QIC890/PMATH950
HOMEWORK SET 2
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1. Problem

Let $M_k \in B(\mathcal{H}, \mathcal{K})), 1 \leq k \leq K$. Show that the map $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$ given by $\Phi(T) = \sum_{k=1}^{K} M_k T M_k^*$ is CP and that $\Phi(\mathcal{C}_1(\mathcal{H})) \subseteq \mathcal{C}_1(\mathcal{K})$. Show that Φ is trace preserving(TP) iff $\sum_{k=1}^{K} M_k^* M_k = I_{\mathcal{H}}$.

2. Problem

Given two matrices $A = (a_{i,j}), B = (b_{i,j})$ their **Schur product** is

$$A \circ B = (a_{i,j} b_{i,j}).$$

Fix $A \in M_n$ and define a linear map $S_A : M_n \to M_n$ by $S_A(B) = A \circ B$. Prove that:

(1) $A \geq 0, B \geq 0 \implies A \circ B \geq 0$.
(2) S_A is CP iff $A \geq 0$.
(3) If $A \geq 0$ then we can always write

$$S_A(X) = \sum_{i=1}^{r} D_i X D_i^*,$$

where the D_i's are diagonal matrices and $r = rank(A)$.

3. Problem

Let $\mathcal{D}_n \subseteq M_n$ denote the set of diagonal matrices, which is a C*-subalgebra. A linear map $\Phi : M_n \to M_n$ is called a $\mathcal{D}_n$**-bimodule map** provided that $D_1, D_2 \in \mathcal{D}_n \implies \Phi(D_1 X D_2) = D_1 \Phi(X) D_2$. Prove that Φ is a $\mathcal{D}_n$-bimodule map iff $\Phi = S_A$ for some $A \in M_n$.

4. Problem

Given two self-adjoint operators, H, K we write $H \leq K$ or $K \geq H$ provided that $K - H$ is a positive operator. Let $X, P \in B(\mathcal{H})$.

(1) Prove that $\begin{pmatrix} I_{\mathcal{H}} & X \\ X^* & P \end{pmatrix} \in B(\mathcal{H} \oplus \mathcal{H})$ is positive if and only if $X^* X \leq P$ in $B(\mathcal{H})$.
(2) Deduce that $\begin{pmatrix} I_{\mathcal{H}} & X \\ X^* & X^* X \end{pmatrix}$ is positive.

(3) Let $\Phi : B(\mathcal{H}) \to B(\mathcal{H})$ be unital and CP. Prove that for any $X \in B(\mathcal{H})$ we have that $\Phi(X)^*\Phi(X) \leq \Phi(X^*X)$.

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