Computation and Analysis for a Constrained Entropy Optimization Problem in Finance *

Changhong He † Thomas F. Coleman ‡ Yuying Li §

Abstract

In [6], an entropy minimization formulation has been proposed to calibrate an uncertain volatility option pricing model (UVM) from market bid and ask prices. To avoid potential infeasibility due to numerical error, a quadratic penalty function approach is applied. In this paper, we show that the solution to the quadratic penalty problem can be obtained by minimizing an objective function which can be evaluated via solving a Hamilton-Jacobian-Bellman (HJB) equation. We prove that the implicit finite difference solution of this HJB equation converges to its viscosity solution. In addition, we provide computational examples illustrating accuracy of calibration.

MSC: 35K55, 54C70, 65M06, 65M12

Keywords: nonlinear PDE, entropy minimization, viscosity solution, uncertain volatility model, option pricing, model calibration.

1 Introduction

In the classical Black-Scholes model for option pricing, the underlying price is modeled as a geometric Brownian motion,

$$\frac{dS_t}{S_t} = (r - q)dt + \sigma dZ_t$$

where the interest rate $r > 0$, the dividend rate $q > 0$, and the volatility $\sigma > 0$ are constants, and Z_t is a standard Brownian motion. Under this assumption, a European option price can be computed easily using an analytic formula. Unfortunately, the market option prices suggest that the Black-Scholes model is often inadequate. Volatility imputed from the market price

*This research was supported by startup grant from University of Waterloo.

†J.P. Morgan Securities Inc., 270 Park Ave, Floor 6, New York, NY, 10017-2070, email: changhong.he@jpmchase.com. Please send correspondence to this author.

‡Faculty of Mathematics, University of Waterloo, 200 University Avenue West, Waterloo, Ontario, Canada N2L 3G1, email: tfcoleman@uwaterloo.ca

§School of Computer Science, University of Waterloo, 200 University Avenue West, Waterloo, Ontario, Canada N2L 3G1, email: yuying@uwaterloo.ca

and Black-Scholes formula varies with different options on the same underlying asset. In fact, volatility is uncertain in practice.

An uncertain volatility model (UVM) is introduced in [3, 4]. In this model, the volatility itself is uncertain and lies within an interval. Consider a set of stochastic processes,

$$\begin{aligned} \frac{dS_t}{S_t} &= (r-q)dt + \sigma_t dZ_t^{\mathbb{Q}}, \\ \text{where} \quad & \underline{\sigma} \le \sigma_t \le \overline{\sigma}. \end{aligned} \tag{1}$$

where $\mathbb{Q}$ is a risk neutral probability measure, $Z_t^{\mathbb{Q}}$ is a standard Brownian motion under $\mathbb{Q}$, r and q are interest rate and dividend yield respectively. Two volatility bounds, $\underline{\sigma}$ and $\overline{\sigma}$, can be pre-determined functions of the underlying price and time; a special case is when both bounds are constants.

Under a UVM model, an option has a lower price and an upper price which can naturally be linked to market bid and ask prices. These two value bounds of an option are described by the nonlinear *Black-Scholes-Barenblatt* (BSB) partial differential equation.

In finance, an option pricing model is typically calibrated to liquid option prices and the resulting model is then used to price more illiquid options and manage option risks. The calibration is accomplished by solving an inverse problem to determine model parameters so that model prices match market prices.

In [6], a constrained entropy minimization formulation is proposed to calibrate an uncertain volatility model (1) from market bid and ask prices. It is illustrated that the calibrated uncertain volatility model yields more realistic price spreads than spreads resulting from an uncertain volatility model from typical constant volatility bounds. To overcome potential infeasibility due to numerical error arising from solving partial differential equations, a quadratic penalty formulation is applied. We refer an interested reader to [6] for more detailed discussion of financial meaning and examples of this calibration method for an uncertain volatility model.

The main objective of this paper is to analyze theoretical properties of the formulations and numerical schemes proposed in [6]. We show in §2 that quadratic penalty minimization can be solved by minimizing a convex function which satisfies a Hamilton-Jacobian-Bellman (HJB) partial differential equation. An implicit finite difference method is applied to the HJB equation to approximate the objective function value while the gradient of the objective function is determined by solving the corresponding Black-Scholes partial differential equation. We prove in §3 that the implicit finite difference scheme is unconditionally stable, consistent and monotone, which are important properties for convergence to a viscosity solution of the HJB equation. Finally we provide in §4 some computational results illustrating the accuracy of the calibration.

2 Mathematical Formulation

Consider a set of probability spaces $\{(\Omega, \mathcal{F}, \mathbb{Q}),\ \mathbb{Q} \in \Theta(\underline{\sigma}, \overline{\sigma})\}$, where $\Theta(\underline{\sigma}, \overline{\sigma})$ denotes the set of all probability measures $\mathbb{Q}$ corresponding to the processes below

$$\frac{dS_t}{S_t} = (r-q)dt + \sigma_t dZ^{\mathbb{Q}}, \quad \text{where } \sigma_{\text{lb}} \le \sigma_t \le \sigma_{\text{ub}} \tag{2}$$

Consider a European option with a payoff function $G(S)$ at the expiry T. Under the assumed uncertain volatility model (1), there are a pair of option values V^- and V^+ , $V^- \leq V^+$, associated with this option. Specifically, the pair of option values satisfies the following equations:

$$V^-(S_t, t) = \inf_{\mathbb{Q}\in\Theta(\underline{\sigma},\overline{\sigma})} \mathbf{E}_t^{\mathbb{Q}}[e^{-r(T-t)}G(S_T)] \tag{3}$$

and

$$V^+(S_t, t) = \sup_{\mathbb{Q}\in\Theta(\underline{\sigma},\overline{\sigma})} \mathbf{E}_t^{\mathbb{Q}}[e^{-r(T-t)}G(S_T)] \tag{4}$$

Based on stochastic control theory, e.g., see [9], these extreme values can be computed by solving a Hamilton-Jacobi-Bellman (HJB) equation. For example V^- satisfies the HJB equation:

$$-\frac{\partial V}{\partial t} + rV + \sup_{\underline{\sigma}\leq\sigma\leq\overline{\sigma}} \left(-(r-q)S\frac{\partial V}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) = 0$$

The HJB equations lead to the *Black-Scholes-Barenblatt* equation (5),

$$\frac{\partial V}{\partial t} + (r-q)S\frac{\partial V}{\partial S} + \frac{1}{2}\left(\sigma\left[\frac{\partial^2 V}{\partial S^2}\right]\right)^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0 \tag{5}$$

See e.g., [3].
The final condition is given by

$$V(S,T) = G(S)$$

where V^- is obtained with $\sigma\Big[\,\cdot\,\Big] = \sigma^-\Big[\,\cdot\,\Big]$,

$$\sigma^-\left[\frac{\partial^2 V}{\partial S^2}\right] \stackrel{\text{def}}{=} \begin{cases} \bar{\sigma} & \text{if } \frac{\partial^2 V}{\partial S^2} \leq 0, \\ \underline{\sigma} & \text{if } \frac{\partial^2 V}{\partial S^2} > 0, \end{cases} \tag{6}$$

and V^+ is obtained with $\sigma\Big[\,\cdot\,\Big] = \sigma^+\Big[\,\cdot\,\Big]$,

$$\sigma^+\left[\frac{\partial^2 V}{\partial S^2}\right] \stackrel{\text{def}}{=} \begin{cases} \bar{\sigma} & \text{if } \frac{\partial^2 V}{\partial S^2} \geq 0, \\ \underline{\sigma} & \text{if } \frac{\partial^2 V}{\partial S^2} < 0. \end{cases} \tag{7}$$

In [6], volatility bounds in a UVM are determined via solving two constrained entropy minimization problems respectively.

