FSD}} Y\} &= \inf\{m \in \mathbb{R} \mid \mathrm{VaR}_\alpha(X - m) \leq \mathrm{VaR}_\alpha(Y) \text{ for all } \alpha \in (0,1)\} \\
&= \inf\{m \in \mathbb{R} \mid m \geq \mathrm{VaR}_\alpha(X) - \mathrm{VaR}_\alpha(Y) \text{ for all } \alpha \in (0,1)\} \\
&= \sup_{\alpha \in (0,1)} \{\mathrm{VaR}_\alpha(X) - \mathrm{VaR}_\alpha(Y)\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}\rho(X) &= \inf_{Y\in\mathcal{A}_\rho} \inf\{m\in\mathbb{R} \mid X-m \succcurlyeq_{\mathcal{FSD}} Y\}\\ &= \inf_{Y\in\mathcal{A}_\rho} \sup_{\alpha\in(0,1)} \{\mathrm{VaR}_\alpha(X)-\mathrm{VaR}_\alpha(Y)\}\\ &= \inf_{Y\in\mathcal{A}_\rho} \sup_{\alpha\in(0,1)} \{\mathrm{VaR}_\alpha(X)-g_Y(\alpha)\} = \inf_{g\in\mathcal{G}} \sup_{\alpha\in(0,1)} \{\mathrm{VaR}_\alpha(X)-g(\alpha)\},\end{aligned}$$

thus showing (25).

(ii) implies (i). Monotonicity, translation invariance, and law-invariance of ρ are obvious. Moreover,

$$\rho(0) = \inf_{g\in\mathcal{G}} \sup_{\alpha\in(0,1)} \{-g(\alpha)\} = \inf_{g\in\mathcal{G}} -g(0+) = -\sup_{g\in\mathcal{G}} g(0+) = 0$$

because star shapedness of $\mathcal{G}$ guarantees $0\in\mathcal{G}$ and $g(0+)\le 0$ for all $g\in\mathcal{G}$, and hence ρ is normalized. This together with monotonicity and translation invariance also shows that ρ takes real values. Hence ρ is a law-invariant risk measure. Finally, if $\lambda>1$ and $X\in\mathcal{X}$, then

$$\begin{aligned}\rho(\lambda X) &= \inf_{g\in\mathcal{G}} \sup_{\alpha\in(0,1)} \{\ \mathrm{VaR}_\alpha(\lambda X)-g(\alpha)\} = \inf_{g\in\mathcal{G}} \sup_{\alpha\in(0,1)} \{\lambda\mathrm{VaR}_\alpha(X)-g(\alpha)\}\\ &= \lambda \inf_{g\in\mathcal{G}} \sup_{\alpha\in(0,1)} \left\{\mathrm{VaR}_\alpha(X)-\frac{g(\alpha)}{\lambda}\right\} = \lambda \inf_{h\in\lambda^{-1}\mathcal{G}} \sup_{\alpha\in(0,1)} \{\mathrm{VaR}_\alpha(X)-h(\alpha)\}\\ &\ge \lambda \inf_{g\in\mathcal{G}} \sup_{\alpha\in(0,1)} \{\mathrm{VaR}_\alpha(X)-g(\alpha)\} = \lambda\rho(X)\end{aligned}$$

because $\lambda^{-1}\mathcal{G}\subseteq\mathcal{G}$. Therefore, ρ is star-shaped.

The final part of the statement regarding inf-convolutions follows from Theorem 2 of Liu, Wang, and Wei [35]. The rest is straightforward. ■

Proof of Proposition 7. Suppose that ρ is an $\mathcal{SSD}$-consistent risk measure dominating VaR_α. We will show $\rho\geqq\mathrm{ES}_\alpha$. Suppose otherwise. By translation invariance, there exists $X\in\mathcal{X}$ such that $\rho(X)\le 0$ and $\mathrm{ES}_\alpha(X)>0$. Since $\alpha\mapsto\mathrm{ES}_\alpha(X)$ is continuous, there exists $\alpha'\in(0,\alpha)$ such that $\mathrm{ES}_{\alpha'}(X)>0$, and denote by $x=\mathrm{ES}_{\alpha'}(X)$. Construct a random variable Y by

$$Y = X\mathbb{1}_{A^c} + x\mathbb{1}_A,$$

where A is an α'-tail event of X as defined by [50], meaning that $P(A)=1-\alpha'$ and $X(\omega)\ge X(\omega')$ for almost every $\omega\in A$ and $\omega'\in A^c$. Note that $X\le 0$ almost surely on A^c because $\mathrm{VaR}_{\alpha'}(X)\le\mathrm{VaR}_\alpha(X)\le\rho(X)\le 0$. Moreover, since $x=\mathrm{ES}_{\alpha'}(X)=\mathbb{E}_P[X|A]$, we have $Y=\mathbb{E}_P[X|Y]$. By Jensen's inequality, we have $Y\succcurlyeq_{\mathcal{SSD}} X$, and $\mathcal{SSD}$-consistency of ρ gives $\rho(Y)\le\rho(X)$. However, since $P(Y>0)=P(A)=1-\alpha'>1-\alpha$, we have $\mathrm{VaR}_\alpha(Y)>0$, which leads to $\rho(Y)<\mathrm{VaR}_\alpha(Y)$, a contradiction. Therefore, we obtain $\rho\geqq\mathrm{ES}_\alpha$. ■

Endnotes

1. *For the connoisseur.* The complete name would be "monetary and normalized risk measures." Of course there are meaningful "risk measures" that do not satisfy these properties, see, e.g., Cerreia-Vioglio et al. [11] and Farkas, Koch-Medina, and Munari [19], but their study goes beyond the scope of this paper.

2. Some possible explanations of this restriction, quoting [29, p. 1147]: "Both parties to a swap need to agree on where the swap will be cleared, and where their optimal allocations may differ. To clear at a given CCP, both parties need to be members of the CCP or trade through members of the CCP. [...] Clearing members clear trades for clients as well as for their own accounts; this limits their ability to subdivide positions. [...] Dealers may prefer one CCP over another for reasons unrelated to margin requirements [...]"

3. Indeed, an implication of our main Theorem 2 is that the preferences obtained by replacing uncertainty aversion with increasing relative ambiguity aversion, in the axiom set of [36], are characterized by representation (1). See also Chandrasekher et al. [14] for a preference representation which is similar, but where the α_a's are not necessarily grounded, hence they lose the interpretation of action-dependent ambiguity indexes, and increasing relative ambiguity aversion is lost too.

4. The normality condition (10) is required because also in the case of two convex risk measures their inf-convolution might fail to be a risk measure (see Barrieu and El Karoui [6]). This condition is obviously satisfied if there exists a linear functional which is dominated by all risk measures in the collection.

5. We thank an anonymous referee for raising this question.

6. We thank an anonymous referee for raising this point.

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In memoriam. I met Erio Castagnoli (Mantova, July 2nd, 1943 — January 9th, 2019), in September, a quarter century ago, at the conference in Urbino where I presented my first paper. Erio gave a lecture on the importance of the Bipolar Theorem of Functional Analysis in Decision Theory: I was hypnotized by the beauty. I knew a bit of the first topic, nothing of the second. We started to work on it that very afternoon.

Erio changed my life, and I will owe him forever.

Fabio Maccheroni