Constant Gap for Self-embezzlement

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Catalytic Production of Entanglement

Suppose that Alice and Bob have their own finite dimensional state spaces, $\mathcal{H}_A$ and $\mathcal{H}_B$ and a shared finite dimensional bipartite resource space $\mathcal{R}_A \otimes \mathcal{R}_B$.
Can we "catalytically produce entanglement"? Say given the Bell state, $b = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$ can we find unitaries

$$U_A : \mathcal{H}_A \otimes \mathcal{R}_A \to \mathcal{H}_A \otimes \mathcal{R}_A \text{ and } U_B : \mathcal{R}_B \otimes \mathcal{H}_B \to \mathcal{R}_B \otimes \mathcal{H}_B$$

and a unit vector $\psi \in \mathcal{R}_A \otimes \mathcal{R}_B$ such that

$$U_A \otimes U_B : (\mathcal{H}_A \otimes \mathcal{R}_A) \otimes (\mathcal{R}_B \otimes \mathcal{H}_B) \to (\mathcal{H}_A \otimes \mathcal{R}_A) \otimes (\mathcal{R}_B \otimes \mathcal{H}_B)$$

satisfies

$$U_A \otimes U_B(|0\rangle \otimes \psi \otimes |0\rangle) = \frac{1}{\sqrt{2}}\big(|0\rangle \otimes \psi \otimes |0\rangle + |1\rangle \otimes \psi \otimes |1\rangle\big) \simeq b \otimes \psi?$$

Hayden and van Dam introduced this question and showed that the answer is no! The proof of this "no-go" fact is a simple argument using Schmidt coefficients.

However, they(together with some later improvements) also showed that given ANY vector

$$\phi = \sum_{i,j} \alpha_{i,j} |i\rangle \otimes |j\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$$

and any $\epsilon > 0$ there exist finite dimensional resource spaces $\mathcal{R}_A, \mathcal{R}_B$(depending on ϵ) and unit vectors $\psi, \psi_\epsilon \in \mathcal{R}_A \otimes \mathcal{R}_B$ with $\|\psi - \psi_\epsilon\| < \epsilon$ and unitaries U_A, U_B such that

$$U_A \otimes U_B(|0\rangle \otimes \psi \otimes |0\rangle) = \sum_{i,j} \alpha_{i,j} \, |i\rangle \otimes \psi_\epsilon \otimes |j\rangle \simeq \phi \otimes \psi_\epsilon.$$

They referred to this as *embezzlement of entanglement.*

They also gave some estimates on the dimensions of $\mathcal{R}_A$ and $\mathcal{R}_B$ needed to carry out this process as a function of ϵ.

However, in Cleve-Liu-P, we showed that one still cannot do this for $\epsilon = 0$ even if one allows $\mathcal{R}_A$ and $\mathcal{R}_B$ to be infinite dimensional.

Thus we have a "task" that can be carried out to an arbitrary degree of accuracy in finite dimensions, but even as we let the dimensions become infinite, we still cannot carry it out exactly. In a sense what happens is that in the limit we must change from the bipartite tensor model to the commuting operator framework. Note that

$$(U_A \otimes I_{\mathcal{R}_B} \otimes I_{\mathcal{H}_B})(I_{\mathcal{H}_A} \otimes I_{\mathcal{R}_A} \otimes U_B) = U_A \otimes U_B = (I_{\mathcal{H}_A} \otimes I_{\mathcal{R}_A} \otimes U_B)(U_A \otimes I_{\mathcal{R}_B} \otimes I_{\mathcal{H}_B}).$$

Theorem (CLP, Harris-P)

Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be finite dimensional. Given any unit vector $\phi = \sum_{i,j} \alpha_{i,j} |i\rangle \otimes |j\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ there exists a Hilbert space $\mathcal{R}$, a unit vector $\psi \in \mathcal{R}$, unitaries

$$U_A : \mathcal{H}_A \otimes \mathcal{R} \to \mathcal{H}_A \otimes \mathcal{R} \text{ and } U_B : \mathcal{R} \otimes \mathcal{H}_B \to \mathcal{R} \otimes \mathcal{H}_B,$$

such that

$$(U_A \otimes I_{\mathcal{H}_B})(I_{\mathcal{H}_A} \otimes U_B) = (I_{\mathcal{H}_A} \otimes U_B)(U_A \otimes I_{\mathcal{H}_B}),$$

with

$$(U_A \otimes I_{\mathcal{H}_B})(I_{\mathcal{H}_A} \otimes U_B)(|0\rangle \otimes \psi \otimes |0\rangle) = \sum_{i,j} \alpha_{i,j} |i\rangle \otimes \psi \otimes |j\rangle \simeq \phi \otimes \psi.$$

Briefly, catalytic production of entanglement is possible in the commuting operator model.