Let $\varepsilon(\mathbb{Q}_1, \mathbb{Q}_0)$ denote the relative entropy between $\mathbb{Q}_1$ and $\mathbb{Q}_0$, i.e.,

$$\varepsilon(\mathbb{Q}_1, \mathbb{Q}_0) = \int ln\Big(\frac{d\mathbb{Q}_1}{d\mathbb{Q}_0}\Big) d\mathbb{Q}_1,$$

where $d\mathbb{Q}_1/d\mathbb{Q}_0$ is the Radon-Nikondym derivative.

To calibrate a UVM which is least biased towards missing information, an entropy minimization formulation is proposed in [6]. Specifically, to determine the lower volatility $\underline{\sigma}$,

$$\begin{aligned} &\inf_{\mathbb{Q}\in\Theta(\sigma_{\rm lb},\sigma_{\rm ub})} \varepsilon(\mathbb{Q},\mathbb{Q}_0) \\ \text{subject to} \quad &\mathbf{E}^{\mathbb{Q}}\left(e^{-rT_i}G_i(S_{T_i})\right) = V_i,\ i=1,2,\ldots,M. \end{aligned} \tag{8}$$

where $\{V_i\}_{i=1}^M$ are given bid prices of liquid call and put options, $\{G_i\}_{i=1}^M$ are associated piecewise linear payoff functions, $\mathbb{Q}_0$ corresponds to some constant minimum volatility. For calibration of the lower volatility $\underline{\sigma}$, the constant volatility prior is also used as $\sigma_{\rm lb}$, and $\sigma_{\rm ub}$ is determined based on the option mid-prices. The upper volatility $\overline{\sigma}$ can be similarly determined with the market prices $\{V_i\}$ corresponding to option ask prices. In addition, the constant volatility prior corresponds to some maximum volatility, the corresponding $\sigma_{\rm ub}$ is set to this constant volatility, and $\sigma_{\rm lb}$ is determined based on the mid-prices.

The constrained entropy problem can be solved, see e.g., [2], based on a Lagrangian approach, i.e.,

$$\inf_{\lambda}\ \sup_{\mathbb{Q}\in\Theta(\sigma_{\rm lb},\sigma_{\rm ub})} \left(-\varepsilon(\mathbb{Q}_1,\mathbb{Q}_0) + \sum_{i=1}^M \lambda_i\left(\mathbf{E}^{\mathbb{Q}}\left(e^{-rT_i}G_i(S_{T_i})\right) - V_i\right)\right). \tag{9}$$

Unfortunately, in the context of calibrating a volatility bound, the numerical approximation of the objective function in the Lagrange formulation (9) is often unbounded from below. This is due to the fact the equality constraints in (8) may be numerically infeasible. For more detailed discussion and computational examples, we refer an interested reader to [6].

An alternative approach considered in [6] is the quadratic penalty function method. Given a weight vector, $\{w_i\}_{i=1}^M$, we consider the quadratic penalty formulation below

$$\inf_{\mathbb{Q}\in\Theta(\sigma_{\rm lb},\sigma_{\rm ub})} \left(\varepsilon(\mathbb{Q},\mathbb{Q}_0) + \frac{1}{2}\sum_{i=1}^M \frac{1}{w_i}\left(\mathbf{E}^{\mathbb{Q}}\left(e^{-rT_i}G_i(S_{T_i})\right) - V_i\right)^2\right) \tag{10}$$

where the dynamics of S_t is described by (2). In contrast to the Lagrangian formulation (9), the objective function of (10) is convex and bounded from below, thus avoiding the potential difficulty of unboundedness. Of course, at a solution to (10), the equality constraints in (8) are only approximately satisfied.

Problem (10) is a stochastic control problem. Unfortunately stochastic control theory cannot be directly applied to it since the second term in the objective function is not linear. Fortunately, an equivalent linear problem can be formulated. We now provide the derivation.

Denote the objective function of the quadratic penalty formulation (10) by $H_0^{\mathbb{Q}}$ and consider

$$\inf_{\mathbb{Q}\in\Theta(\sigma_{\rm lb},\sigma_{\rm ub})} \left\{H_0^{\mathbb{Q}} \stackrel{\rm def}{=} \left(\varepsilon(\mathbb{Q},\mathbb{Q}_0) + \frac{1}{2}\sum_{i=1}^M \frac{1}{w_i}\left(\mathbf{E}^{\mathbb{Q}}\left(e^{-rT_i}G_i(S_{T_i})\right) - V_i\right)^2\right)\right\} \tag{11}$$

Similar to [1], we can consider the min-max formulation to eliminate the nonlinear term involving $\mathbf{E}^{\mathbb{Q}}$,

$$\sup_{\lambda}\ \inf_{\mathbb{Q}\in\Theta(\sigma_{\rm lb},\sigma_{\rm ub})} \left\{H_1^{\mathbb{Q},\lambda} \stackrel{\rm def}{=} \left(\varepsilon(\mathbb{Q},\mathbb{Q}_0) - \sum_{i=1}^M \lambda_i(\mathbf{E}^{\mathbb{Q}}\left(e^{-rT_i}G_i\right) - V_i) - \frac{1}{2}\sum_{i=1}^M \lambda_i^2 w_i\right)\right\}. \tag{12}$$

We demonstrate next why (11) can be solved via the min-max formulation (12) based on the continuous problem formulations. We show that, if there exists $\mathbb{Q}^*$ such that $H_0^{\mathbb{Q}^*} = \inf_{\mathbb{Q}\in\Theta(\sigma_{\rm lb},\sigma_{\rm ub})} H_0^{\mathbb{Q}}$, then

$$\inf_{\mathbb{Q}\in\Theta(\sigma_{\rm lb},\sigma_{\rm ub})} H_0^{\mathbb{Q}} = \sup_{\lambda} \inf_{\mathbb{Q}\in\Theta(\sigma_{\rm lb},\sigma_{\rm ub})} H_1^{\mathbb{Q},\lambda}.$$

First, by the Cauchy inequality, for any $w = (w_1, w_2, \cdots, w_M) > 0$, we have

$$\frac{1}{2}\lambda_i^2 w_i + \frac{1}{2w_i}(\mathbf{E}^{\mathbb{Q}}(e^{-rT_i}G_i(S_{T_i})) - V_i)^2 \geq -\lambda_i(\mathbf{E}^{\mathbb{Q}}(e^{-rT_i}G_i(S_{T_i})\} - V_i), \quad \forall \lambda_i. \tag{13}$$

Thus, for any $\{\lambda_i\}_{i=1}^M$,

$$\frac{1}{2}\sum_{i=1}^{M}\frac{1}{w_i}\left(\mathbf{E}^{\mathbb{Q}}\left(e^{-rT_i}G_i(S_{T_i})\right) - V_i\right)^2 \geq -\sum_{i=1}^{M}\lambda_i\left(\mathbf{E}^{\mathbb{Q}}\left(e^{-rT_I}G_i - V_i\right)\right) - \frac{1}{2}\sum_{i=1}^{M}\lambda_i^2 w_i. \tag{14}$$

Hence, by the definition of $H_0^{\mathbb{Q}}$ and $H_1^{\mathbb{Q},\lambda}$, we have

$$H_0^{\mathbb{Q}} \geq H_1^{\mathbb{Q},\lambda} \quad \forall \{\lambda_i\}_{i=1}^M.$$

Thus

$$\inf_{\mathbb{Q}\in\Theta(\sigma_{\rm lb},\sigma_{\rm ub})} H_0^{\mathbb{Q}} \geq \sup_{\lambda} \inf_{\mathbb{Q}\in\Theta(\sigma_{\rm lb},\sigma_{\rm ub})} H_1^{\mathbb{Q},\lambda}.$$

Therefore we only need to show

$$\inf_{\mathbb{Q}\in\Theta(\sigma_{\rm lb},\sigma_{\rm ub})} H_0^{\mathbb{Q}} \leq \sup_{\lambda} \inf_{\mathbb{Q}\in\Theta(\sigma_{\rm lb},\sigma_{\rm ub})} H_1^{\mathbb{Q},\lambda}.$$

We consider the necessary condition for the minimiziers $\inf_{\mathbb{Q}} H_0^{\mathbb{Q}}$ and $\inf_{\mathbb{Q}} H_1^{\mathbb{Q},\lambda}$.

