1. Problem

Let $M_k \in B(\mathcal{H}, \mathcal{K})$, $1 \leq k \leq K$. Show that the map $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$ given by $\Phi(T) = \sum_{k=1}^{K} M_k T M_k^*$ is CP and that $\Phi(\mathcal{C}_1(\mathcal{H})) \subseteq \mathcal{C}_1(\mathcal{K})$. Show that $\Phi$ is trace preserving (TP) iff $\sum_{k=1}^{K} M_k^* M_k = I_\mathcal{H}$.

2. Problem

Given two matrices $A = (a_{i,j}), B = (b_{i,j})$ their Schur product is

$$A \circ B = (a_{i,j}b_{i,j}).$$

Fix $A \in M_n$ and define a linear map $S_A : M_n \to M_n$ by $S_A(B) = A \circ B$. Prove that:

1. $A \geq 0, B \geq 0 \implies A \circ B \geq 0$.
2. $S_A$ is CP iff $A \geq 0$.
3. If $A \geq 0$ then we can always write

$$S_A(X) = \sum_{i=1}^{r} D_i X D_i^*,$$

where the $D_i$'s are diagonal matrices and $r = \text{rank}(A)$.

3. Problem

Let $\mathcal{D}_n \subseteq M_n$ denote the set of diagonal matrices, which is a $C^*$-subalgebra. A linear map $\Phi : M_n \to M_n$ is called a $\mathcal{D}_n$-bimodule map provided that $D_1, D_2 \in \mathcal{D}_n \implies \Phi(D_1 X D_2) = D_1 \Phi(X) D_2$. Prove that $\Phi$ is a $\mathcal{D}_n$-bimodule map iff $\Phi = S_A$ for some $A \in M_n$.

4. Problem

Given two self-adjoint operators, $H, K$ we write $H \leq K$ or $K \geq H$ provided that $K - H$ is a positive operator. Let $X, P \in B(\mathcal{H})$.

1. Prove that $\begin{pmatrix} I_\mathcal{H} & X \\ X^* & P \end{pmatrix} \in B(\mathcal{H} \oplus \mathcal{H})$ is positive if and only if $X^* X \leq P$ in $B(\mathcal{H})$.
2. Deduce that $\begin{pmatrix} I_\mathcal{H} & X \\ X^* & X^* X \end{pmatrix}$ is positive.
(3) Let $\Phi : B(\mathcal{H}) \to B(\mathcal{H})$ be unital and CP. Prove that for any $X \in B(\mathcal{H})$ we have that $\Phi(X^*X) \leq \Phi(XX^*)$.