Quantum probabilities, synchronous games and C*-algebras

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joint work with many people references at end

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Finite Input-Output Games

These are games where two *cooperating* but *non-communicating* players, Alice and Bob try to give *correct* answers to questions posed by the Referee.

For each *round* of the game, the cooperating players each receive an input, i.e., a question, from the Referee from some finite set of inputs I_A, I_B.

They must each produce an output, i.e., an answer, belonging to some finite set O_A, O_B.

The game $\mathcal{G}$ has *rules* given by a function

$$\lambda : I_A \times I_B \times O_A \times O_B \to \{0, 1\}$$

where $\lambda(x, y, a, b) = 1$ means that if Alice and Bob receive inputs x, y, respectively and produce respective outputs a, b, then they win. If $\lambda(x, y, a, b) = 0$, they lose.

They both know the rule function and can jointly create a strategy for winning, but once the game starts Alice and Bob must produce their outputs without knowing what input the other received and without knowing what output the other produced. This is what is meant by *non-communicating*.

A *deterministic strategy* is a pair of functions, $f_A : I_A \to O_A$, $f_B : I_B \to O_B$ so that when Alice and Bob receive inputs x, y then they give outputs, $f_A(x), f_B(y)$.

A deterministic strategy is called *perfect* if it always wins, i.e., $\lambda(x, y, f_A(x), f_B(y)) = 1, \forall x, y$.

A *random strategy* just means that each time they receive the input pair (x, y) they do not necessarily produce the same output. In this case there is a conditional probability density $p(a, b|x, y)$ that represents the probabiity that they output the pair (a, b) given that they received input (x, y).

A random strategy is called *perfect* if

$$\lambda(x, y, a, b) = 0 \implies p(a, b|x, y) = 0,$$

so there is 0 probability of them producing an incorrect output.
It turns out that there are many games with no perfect deterministic strategy and no perfect classical random strategy, but they do have perfect *quantum strategies*.
The goal of this talk is to show that for a certain family of such games, called *synchronous games*, we can construct a *-algebra whose representation theory completely characterizes these behaviours.

Synchronous games

A game is called synchronous if $I_A = I_B$, $O_A = O_B$ and whenever the players receive the same input(question) they must produce the same output(answer). If $p(a, b|x, y)$ represents the probability that when receiving inputs x, y the players produce outputs a, b, respectively, then to always win a synchronous game this must satisfy,

$$\forall x, p(a, b|x, x) = 0 \text{ whenever } a \neq b$$

such densities are called *synchronous*.

Graph Homomorphisms

Given two graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ a **homomorphism** from G to H is a function $f : V(G) \to V(H)$ such that

$$(v, w) \in E(G) \implies (f(v), f(w)) \in E(H),$$

we write $G \to H$ to indicate that there is a graph homomorphism from G to H.
Many graph parameters are defined using graph homomorphisms and the complete graph, K_c, on c vertices.

- $\chi(G) = min\{c : G \to K_c\}$(chromatic number)
- $\omega(G) = max\{c : K_c \to G\}$(clique number)
- $\alpha(G) = max\{c : K_c \to \overline{G}\}$(independence number), where $\overline{G}$ is the graph with the same vertex set but the opposite edges.

The Graph Homomorphism Game

Given graphs G and H on n and m vertices, the game goes as follows: The Referee gives Alice and Bob a pair of vertices x, y from $V(G)$ and they reply with a pair of vertices a, b from $V(H)$ .They win provided:

- $x = y \implies a = b$,
- $(x, y) \in E(G) \implies (a, b) \in E(H)$

This game has a perfect deterministic strategy iff there is a graph homomorphism from G to H. In a similar fashion, the graph homomorphism game from G to K_c has a perfect deterministic strategy iff $\chi(G) \leq c$. In this manner the above graph parameters all have game theoretic interpretations.
Also given a pair of graphs, there is a *graph isomorphism game*, which has a perfect deterministic strategy iff the two graphs are isomorphic.