We start with the definition of the first variation, see e.g., [8]. Let us denote $\mathcal{G}$ as a set of probability density functions on Ω. Let $\mathcal{L}$ be a smooth function

$$\mathcal{L} : R \to R.$$

Given a density function $q(x) \in \mathcal{G}$, let us define functional $y(q)$,

$$y(q) \stackrel{\rm def}{=} \int_{\Omega} \mathcal{L}(q(x))dx.$$

Suppose $y(q)$ achieves minimum in $\mathcal{G}$ at q^*, i.e.,

$$y(q^*) = \inf_{q\in\mathcal{G}} y(q).$$

Then, for a real number ϵ, and an admissible function $h(x)$ such that $q^* + \epsilon h(x) \in \mathcal{G}$ when $|\epsilon|$ is small enough, we consider a real valued function

$$I(\epsilon) = y(q^* + \epsilon h).$$

Since $I(\cdot)$ has a minimum at $\epsilon = 0$, we have

$$I'(0) = 0,$$

i.e.,

$$\int_\Omega \mathcal{L}'(q^*(x))h(x)dx = 0. \tag{15}$$

Equation (15) is the necessary condition of the optimal solution of y. The left hand side of (15) represents the first variation of y. For notational simplicity, the first variation is represented as $\delta y^{q^*,h}$.

For simplicity, assume that the densities for the joint distribution of the underlying prices at relevant and distinct maturity dates of $\{T_i\}$ under measures $\mathbb{Q}$ and $\mathbb{Q}_0$ exist. Let $q(x)$ and $q_0(x)$ denote these density functions respectively. Then, the first variations of $H_0^{\mathbb{Q}}$ and $H_1^{\mathbb{Q}}$ are

$$\begin{aligned} \delta H_0^{\mathbb{Q},\delta q} &= \int_\Omega (1 + ln(q(x)) - ln(q_0(x)))\delta q(x)dx + \\ &\quad \sum_i \Big[\frac{1}{w_i}(\mathbf{E}^{\mathbb{Q}}(e^{-rT_i}G_i) - V_i)\int_\Omega e^{-rT_i}G_i\delta q(x)dx\Big], \end{aligned} \tag{16}$$

$$\begin{aligned} \delta H_1^{\mathbb{Q},\lambda,\delta q} &= \int_\Omega (1 + ln(q(x)) - ln(q_0(x)))\delta q(x)dx - \\ &\quad \sum_i \lambda_i \int_\Omega e^{-rT_i}G_i\delta q(x)dx. \end{aligned} \tag{17}$$

Assume that $H_0^{\mathbb{Q}}$ obtains an infimium at $\mathbb{Q}^*$ with q^* the density of the joint distribution of the underlying prices at relevant and distinct maturity dates of $\{T_i\}$. Its first variation is zero at q^*, i.e.,

$$\delta H_0^{\mathbb{Q}^*,\delta q} = 0$$

for all admissible δq, see e.g., [10, 12]. Let us define

$$\lambda_i^* = -\frac{1}{w_i}(\mathbf{E}^{\mathbb{Q}^*}\left(e^{-rT_i}G_i\right) - V_i). \tag{18}$$

Then, the first variation of $H_1^{\mathbb{Q}^*,\lambda^*,\delta q}$ and $H_0^{\mathbb{Q}^*,\delta q}$ are both zero for all admissible δq, i.e.,

$$\delta H_1^{\mathbb{Q}^*,\lambda^*,\delta q} = \delta H_0^{\mathbb{Q}^*,\delta q} = 0$$

Recall that the relative entropy, $\varepsilon(\mathbb{Q},\mathbb{Q}_0)$, is a convex functional, see e.g., [7]. By definition, $H_1^{\mathbb{Q},\lambda}$ is a convex functional with respect to $\mathbb{Q}$. Thus

$$H_1^{\mathbb{Q}^*,\lambda^*} = \inf_{\mathbb{Q}\in\Theta(\sigma_{\text{lb}},\sigma_{\text{ub}})} H_1^{\mathbb{Q},\lambda^*}.$$

By (11), (12), and (18), we have

$$\sup_{\lambda} \inf_{\mathbb{Q}\in\Theta(\sigma_{\rm lb},\sigma_{\rm ub})} H_1^{\mathbb{Q},\lambda} \geq \inf_{\mathbb{Q}\in\Theta(\sigma_{\rm lb},\sigma_{\rm ub})} H_1^{\mathbb{Q},\lambda^*} = H_1^{\mathbb{Q}^*,\lambda^*} = H_0^{\mathbb{Q}^*} = \inf_{\mathbb{Q}\in\Theta(\sigma_{\rm lb},\sigma_{\rm ub})} H_0^{\mathbb{Q}}.$$

The proceeding discussion suggests that the quadratic penalty problem (10) can, alternatively, be solved via (12) in which the associated term $\mathbf{E}^{\mathbb{Q}}$ appears linearly.

To provide a meaningful prior, the relative entropy can be represented in an alternative form. The relative entropy can be approximated by an expectation on the integral of a function of volatilities. The approximate integrand is not unique, see e.g., [2]. Therefore an optimization problem is derived from a general class of pseudo-entropy (PE) functions. The choice of a PE function does not in general qualitatively affect the result in calibrating volatility function [2]. In [2], a simple PE function is suggested:

$$\eta(\sigma^2(S,t)) \stackrel{\rm def}{=} \frac{1}{2}(\sigma^2(S,t)-\sigma_0^2)^2, \tag{19}$$

where σ_0, $\sigma_{\rm lb} \leq \sigma_0 \leq \sigma_{\rm ub}$, is the prior volatility, and the unknown volatility is assumed to be a function of the underlying, S, and time, t.

Following [2], the pseudo relative entropy below is used:

$$\varepsilon(\sigma,\sigma_0) = \mathbf{E}^{\mathbb{Q}}\left[\int_0^T \eta(\sigma^2(S,t))dt\right]. \tag{20}$$

Hence, the entropy minimization problem (8) can be approximated by the optimal control problem

$$\inf_{\lambda} \sup_{\mathbb{Q}\in\Theta(\sigma_{\rm lb},\ \sigma_{\rm ub})} \left(-\varepsilon(\sigma,\sigma_0) + \sum_{i=1}^{M} \lambda_i(\mathbf{E}^{\mathbb{Q}}\left(e^{-rT_i}G_i\right) - V_i) + \frac{1}{2}\sum_{i=1}^{M}\lambda_i^2 w_i\right), \tag{21}$$

where $\mathbb{Q}$ is the measure defined by (2) describing the dynamics of S_t, σ is the volatility associated to $\mathbb{Q}$.

Notice that the terms associated with the expectation $\mathbf{E}^{\mathbb{Q}}(\cdot)$ in problem (21) are now linear. Thus we can apply stochastic control theory to derive a Hamilton-Jacobi-Bellman (HJB) equation.