Self-embezzlement

Suppose that $\mathcal{R} = \mathcal{H}_A \otimes \mathcal{H}_B$. How "nearly" can we catalytically produce the catalytic state?
We have the following "constant gap" theorem.

Theorem (Cleve-Collins-Liu-P)

Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be finite dimensional. If $\psi = \sum_{i,j} \beta_{i,j} |i\rangle \otimes |j\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ and its highest Schmidt coefficient satisfies $\lambda_1 \leq \sqrt{\frac{2}{3}}$, and $U_A \in B(\mathcal{H}_A \otimes \mathcal{H}_A)$, $U_B \in B(\mathcal{H}_B \otimes \mathcal{H}_B)$ are unitaries then

$$\|U_A \otimes U_B(|0\rangle \otimes \psi \otimes |0\rangle) - \sum_{i,j} \beta_{i,j} |i\rangle \otimes \psi \otimes |j\rangle\| \geq \frac{2}{3}(3 - 2\sqrt{2})$$

and this bound is independent of the dimension of $\mathcal{H}_A$ and $\mathcal{H}_B$.

This gap vanishes in the commuting operator model.

Theorem (CCLP)

Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be infinite dimensional, set $\mathcal{R} = \mathcal{H}_A \otimes \mathcal{H}_B$, and let $\psi \in \mathcal{R}$ be a unit vector as before. Then there exist unitaries $U_A \in B(\mathcal{H}_A \otimes \mathcal{R})$ and $U_B \in B(\mathcal{R} \otimes \mathcal{H}_B)$ such that $(U_A \otimes I_{\mathcal{H}_B})$ and $(I_{\mathcal{H}_A} \otimes U_B)$ commute and

$$(U_A \otimes I_{\mathcal{H}_B})(I_{\mathcal{H}_A} \otimes U_B)(|0\rangle \otimes \psi \otimes |0\rangle) = \sum_{i,j} \beta_{i,j} |i\rangle \otimes \psi \otimes |j\rangle \simeq \psi \otimes \psi.$$

Sketch of the proof. Different from the one found in CCLP. Suppose that we have $\gamma \in \mathcal{R}$, $U_A \in B(\mathcal{H}_A \otimes \mathcal{R})$ and $U_B \in B(\mathcal{R} \otimes \mathcal{H}_B)$, with $\mathcal{R} = \mathcal{H}_A \otimes \mathcal{H}_B$. Such that $U_A \otimes I_{\mathcal{H}_B}$ commutes with $I_{\mathcal{H}_A} \otimes U_B$
and

$$(U_A \otimes I_{\mathcal{H}_B})(I_{\mathcal{H}_A} \otimes U_B)(|0\rangle \otimes \gamma \otimes |0\rangle) \simeq \psi \otimes \gamma.$$

Choose a unitary $W \in B(\mathcal{R})$ with $W\psi = \gamma$ set
$\widetilde{U_A} = (I_{\mathcal{H}_A} \otimes W)^* U_{\mathcal{H}_A}(I_{\mathcal{H}_A} \otimes W)$ and
$\widetilde{U_B} = (W \otimes I_{\mathcal{H}_B})^* U_B(W \otimes I_{\mathcal{H}_B})$ then $\widetilde{U_A} \otimes I_{\mathcal{H}_B}$ commutes with $I_{\mathcal{H}_A} \otimes \widetilde{U_B}$ and
$(\widetilde{U_A} \otimes I_{\mathcal{H}_B})(I_{\mathcal{H}_A} \otimes \widetilde{U_B})(|0\rangle \otimes \psi \otimes |0\rangle) \simeq \psi \otimes \psi.$

Thanks!