The syncBCS game

Suppose $Ax = b$ is an $m \times n$ linear system over $\mathbb{Z}/2$; that is, $A = (a_{i,j}) \in \mathbb{M}_{m,n}(\mathbb{Z}/2)$ and $b \in (\mathbb{Z}/2)^n$.
Idea of the game is that Alice and Bob want to convince the Referee that they have a solution x to $Ax = b$.
Let R_i denote the i-th row of A so that a solution to $Ax = b$ would need to satisfy $R_i \cdot x = b_i$.
In the syncBCS game a Referee chooses rows i, j, Alice receives i, Bob receives j and they each produce an output vector, $v, w \in (\mathbb{Z}/2)^n$.
They win if:

- when $i = j \implies x = y$,
- $R_i \cdot v = b_i$ and $R_j \cdot w = b_j$,
- $a_{i,k} = 0 \implies v_k = 0$ and $a_{j,k} = 0 \implies w_k = 0$,
- whenever $a_{i,k} = a_{j,k} = 1$, then $v_k = w_k$.

Conditional Quantum Probabilities: Tsirelson and Connes

Suppose that Alice and Bob each have n quantum experiments and each experiment has m outcomes. We let $p(a,b|x,y)$ denote the conditional probability that Alice gets outcome a and Bob gets outcome b given that they perform experiments x and y respectively. There are several possible models for describing the set of all such tuples.

One model is that Alice and Bob have finite dimensional state spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. For each experiment x, Alice has projections $\{E_{x,a}, 1 \le a \le m\}$ such that $\sum_a E_{x,a} = I_A$. Similarly, for each y, Bob has projections $\{F_{y,b} : 1 \le b \le m\}$ such that $\sum_b F_{y,b} = I_B$. They share an entangled state $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$ and

$$p(a,b|x,y) = \langle \psi | E_{x,a} \otimes F_{y,b} | \psi \rangle.$$

We let $C_q(n,m) = \{p(x,y|a,b) : \text{ obtained as above } \} \subseteq \mathbb{R}^{n^2m^2}$.
We let $C_{qs}(n,m)$ denote the possibly larger set that we could obtain if we allowed the spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ to also be infinite dimensional.
We let $C_{qc}(n,m)$ denote the possibly larger set that we could obtain if instead of requiring the common state space to be a tensor product, we just required one common state space, and demanded that $E_{a,x}F_{y,b} = F_{y,b}E_{x,a}$ for all a, b, x, y, this is called the *commuting model.*
Tsirelson was the first to examine these sets and study the relations between them. In fact, he wondered if they could all be equal.
Here are some of the things that we know/don't know about these sets.

- $C_q(n,m) \subseteq C_{qs}(n,m) \subseteq C_{qc}(n,m)$.
- $C_q(n,m)^- = C_{qs}(n,m)^- := C_{qa}(n,m) \subseteq C_{qc}(n,m)$ and this can be identified with the states on a minimal tensor product.
- $C_{qc}(n,m)$ is closed and can be identified with the states on a maximal tensor product.
- (JNPPSW + Ozawa)$C_q(n,m)^- = C_{qc}(n,m), \forall n,m$ iff Connes' Embedding conjecture has an affirmative answer.
- (Slofstra, March 2017) there exists a $n \sim 100$, such that $C_q(n,8)$ is not closed.
- (Dykema, P, Prakash) $C_q(n,m)$ and $C_{qs}(n,m)$ are not closed $\forall n \geq 5, m \geq 2$.
- (Coladangelo, Stark) $C_q(4,3) \neq C_{qs}(4,3)$.

For $t = q, qs, qa, qc$ we will say that a game with n inputs and m outputs has a *perfect t-strategy* if there exists $p(a, b|x, y) \in C_t(n, m)$ such that

$$\lambda(x, y, a, b) = 0 \implies p(a, b|x, y) = 0.$$

Slofstra's result came from his construction of a binary constraint system(BCS) game with a perfect qa-strategy but no perfect q-strategy.
Our work characterizes the existence of perfect t-strategies in terms of a *-algebra constructed from the game.

The *-algebra of a synchronous game

Let $\mathcal{G} = (I, O, \lambda)$ be a synchronous game. By the **-algebra of the game*, $\mathcal{A}(\mathcal{G})$, we mean the "universal" *-algebra generated by projections $\{e_{x,a} : x \in I, a \in O\}$ satisfying:

- $\forall x \in I, \sum_{a \in O} e_{x,a} = I,$
- $\lambda(a, b, x, y) = 0 \implies e_{x,a} e_{y,b} = 0$

Theorem (HMPS, KPS)

Let $\mathcal{G}$ be a synchronous game then:

- *$\mathcal{G}$ has a perfect deterministic strategy iff $\mathcal{A}(\mathcal{G})$ has a non-trivial one-dimensional representation,*
- *$\mathcal{G}$ has a perfect q-strategy iff $\mathcal{G}$ has a perfect qs-strategy iff $\mathcal{A}(\mathcal{G})$ has a non-trivial finite dimensional representation.*
- *$\mathcal{G}$ has a perfect qc-strategy iff $\mathcal{A}(\mathcal{G})$ has a trace.*
- *$\mathcal{G}$ has a perfect qa-strategy iff $\mathcal{A}(\mathcal{G})$ has a hyperlinear trace.*

The graph isomorphism algebra

If G, H are two graphs with $|V(G)| = |V(H)| = N$, then $\mathcal{A}(Iso(G,H))$ is generated by N^2 projections $e_{g,h}$, $g \in V(G)$, $h \in V(H)$ and the relations induced by the rule function are equivalent to:

- $\sum_h e_{g,h} = 1, \ \sum_g e_{g,h} = 1$, so that we have the relations for the generators of the *quantum symmetric group* S_N^+,
- $\forall g \in V(G), h \in V(H), \ \sum_{g' \sim g} e_{g',h} = \sum_{h' \sim h} e_{g,h'}$, where we write $x \sim y$ to mean that (x,y) is an edge.

The second relation can also be written as $A_G \circ (e_{g,h}) = (e_{g,h}) \circ A_H$, where A_X is the adjacency matrix of the graph X.

Equivalences among games

Given $A \in M_{m,n}(\mathbb{Z}/2)$ and $b \in (\mathbb{Z}/2)^n$ there is a graph on $2^k m$ vertices, $G_{A,b}$ with the following properties:

Theorem (KPS)

Let A and b be as above and let $t \in \{deterministic, q, qa, qc\}$ then the following are equivalent:

- *$synBCS(A, b)$ has a perfect t-strategy,*
- *$Iso(G_{A,0}, G_{A,b})$ has a perfect t-strategy,*
- *$\omega_t(G_{A,b}) = m$.*

Finally, using Slofstra's construction of a BCS game with a perfect qa-strategy but no perfect q-strategy, we are able to prove that there exists a $m \times n$ matrix with $m \sim 100$ and a vector b such that $syncBCS(A, b)$ has a perfect qa-strategy but no perfect q-strategy.

This implies that there exist a sequence of integers n_k, and for each g, h, a sequence of projections $E_{g,h,k} \in M_{n_k}$ such that:

- $\| \sum_h E_{g,h,k} - I_{n_k} \|_2 \to 0$, $\| \sum_g E_{g,h,k} - I_{n_k} \|_2 \to 0$ as $k \to \infty$,
- $\forall g, h$, $\| \sum_{g' \sim g} E_{g',h,k} - \sum_{h' \sim h} E_{g,h',k} \|_2 \to 0$,

but one can not have exact solutions to both these equations in M_n for any n.

This suggests the following problem: *Are nearly magic permutations, near to magic permutations?*

Explicitly: for a fixed N, does there exist $\delta = \delta(\epsilon) \to 0$ as $\epsilon \to 0$, independent of n, such that if $E_{g,h} \in M_n, 1 \le g, h \le N$ are projections with $\| \sum_h E_{g,h} - I_n \|_2 < \epsilon$ and $\| \sum_g E_{g,h} - I_n \|_2 < \epsilon$ then there exist projections $F_{g,h}$ with $\sum_h F_{g,h} = \sum_g F_{g,h} = I_n$ and $\| E_{g,h} - F_{g,h} \|_2 < \delta$?

Further results and problems

- HMPS prove that $\mathcal{A}(Hom(K_4, K_3)) = (0)$. Hence algebras of games can be 0.
- HMPS also show that $\mathcal{A}(Hom(K_5, K_4)) \neq (0)$,but this *-algebra can have no representation on a Hilbert space because it contains 4 projections whose sum is $-I$.
- Does there exist a synchronous game such that its C*-algebra is non-zero but has no traces?
- Tobias Fritz proves that every *-algebra of a synchronous game is a hypergraph *-algebra and conversely.
- Fritz proves that if ZFC is consistent, then there is a synchronous game $\mathcal{G}$ such that whether or not $\mathcal{A}(\mathcal{G})$ has a representation on a Hilbert space is undecidable.

Thanks!

References

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