Given S, t, λ, consider the following function $W(S,t,\lambda)$

$$\begin{aligned} & W(S,t,\lambda) \\ = \ & \sup_{\mathbb{Q}\in\Theta} \mathbf{E}_t^{\mathbb{Q}}\left[-e^{rt}\int_t^T \eta(\sigma_s^2)ds\right] + \sum_{t<T_i\leq T} \mathbf{E}_t^{\mathbb{Q}}\left[\lambda_i\left(e^{-r(T_i-t)}G_i - e^{rt}V_i + \frac{1}{2}e^{rt}\lambda_i w_i\right)\right]. \end{aligned} \tag{22}$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_M)$. Assume that currently $t=0$ and the underlying S_0 and the option bid and ask prices are given. Then the optimization problem (21) becomes an unconstrained convex minimization problem

$$\inf_{\lambda\in\Re^M} W(S_0, 0, \lambda). \tag{23}$$

It can be shown that $W(S,t,\lambda)$ satisfies the following HJB equation, [2] and [6]:

$$
\begin{aligned}
W_t \quad + \quad & e^{rt}\Phi\Big(\frac{e^{-rt}}{2}S^2 W_{SS}\Big) + (r-q)SW_S - rW = \\
- \quad & \sum_{t<T_i\le T} \lambda_i \delta(t-T_i)\left(G_i(S) - e^{rT_i}V_i + \frac{1}{2}e^{rT_i}\lambda_i w_i\right), \quad s>0, \quad t\le T
\end{aligned}
\tag{24}
$$

where

$$
\Phi(X) = \begin{cases} \frac{1}{2}X^2 + \sigma_0^2 X, & \text{if } \sigma_{\text{lb}}^2 - \sigma_0^2 < X < \sigma_{\text{ub}}^2 - \sigma_0^2, \\ \sigma_{\text{lb}}^2 X - \frac{1}{2}(\sigma_{\text{lb}}^2 - \sigma_0^2)^2, & \text{if } X \le \sigma_{\text{lb}}^2 - \sigma_0^2, \\ \sigma_{\text{ub}}^2 X - \frac{1}{2}(\sigma_{\text{ub}}^2 - \sigma_0^2)^2, & \text{if } X \ge \sigma_{\text{ub}}^2 - \sigma_0^2, \end{cases}
\tag{25}
$$

and

$$
X = \frac{e^{-rt}}{2}S^2 W_{SS}.
\tag{26}
$$

with the final condition $W(S, T+0, \lambda) = 0$; here $\delta(\cdot)$ is a Dirac function.

Let the derivative of W with respect to λ_i be denoted as

$$
W_i(S,t,\lambda) = \frac{\partial W}{\partial \lambda_i}.
$$

Similar to [2], $W_i(S,t,\lambda)$ can be determined by taking derivative with respect to λ_i in equation (24), i.e.,

$$
\begin{aligned}
(W_i)_t + \frac{1}{2}\Phi'\left(\frac{e^{-rt}}{2}S^2 W_{ss}\right) S^2 (W_i)_{ss} + (r-q)S(W_i)_s - rW_i = \\
-\delta(t-T_i)\left(G_i(S) - e^{rT_i}V_i + e^{rT_i}\lambda_i w_i\right).
\end{aligned}
\tag{27}
$$

where

$$
\Phi'(X) = \begin{cases} X + \sigma_0^2, & \text{if } \sigma_{\text{lb}}^2 - \sigma_0^2 < X < \sigma_{\text{ub}}^2 - \sigma_0^2, \\ \sigma_{\text{lb}}^2, & \text{if } X \le \sigma_{\text{lb}}^2 - \sigma_0^2, \\ \sigma_{\text{ub}}^2, & \text{if } X \ge \sigma_{\text{ub}}^2 - \sigma_0^2. \end{cases}
\tag{28}
$$

From the definition of $W(S,t,\lambda)$, it can be easily shown that the function $W(S,t,\lambda)$ is convex with respect to the last argument, λ.

3 Convergence Analysis for the HJB Equation

Computationally, we solve an unconstrained convex minimization problem (23) to calibrate each volatility bound. This requires evaluating function and gradient for $W(S_0, 0, \lambda)$. The function value $W(S_0, 0, \lambda)$ is computed by solving the Hamilton-Jacobian equation (24) and the gradient is computed by solving a Black-Scholes equation (27), given $W(S,t,\lambda)$.

We now present a numerical scheme to solve equation (24). Equation (24) is a nonlinear HJB equation. In [2, 3], a trinomial method is used. A trinomial method is an explicit finite difference method. Unfortunately, to guarantee convergence to the correct (viscosity) solution, a trinomial method may need very small time step sizes and can be computationally inefficient in general. Moreover, the trinomial method only provides volatility information over a triangular

region in space (S,t). We propose a monotone, unconditionally stable, convergent implicit finite difference scheme.

Equation (24) can be written as

$$W_t + \frac{1}{2}\rho[W_{SS}]S^2 W_{SS} + (r-q)SW_S - rW = \\ -\sum_{t<T_i\le T} \lambda_i \delta(t-T_i)(G_i(S) - e^{rT_i}V_i + \frac{1}{2}e^{rT_i}\lambda_i w_i), \quad S>0, \quad t\le T \tag{29}$$

where

$$\rho[W_{SS}] = \frac{\Phi(X)}{X}, \tag{30}$$

and, as in (26),

$$X = \frac{e^{-rt}}{2}S^2 W_{SS}.$$

Equation (29) is a nonlinear parabolic partial differential equation. Thus there is a question of the existence and uniqueness of the solution.

A viscosity solution, which is a weak solution to a nonlinear parabolic PDE, has been studied in [9]. It has been shown that the viscosity solution for a nonlinear parabolic PDE is the correct solution for financial applications, see e.g., [5, 9]. For any given λ, the solution $W(S,t,\lambda)$ to the optimal control problem (22) satisfies the dynamic programming property ([9], pp. 176). Therefore the solution W is the viscosity solution of the HJB equation (29), see e.g., Corollary 3.1, pp. 209, in [9]. Moreover, there is at most one viscosity solution $W(S,t,\lambda)$ for a given λ, see e.g., Corollary 8.1, pp. 221, in [9].

In [5], it is shown that a stable, consistent and monotonic discretization of nonlinear parabolic PDE problems must converge to the desired viscosity solution. We now discuss whether the solution of algebraic equations arising from finite difference converges to the viscosity solution of the nonlinear PDE (29).

In [11], the implicit finite difference method and the Crank-Nicolson method are used to solve the *Black-Scholes-Barenblatt* (BSB) equation (5). They propose a monotone fixed-point iterative method to solve the algebraic equation at each time step. Moreover, they prove that the solution of the algebraic equation converges to the viscosity solution of the BSB equation (5).

Comparing the nonlinear HJB equation (29) to the BSB equation (5), the coefficient of the gamma term W_{SS} involves $\Phi(\cdot)$ and this coefficient becomes much more complicated. Because of this, the fixed-point method proposed in [11] is not monotone when applied to the HJB equation (29). Given that the algebraic solution at time t_k is often a good starting point for the solution at t_{k-1}, we apply an iterative Newton method, which converges locally quadratically to the solution to the implicit finite difference equation.

Given that the nonlinear HJB equation (29) is more complicated than the BSB equation (5) in the coefficient of the gamma term W_{SS}, we need to verify that solutions to the implicit finite difference equations converge to the weak solution of HJB equation (24), which is the viscosity solution [9]. Although the convergence analysis is similar to that in [11], the proof is more

complicated because the coefficient $\Phi(\cdot)$ in the HJB equation (24) becomes piecewise quadratic rather than piecewise linear.

Denote the grid points along S as $\{S_1, \cdots, S_m\}$ where $S_1 = 0$. Consider a uniform spacing in time with a step size $\triangle t$. The standard fully implicit finite difference method for the equation (24) gives

$$W_i^{n+1} - W_i^n + \triangle t e^{rt}\Phi(X_{h,i}^n) + \triangle t(\alpha_i W_{i+1}^n + \beta_i W_{i-1}^n + \gamma_i W_i^n) - rW_i^n \triangle t = 0 \qquad i = 2, \cdots, m-1, \tag{31}$$

where supscript n indicates the discretization at $n\triangle t$, subscript i indicates the discretization at S_i, and $X_{h,i}^n$ is the following finite difference approximation at X_i^n:

$$\begin{aligned} X_{h,i}^n &\stackrel{\text{def}}{=} X_{h,i}(W_{i+1}^n, W_i^n, W_{i-1}^n) \\ &= e^{-rt} S_i^2 \left[\frac{W_{i+1}^n - W_i^n}{(S_{i+1} - S_{i-1})(S_{i+1} - S_i)} + \frac{W_{i-1}^n - W_i^n}{(S_{i+1} - S_{i-1})(S_i - S_{i-1})} \right]. \end{aligned} \tag{32}$$

To ensure that discretization leads to monotonicity and stability of the fully implicit scheme (see following Lemma 3.2 and Lemma 3.3), we choose $\alpha_i, \beta_i, \gamma_i$ as follows

$$\begin{cases} \alpha_i = \alpha_{i,central}, & \beta_i = \beta_{i,central}, \quad \gamma_i = \gamma_{i,central} \quad \text{if} \frac{S_i}{S_i - S_{i-1}} \sigma_{\text{lb}}^2 \geq (r-q) \\ \alpha_i = \alpha_{i,forward}, & \beta_i = \beta_{i,forward}, \quad \gamma_i = \gamma_{i,forward} \quad \text{otherwise.} \end{cases} \tag{33}$$

where,

$$\alpha_{i,central} = \frac{(r-q)S_i}{S_{i+1} - S_{i-1}}, \quad \beta_{i,central} = -\alpha_{i,central}, \quad \gamma_{i,central} = 0.$$

corresponding to a central difference scheme and

$$\alpha_{i,forward} = \frac{(r-q)S_i}{S_{i+1} - S_i}, \quad \beta_{i,forward} = 0, \quad \gamma_{i,forward} = -\alpha_{i,forward}.$$

corresponding to the forward difference scheme.

