QIC890/PMATH950
HOMEWORK SET 3
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1. Problem

Let $\mathcal{A}$ and $\mathcal{B}$ be unital C*-algebras and let $\Phi : \mathcal{A} \to \mathcal{B}$ be a positive map(not necessarily CP). Prove that

$$\Phi(X^*)^* = \Phi(X).$$

Show that this equation also holds for differences of positive maps.

2. Problem

Let $L : M_2 \to M_r$ be a linear map, and let $L(E_{1,1}) = A = (a_{i,j}),\ L(E_{1,2}) = B = (b_{i,j}),\ L(E_{2,1}) = C = (c_{i,j}),\ L(E_{2,2}) = D = (d_{i,j})$ so that the Choi matrix of this map is

$$C_L = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_2(M_n).$$

Write out $C_{L^d}, C_{L^\dagger} \in M_n(M_2)$ and note that they are not neccessariiy equal. Use Choi's theorem(not Stinespring as in class) to show that $C_{L^d} = C_{L^\dagger}$ when L is CP.

3. Problem

Let $S : \ell^2_{\mathbb{N}} \to \ell^2_{\mathbb{N}}$ denote the unilateral shift, $Se_n = e_{n+1}$. Set $V_k = \frac{1}{2^{k/2}} S^k$. Use our theorems to conclude that there is a CPTP map $\Phi : \mathcal{C}_1(\ell^2_{\mathbb{N}}) \to \mathcal{C}_1(\ell^2_{\mathbb{N}})$ given by $\Phi(X) = \sum_k V_k X V_k^*$. For all i, j describe the matrix $\Phi(E_{i,j})$.

4. Problem

Let V, W be vector spaces, let $\{v_1, ..., v_n\} \subseteq V, \{w_1, ..., w_n\} \subseteq W$ and let $V_1 = span\{v_1, ..., v_n\}$.

- Prove that if $\sum_{i=1}^n \alpha_i v_i = 0 \implies \sum_{i=1}^n \alpha_i w_i = 0$, then there exists a linear map $L : V_1 \to W$ with $L(v_i) = w_i$.
- Let $\mathcal{S}$ be the subspace from the quantum marginals problem, and let $\{z_1, ..., z_K, w_1, ..., w_J\} \subseteq W$. Prove that if $\sum_{k=1}^K z_k = \sum_{j=1}^J w_j$, then there exists a linear map $L : \mathcal{S} \to W$ with $L(f_k) = z_k, L(g_j) = w_j$.

5. Problem

Let $P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, Q_1 = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}, Q_2 = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$.
Prove that these cannot be the quantum marginals of a joint distribution, directly and by applying the theorem.

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