When each $G_i(S)$ is either a call or a put payoff, the right hand side of (24) is piecewise linear in S and the solution of (24) is asymptotically linear as $S \to +\infty$. Thus we incorporate the linear boundary conditions below

$$W_t - rW = 0 \qquad \text{at } S = 0 \tag{34}$$

$$W(S,t) \approx A(t)S + B(t) \quad \text{as } S \to \infty \tag{35}$$

If we substitute (35) into the equation (24), $A(t)$ and $B(t)$ can be determined. Thus W_1^n and W_m^n can be determined from the boundary condition at $S = 0$ and the asymptotic boundary condition (35).

Finite difference equation (31) can be written as $F(\vec{W}^n) = 0$ where $\vec{W}^n$ denotes the vector of unknowns $[W_2^n, \cdots, W_{m-1}^n]$ and $F : \Re^{m-2} \to \Re^{m-2}$ denote the left hand side of (31).

The Jacobian ∇F of F is

$$\nabla F = \begin{bmatrix} \mu_2 & \nu_2 & & & \\ \kappa_3 & \mu_3 & \nu_3 & & \\ & \ddots & \ddots & \ddots & \\ & & \kappa_{m-2} & \mu_{m-2} & \nu_{m-2} \\ & & & \kappa_{m-1} & \mu_{m-1} \end{bmatrix} \tag{36}$$

where the subscript i denotes the descritization associated with ith grid and

$$\begin{aligned}
\kappa_i &= \beta_i \triangle t + \Phi'(X^n_{h,i}) \triangle t \frac{S_i^2}{(S_{i+1} - S_{i-1})(S_i - S_{i-1})}, \\
\mu_i &= -1 + (\gamma_i - r)\triangle t - \Phi'(X^n_{h,i}) S_i^2 \triangle t \\
&\quad \left(\frac{1}{(S_{i+1} - S_{i-1})(S_{i+1} - S_i)} + \frac{1}{(S_{i+1} - S_{i-1})(S_i - S_{i-1})} \right), \\
\nu_i &= \alpha_i \triangle t + \Phi'(X^n_{h,i}) \triangle t \frac{S_i^2}{(S_{i+1} - S_{i-1})(S_{i+1} - S_i)},
\end{aligned}$$

Let us apply a Newton method at each time step as follows,

1. $(W^n)^0 = W^{n+1}$
2. For $k = 0, 1, 2, \cdots$
   Solve $\delta w_k = -(\nabla F)_k^{-1} \cdot F(W^{n+1}, (W^n)^k)$
   $(W^n)^{k+1} = (W^n)^k + \delta w_k$.
   Endfor

It can be easily verified that ∇F is Lipschitz continuous. Thus the Newton iterations converge locally quadratically. Assuming convergence to the implicit finite difference equation can be achieved, questions remain on convergence to the viscosity solution. It has been shown in [5] that a stable, consistent and monotone discretization of option pricing problems must converge to the desired viscosity solution.

We now verify monotone property and stability of the fully implicit scheme for the HJB equation (24). Let $F_i(W_i^{n+1}, W_{i+1}^n, W_i^n, W_{i-1}^n) = 0$ correspond to the FD equation (31). Similar to [11], for discretization equation $F_i(W_i^{n+1}, W_{i+1}^n, W_i^n, W_{i-1}^n) = 0$, we define the monotonicity property as follows.

Definition 3.1. *(Monotone Discretizations) A discretization of the form*

$$F_i(W_i^{n+1}, W_{i+1}^n, W_i^n, W_{i-1}^n) = 0$$

is monotone if either

$$\begin{aligned}
F_i(W_i^{n+1} + \rho_i^{n+1}, W_{i+1}^n + \rho_{i+1}^n, W_i^n, W_{i-1}^n + \rho_{i-1}^n) &\geq F_i(W_i^{n+1}, W_{i+1}^n, W_i^n, W_{i-1}^n) \\
&\quad \forall \rho_{i+1}^n \geq 0, \rho_i^{n+1} \geq 0, \rho_{i-1}^n \geq 0 \\
F_i(W_i^{n+1}, W_{i+1}^n, W_i^n + \rho_i^n, W_{i-1}^n) &\leq F_i(W_i^{n+1}, W_{i+1}^n, W_i^n, W_{i-1}^n) \\
&\quad \forall \rho_i^n \geq 0
\end{aligned}$$

or

$$\begin{aligned} F_i(W_i^{n+1}+\rho_i^{n+1}, W_{i+1}^n+\rho_{i+1}^n, W_i^n, W_{i-1}^n+\rho_{i-1}^n) &\leq F_i(W_i^{n+1}, W_{i+1}^n, W_i^n, W_{i-1}^n) \\ &\quad \forall \rho_{i+1}^n \geq 0, \rho_i^{n+1} \geq 0, \rho_{i-1}^n \geq 0 \\ F_i(W_i^{n+1}, W_{i+1}^n, W_i^n+\rho_i^n, W_{i-1}^n) &\geq F_i(W_i^{n+1}, W_{i+1}^n, W_i^n, W_{i-1}^n) \\ &\quad \forall \rho_i^n \geq 0 \end{aligned}$$

We first extablish the following auxiliary lemmas.

Lemma 3.1. *Denote*

$$\begin{aligned} \tilde{X} &= X_{h,i}(W_{i+1}+\epsilon, W_i, W_{i-1}), \\ \breve{X} &= X_{h,i}(W_{i+1}, W_i, W_{i-1}+\epsilon), \\ \widehat{X} &= X_{h,i}(W_{i+1}, W_i, W_{i-1}) \end{aligned}$$

where $X_{h,i}$ is defined by (32) Then, for any $\epsilon > 0$,

$$\begin{aligned} \Phi(\tilde{X}) - \Phi(\widehat{X}) &\geq \sigma_{lb}^2(\tilde{X} - \widehat{X}) \\ \Phi(\breve{X}) - \Phi(\widehat{X}) &\geq \sigma_{lb}^2(\breve{X} - \widehat{X}) \end{aligned} \tag{37}$$

where Φ is defined by (25).

Proof: Following definition (32) for $X_{h,i}$, it can be easily verified that $\tilde{X} > \widehat{X}$ for $\epsilon > 0$,. There are six possibilities for $\Phi(\tilde{X}) - \Phi(\widehat{X})$.

1. If $\widehat{X} < \sigma_{\text{lb}}^2 - \sigma_0^2$ and $\tilde{X} < \sigma_{\text{lb}}^2 - \sigma_0^2$, then $\Phi(\tilde{X}) - \Phi(\widehat{X}) = \sigma_{\text{lb}}^2(\tilde{X} - \widehat{X})$.

2. If $\widehat{X} < \sigma_{\text{lb}}^2 - \sigma_0^2$ and $\sigma_{\text{lb}}^2 - \sigma_0^2 \leq \tilde{X} \leq \sigma_{\text{ub}}^2 - \sigma_0^2$, then

$$\Phi(\tilde{X}) - \Phi(\widehat{X}) = \frac{1}{2}\tilde{X}^2 + \sigma_0^2\tilde{X} - \sigma_{\text{lb}}^2\widehat{X} + \frac{1}{2}(\sigma_{\text{lb}}^2 - \sigma_0^2)^2 > \sigma_0^2\tilde{X} - \sigma_{\text{lb}}^2\widehat{X}.$$

   Because $\sigma_{\text{lb}} \leq \sigma_0 \leq \sigma_{\text{ub}}$,

$$\Phi(\tilde{X}) - \Phi(\widehat{X}) \geq \sigma_{\text{lb}}^2(\tilde{X} - \widehat{X}).$$

3. If $\widehat{X} < \sigma_{\text{lb}}^2 - \sigma_0^2$ and $\tilde{X} > \sigma_{\text{ub}}^2 - \sigma_0^2$

$$\begin{aligned} \Phi(\tilde{X}) - \Phi(\widehat{X}) &= \sigma_{\text{ub}}^2\tilde{X} - \frac{1}{2}(\sigma_{\text{ub}}^2 - \sigma_0^2)^2 - \sigma_{\text{lb}}^2\widehat{X} + \frac{1}{2}(\sigma_{\text{lb}}^2 - \sigma_0^2)^2 \\ &\geq \sigma_0^2\tilde{X} + (\sigma_{\text{ub}}^2 - \sigma_0^2)\tilde{X} - \frac{1}{2}(\sigma_{\text{ub}}^2 - \sigma_0^2)^2 - \sigma_{\text{lb}}^2\widehat{X} \\ &\geq \sigma_0^2\tilde{X} - \sigma_{\text{lb}}^2\widehat{X} \\ &\geq \sigma_{\text{lb}}^2(\tilde{X} - \widehat{X}). \end{aligned}$$

4. If both $\widehat{X}$ and $\tilde{X}$ are between $[\sigma_{\text{lb}}^2 - \sigma_0^2, \sigma_{\text{ub}}^2 - \sigma_0^2]$,

$$\begin{aligned} \Phi(\tilde{X}) - \Phi(\widehat{X}) &= \frac{1}{2}(\tilde{X}^2 - \widehat{X}^2) + \sigma_0^2(\tilde{X} - \widehat{X}) \\ &\geq \sigma_0^2(\tilde{X} - \widehat{X}) \\ &\geq \sigma_{\text{lb}}^2(\tilde{X} - \widehat{X}). \end{aligned}$$

5. If $\sigma_{\text{lb}}^2 - \sigma_0^2 \le \widehat{X} \le \sigma_{\text{ub}}^2 - \sigma_0^2$ and $\tilde{X} > \sigma_{\text{ub}}^2 - \sigma_0^2$,

$$\begin{aligned}
\Phi(\tilde{X}) - \Phi(\widehat{X}) &= \sigma_{\text{ub}}^2 \tilde{X} - \frac{1}{2}(\sigma_{\text{ub}}^2 - \sigma_0^2)^2 - \frac{1}{2}\widehat{X}^2 - \sigma_0^2 \widehat{X} \\
&= (\sigma_{\text{ub}}^2 - \sigma_0^2)\tilde{X} - \frac{1}{2}(\sigma_{\text{ub}}^2 - \sigma_0^2)^2 - \frac{1}{2}\widehat{X}^2 + \sigma_0^2(\tilde{X} - \widehat{X}) \\
&\ge \frac{1}{2}(\sigma_{\text{ub}}^2 - \sigma_0^2)^2 - \frac{1}{2}\widehat{X}^2 + \sigma_0^2(\tilde{X} - \widehat{X}) \\
&\ge \sigma_0^2(\tilde{X} - \widehat{X}) \\
&\ge \sigma_{\text{lb}}^2(\tilde{X} - \widehat{X}).
\end{aligned}$$

6. If $\widehat{X} \ge \sigma_{\text{ub}}^2 - \sigma_0^2$, $\Phi(\tilde{X}) - \Phi(\widehat{X}) = \sigma_{\text{ub}}^2(\tilde{X} - \widehat{X}) > \sigma_{\text{lb}}^2(\tilde{X} - \widehat{X})$.

Similarly we can prove that

$$\Phi(\breve{X}) - \Phi(\widehat{X}) \ge \sigma_{\text{lb}}^2(\breve{X} - \widehat{X}).$$

■

Lemma 3.2. *The fully implicit discretization (31) is monotone.*

Proof: $\forall \epsilon > 0$, let $\tilde{X}$, $\breve{X}$, and $\widehat{X}$ be the same to those defined in Lemma 3.1. Let us first consider positive perturbation on W_{i+1}^n. By Lemma 3.1, we have

$$\begin{aligned}
& F_i(W_i^{n+1}, W_{i+1}^n + \epsilon, W_i^n, W_{i-1}^n) \\
= \; & F_i(W_i^{n+1}, W_{i+1}^n, W_i^n, W_{i-1}^n) + \alpha_i \epsilon \triangle t + e^{rt} \triangle t(\Phi(\tilde{X}) - \Phi(\widehat{X})) \\
\ge \; & F_i(W_i^{n+1}, W_{i+1}^n, W_i^n, W_{i-1}^n) + \alpha_i \epsilon \triangle t + e^{rt} \sigma_{\text{lb}}^2 \triangle t(\tilde{X} - \widehat{X}) \\
= \; & F_i(W_i^{n+1}, W_{i+1}^n, W_i^n, W_{i-1}^n) + \alpha_i \epsilon \triangle t + \sigma_{\text{lb}}^2 S_i^2 \triangle t \frac{\epsilon}{(S_{i+1} - S_{i-1})(S_{i+1} - S_i)} \\
\ge \; & F_i(W_i^{n+1}, W_{i+1}^n, W_i^n, W_{i-1}^n).
\end{aligned}$$

Similarly, let us consider positive perturbation on W_{i-1}^n. By the definition of β_i, we have

$$\begin{aligned}
& F_i(W_i^{n+1}, W_{i+1}^n, W_i^n, W_{i-1}^n + \epsilon) \\
= \; & F_i(W_i^{n+1}, W_{i+1}^n, W_i^n, W_{i-1}^n) + \beta_i \epsilon \triangle t + e^{rt} \triangle t(\Phi(\breve{X}) - \Phi(\widehat{X})) \\
\ge \; & F_i(W_i^{n+1}, W_{i+1}^n, W_i^n, W_{i-1}^n) + \beta_i \epsilon \triangle t + \sigma_{\text{lb}}^2 S_i^2 \triangle t \frac{\epsilon}{(S_{i+1} - S_{i-1})(S_i - S_{i-1})} \\
\ge \; & F_i(W_i^{n+1}, W_{i+1}^n, W_i^n, W_{i-1}^n).
\end{aligned}$$

From equation (28), $\Phi'(\cdot)$ is positive. Therefore $\Phi(\cdot)$ is monotone increasing. It is then easy to verify that

$$F_i(W_i^{n+1}, W_{i+1}^n, W_i^n + \epsilon, W_{i-1}^n) \le F_i(W_i^{n+1}, W_{i+1}^n, W_i^n, W_{i-1}^n).$$

From equation (31), it is obvious that

$$F_i(W_i^{n+1} + \epsilon, W_{i+1}^n, W_i^n, W_{i-1}^n) \ge F_i(W_i^{n+1}, W_{i+1}^n, W_i^n, W_{i-1}^n).$$

By Definition 3.1, the fully implicit discretization is monotone.

■

Similar to [11], we establish the stability property for the implicit finite difference scheme for the nonlinear HJB equation (29).

Lemma 3.3. *The fully implicit discretization (31) is unconditionally stable.*

Proof: Let us denote $\tilde{W}$ as the boundary values of W associated with all time grids. Denote

$$\begin{aligned}\overline{W}^{n+1} &= \max(\max_{1<i<m} W_i^{n+1}, \tilde{W}) \\ \underline{W}^{n+1} &= \min(\min_{1<i<m} W_i^{n+1}, \tilde{W}).\end{aligned}$$

Assume that the maximum values of W_i^n is achieved at i_0 ,

$$W_{i_0}^n = \overline{W}^n.$$

Without loss of generality, let us assume $1 < i_0 < m$. From Lemma 3.2, we have

$$\begin{aligned}0 &= F_{i_0}(W_{i_0}^{n+1}, W_{i_0+1}^n, W_{i_0}^n, W_{i_0-1}^n) \\ &\leq F_{i_0}(\overline{W}^{n+1}, W_{i_0+1}^n, W_{i_0}^n, W_{i_0-1}^n) \\ &\leq F_{i_0}(\overline{W}^{n+1}, W_{i_0}^n, W_{i_0}^n, W_{i_0}^n) \\ &= \overline{W}^{n+1} - W_{i_0}^n - r\triangle t W_{i_0}^n\end{aligned}$$

Then we have

$$W_{i_0}^n \leq \frac{\overline{W}^{n+1}}{1 + r\triangle t} \leq \overline{W}^{n+1} \tag{38}$$

Similarly, let us assume that the minimum value of W_i^n is achieved at j_0,

$$W_{j_0}^n = \underline{W}^n.$$

Without loss of generality, let us assume $1 < j_0 < m$. From Lemma 3.2, we have

$$\begin{aligned}0 &= F_{j_0}(W_{j_0}^{n+1}, W_{j_0+1}^n, W_{j_0}^n, W_{j_0-1}^n) \\ &\geq F_{j_0}(\underline{W}^{n+1}, W_{j_0+1}^n, W_{j_0}^n, W_{j_0-1}^n) \\ &\geq F_{j_0}(\underline{W}^{n+1}, W_{j_0}^n, W_{j_0}^n, W_{j_0}^n) \\ &= \underline{W}^{n+1} - W_{j_0}^n - r\triangle t W_{j_0}^n\end{aligned}$$

$$W_{j_0}^n \geq \frac{\underline{W}^{n+1}}{1 + r\triangle t}. \tag{39}$$

Hence, from (38) and (39), we have

$$\overline{W}^{n+1} \geq W_i^n > \frac{\underline{W}^{n+1}}{1 + r\triangle t} \quad \forall i = 1, \cdots, m.$$

■

Following Lemma 3.2 and Lemma 3.3, we conclude that the implicit finite difference method is consistent, monotone, and stable.

4 Computational Examples

We now present some computational examples to illustrate calibration of an uncertain volatility model using the proposed quadratic penalty formulation. Our main focus here is on calibration accuracy and sensitivity to different weights and prior settings. We note that, in [6], additional computational examples and discussions are provided for calibrated volatility surfaces, including using real market data.

Recall that our objective is to solve two entropy minimization problems to calibrate an uncertain volatility model,

$$\begin{aligned} &\frac{dS_t}{S_t} = (r-q)dt + \sigma_t dZ_t^{\mathbb{Q}}, \\ \text{where} \quad &\underline{\sigma} \le \sigma_t \le \overline{\sigma}. \end{aligned}$$

where $\mathbb{Q}$ is the risk neutral probability measures, $Z_t^{\mathbb{Q}}$ is a standard Brownian motion under $\mathbb{Q}$, r and q are interest rate and dividend yield respectively.

Each volatility bound, $\underline{\sigma}$ or $\overline{\sigma}$, is determined by approximately solving an entropy minimization problem,

$$\begin{aligned} &\inf_{\mathbb{Q}\in\Theta(\sigma_{\text{lb}},\sigma_{\text{ub}})} \varepsilon(\mathbb{Q},\mathbb{Q}_0) \\ \text{subject to} \quad &\mathbf{E}^{\mathbb{Q}}\left(e^{-rT_i}G_i(S_{T_i})\right) = V_i,\ i=1,2,\ldots,M. \end{aligned}$$

where $\{V_i\}_{i=1}^M$ are given standard European option prices. This entropy problem is approximately solved based on its quadratic penalty formulation,

$$\inf_{\mathbb{Q}\in\Theta(\sigma_{\text{lb}},\sigma_{\text{ub}})} \left(\varepsilon(\mathbb{Q},\mathbb{Q}_0) + \frac{1}{2}\sum_{i=1}^{M} \frac{1}{w_i}\left(\mathbf{E}^{\mathbb{Q}}\left(e^{-rT_i}G_i(S_{T_i})\right) - V_i\right)^2 \right)$$

where the dynamics of S_t are described by (2) and $\{w_i\}_{i=1}^M$ are the specified weights. Prices $\{V_i\}$ correspond to bid (or ask) prices of specified liquid call and put options, $\{G_i\}_{i=1}^M$ are associated payoff functions, $\mathbb{Q}_0$ corresponds to a constant volatility pricing measure.

Using a pseudo entropy function in the continuous setting, we have shown that the solution to the quadratic penalty formulation can be approximated by solving

$$\min_{\lambda} W(S_0, 0, \lambda).$$

We compute the objective function value $W(S_0, 0, \lambda)$ in the above optimization problem by solving the nonlinear HJB equation (24) with the fully implicit method presented in §3.

In subsequent computational examples, we assume that the mid-prices are generated by the model option prices assuming the underlying price follows a CEV process

$$\frac{dS_t}{S_t} = (r-q)dt + \frac{\alpha}{S_t} dZ_t^{\mathbb{Q}} \tag{40}$$

with $\alpha = 15$, where $Z_t^{\mathbb{Q}}$ is a standard Brownian motion under the pricing probability measure $\mathbb{Q}$. Assume the initial underlying price $S_0 = 100$, the risk free interest rate $r = 0.05$, and the dividend rate $q = 0.01$.

Suppose that a vector V of 35 European call and put option mid-prices are given. Table 5 displays their strikes, maturities, values, and associated implied volatilities denoted as $\breve{\sigma}$.

In addition, we assume that a vector $\underline{V}$ of 35 option bid prices and a vector $\overline{V}$ of 35 option ask prices are generated from a fixed spread from the middle price V as follows

$$\underline{V} = V - \frac{1}{2}\text{spread},$$

and

$$\overline{V} = V + \frac{1}{2}\text{spread}.$$

In following examples, we assume that the bid and ask spread level is a monotone function of maturity as listed in Table 1; these spreads are similar to the empirical observations on the average spread level of S&P500 index options on April 20, 1999.

To illustrate the accuracy of the quadratic penalty function calibration, we consider a set of six tests, corresponding to different priors σ_0 and weights w. We report computation results for different values of σ_0 to test the accuracy of calibration; the values of the prior considered are listed in Table 2, where $\breve{\sigma}$ denotes implied volatilities of the mid-prices.

For calibration of the lower volatility $\underline{\sigma}$, $\{V_i\}_{i=1}^M$ are the given option bid prices. The prior volatility constant σ_0 corresponds to a prior measure $\mathbb{Q}_0$. For the lower volatility $\underline{\sigma}$ calibration, the prior σ_0 in general corresponds to estimation of the lowest volatility. For lower volatility calibration, the constant volatility prior σ_0 is also used as σ_{lb}. To stress test the calibration method, we also set

$$\sigma_{\text{ub}} = \max(\sigma_0, \alpha/S),$$

since this makes calibration problem more difficult; note that V is the model price using the volatility function α/S. Recall that the prior σ_0 for the lower volatility $\underline{\sigma}$ is listed in the second column of Table 2.

For calibration of the upper bound $\overline{\sigma}$, $\{V_i\}_{i=1}^M$ are given option ask prices, the constant volatility prior σ_0 corresponds to estimation or subjective view on the maximum volatility. We set σ_{ub} to this constant volatility and σ_{lb} is now set to

$$\sigma_{\text{lb}} = \min(\sigma_0, \alpha/S),$$

where σ_0 is a prior for the upper volatility bound calibration. When calibrating the upper volatility bound $\overline{\sigma}$, the prior σ_0 values considered in our tests are given in the third column of Table 2.

In Table 3, calibration errors for lower volatility bound $\underline{\sigma}$ and the upper volatility bound $\underline{\sigma}$ are reported. In Test 1 - Test 4, the maximum calibration error is less than 1 cent and the 2-norm of the calibration error is about 0.01. Comparing to Test 1 - Test 4, we observe slightly larger calibration errors in Test 5 and Test 6. The large calibration errors are associated with deeper out-the-money options and the calibration errors for other options remain relatively small. The averages of absolute calibration errors for the lower and upper volatility function bounds in Test 5 are 8.8191e-04 and 0.0026 respectively.

Table 1: Spread Assumptions: spread is a monotonic increasing function of the maturity

Maturity(T)	Spread($)
0.1	0.10
0.25	0.12
0.5	0.14
0.75	0.18
1	0.20

Table 2: Priors and Weights Considered: a uniform weighting is assumed for each test.

Test	σ_0 for $\underline{\sigma}$	σ_0 for $\overline{\sigma}$	weight w_i
1	$0.2\min(\breve{\sigma})$	$2\max(\breve{\sigma})$	1
2	$0.2\min(\breve{\sigma})$	$2\max(\breve{\sigma})$	0.01
3	$0.5\min(\breve{\sigma})$	$1.5\max(\breve{\sigma})$	1
4	$0.5\min(\breve{\sigma})$	$1.5\max(\breve{\sigma})$	0.01
5	$0.8\min(\breve{\sigma})$	$1.2\max(\breve{\sigma})$	1
6	$0.8\min(\breve{\sigma})$	$1.2\max(\breve{\sigma})$	0.01

Finally, we discuss the impact of the weights on the calibration. Theoretically the calibration error decreases as weights decrease. This can be observed from results in Table 4. For this example, the calibration error is acceptably small when weights are equal to 1. There is no significant improvement in calibration accuracy for smaller weights considered in our experiments, see Table 3 & 4.

Table 3: Calibration Errors for Lower and Upper Volatility Bounds: $\underline{\text{Error}}$ and $\overline{\text{Error}}$ denote calibration errors of the lower volatility bound and upper volatility bound respectively.

Test	$\lVert\underline{\text{Error}}\rVert_2$	$\lVert\underline{\text{Error}}\rVert_\infty$	$\lVert\overline{\text{Error}}\rVert_2$	$\lVert\overline{\text{Error}}\rVert_\infty$
1	0.0101	0.0050	0.0048	0.0023
2	0.0127	0.0045	0.0073	0.0041
3	0.0094	0.0043	0.0016	0.0011
4	0.0087	0.0031	0.0003	0.0001
5	0.0086	0.0063	0.0461	0.0425
6	0.0027	0.0010	0.0461	0.0424

Table 4: Sensitivity of Calibration Error to Weights: uniform weighting is used for each test. Prior values are the same as in Test 1 in Table 2.

Weight	$\|\underline{\text{Error}}\|_2$	$\|\underline{\text{Error}}\|_\infty$
10^5	2.6972	0.9731
10^4	0.5382	0.2280
10^3	0.0824	0.0406
100	0.0162	0.0080
10	0.0095	0.0042
1	0.0101	0.0050
0.01	0.0127	0.0045

5 Conclusion

In option pricing, volatility is a crucial parameter since it is the only variable that is not directly observable. An uncertain volatility model, proposed by [3, 4], is a potentially promising model to address volatility uncertainty.

In order for an uncertain volatility model to be practically useful for option pricing and risk management, an appropriate uncertain volatility model which is consistent with market observations needs to be calibrated. Following typical practice in derivative pricing, calibrating such an model from market liquid option bid and ask prices directly is both intuitive and reasonable.

In [6], entropy optimization formulations are proposed for calibrating an uncertain volatility model to the bid and ask option prices.

The main objective of this paper is to provide mathematical justification for the proposed entropy formulations and computational methods. We explain that the quadratic penalty formulation for the constrained entropy minimization problem can be solved by minimizing a convex function. We show that the objective function can be evaluated by solving a Hamilton-Jacobian-Bellman equation. We propose to solve the resulting HJB equation using an implicit finite difference method. Moreover we prove that the solutions to the implicit finite difference equations converge to the viscosity solution of the Hamilton-Jacobian-Bellman equation. Finally computational examples are provided to illustrate the accuracy of calibration.

References

[1] M. Avellaneda, R. Buff, C. Friedman, N. Grandchamp, L. Kruk, and J. Newman. Weighted monte carlo: A new technique for calibrating asset-pricing models. *International Journal of Theoretical and Applied Finance*, 4(1):91–119, 2001.

[2] M. Avellaneda, C. Friedman, R. Holemes, and D. Samperi. Calibrating volatility surfaces via relative entropy minimization. *Applied Mathematical Finance*, 4:37–64, 1997.

[3] M. Avellaneda, A. Levy, and A. Paras. Pricing and hedging derivative securities in markets with uncertain volatilities. *Applied Mathematical Finance*, 2:73–88, 1995.

[4] M. Avellaneda and A. Paras. Managing the volatility risk of portfolios of derivative securities: the Lagrangian uncertain volatility model. *Applied Mathematical Finance*, 3:21–52, 1996.

[5] G. Barles. Convergence of numerical schemes for degenerate parabolic equations arising in finance. In L. C. G. Rogers and D. Talay, editors, *Numerical Methods in Finance*. Cambridge University Press, Cambridge, 1997.

[6] T. Coleman, C. He, and Y. Li. Calibrating volatility function bounds for a uncertain volatility model. *Journal of Computational Finance*, submitted to, 2006.

[7] T. M. Cover and J. A. Thomas. *Elements of Information Theory*. John Wiley & Sons, Inc., 1991.

[8] L. C. Evans. *Partial Differential Equations*. American Mathematical Society, Providence, Rhode Island, 1998.

[9] W. H. Fleming and H. M. Soner. *Controlled Markov Processes and Viscosity Solutions*. Springer-Verlag, The second edition, 2006.

[10] I. M. Gelfand and S. V. Fomin. *Calculus of Variations*. Prentice-Hall Inc., 1963.

[11] D. M. Pooley, P. A. Forsyth, and K. R. Vetzal. Numerical convergence properties of option pricing pdes with uncertain volatility. *IMA. Journal of Numerical Analysis*, 23(2):241–267, 2003.

[12] N. Wu. *The Maximum Entropy Method*. Springer-Verlag, 1997.

Table 5: European option middle prices and associated implied volatilities of computation examples. The underlying price is assume to follow process (40). $S_0 = 100$, $r = 0.05$, $q = 0.01$.

Maturity (T_i)	Type	Strike (K_i)	Middle Price	implied vol $\breve{\sigma}$
0.1	Call	105	0.4198	0.1464
	Call	100	2.0928	0.1500
	Put	95	0.3019	0.1539
0.25	Call	110	0.4234	0.1430
	Call	105	1.4047	0.1464
	Call	100	3.4924	0.1500
	Put	95	0.9001	0.1539
	Put	90	0.2388	0.1581
0.5	Call	115	0.5725	0.1399
	Call	110	1.3828	0.1431
	Call	105	2.8756	0.1465
	Call	100	5.2276	0.1501
	Put	95	1.6094	0.1539
	Put	90	0.6864	0.1581
	Put	85	0.2494	0.1626
0.75	Call	120	0.5962	0.1369
	Call	115	1.2606	0.1399
	Call	110	2.4069	0.1431
	Call	105	4.1817	0.1465
	Call	100	6.6687	0.1501
	Put	95	2.1077	0.1540
	Put	90	1.0802	0.1581
	Put	85	0.4992	0.1626
	Put	80	0.2067	0.1674
1.0	Call	125	0.5741	0.1341
	Call	120	1.1231	0.1369
	Call	115	2.0333	0.1400
	Call	110	3.4219	0.1431
	Call	105	5.3815	0.1465
	Call	100	7.9543	0.1502
	Put	95	2.4823	0.1540
	Put	90	1.4092	0.1582
	Put	85	0.7415	0.1626
	Put	80	0.3602	0.1674
	Put	75	0.1609	0.1727