Mathematical Programming 19 (1980) 178–185.
North-Holland Publishing Company

SECOND-ORDER CONDITIONS FOR AN EXACT PENALTY FUNCTION*

T.F. COLEMAN

Applied Mathematics Division, Argonne National Laboratory, Argonne, IL 60439, U.S.A.

A.R. CONN

Department of Computer Science, University of Waterloo, Waterloo, Ont., Canada N2L 3G1

Received 18 September 1978
Revised manuscript received 7 March 1980

In this paper we give first- and second-order conditions to characterize a local minimizer of an exact penalty function. The form of this characterization gives support to the claim that the exact penalty function and the nonlinear programming problem are closely related.

In addition, we demonstrate that there exist arguments for the penalty function from which there are no descent directions even though these points are not minimizers.

Key words: Nonlinear Programming, Exact Penalty Function, Constrained Optimization, Piecewise Differentiable.

1. Introduction

The nonlinear programming problem can be written as

$$\begin{array}{lll} \text{minimize} & f(x), & \\ \text{subject to} & \phi_i(x) \geq 0, & i \in M_1, \\ & \phi_j(x) = 0, & j \in M_2, \end{array} \tag{1}$$

where M_1 and M_2 are index sets and the functions f, ϕ_i, $i \in M_1 \cup M_2$ are continuous and map $\mathbf{R}^n$ to $\mathbf{R}^1$. Problem (1) is closely related to the exact penalty function

$$p(x, \mu) = \mu \cdot f(x) - \sum_{i \in M_1} \min(0, \phi_i(x)) + \sum_{j \in M_2} |\phi_j(x)|. \tag{2}$$

Recently, a number of nonlinear programming algorithms have been proposed which generate descent directions for p (see [3, 6, 9, 12]). By ensuring a sufficient decrease in p, at each step, global convergence properties can be attained.

The major purpose of this paper is to present optimality conditions for the penalty function p. These conditions emphasize and elucidate the close relation-

* This research is partially supported by the Natural Science and Engineering Research Council Grant No. A8639 and the U.S. Department of Energy.

ship between p and problem (1). Unfortunately, perhaps, the optimality characterizations do not include feasibility to (1). Clearly, a consequence of this fact is that we may be optimal to p but not even feasible to (1). Indeed, we give an example, in Section 3, of a problem which contains a point x^0 with the following properties:

(1) x^0 satisfies the first-order optimality characterization for p,
(2) x^0 is not a local minimizer of p,
(3) x^0 is not feasible to (1),
(4) there does not exist a descent direction for p, at x^0.

Since x^0 is not feasible, it is obviously not an ideal terminating point for an algorithm designed to solve (1); however, a p-descent direction algorithm cannot leave x^0.

2. Necessary and sufficient conditions

In this section we present optimality characterizations for p. That is, we ask (and answer) the question: when is a point x^0 a minimizer of p?

First we introduce a few definitions and some notation.

Define

$$\begin{aligned}
A_1 &= \{i \in M_1 \mid \phi_i(x^0) = 0\},\\
A_2 &= \{i \in M_2 \mid \phi_i(x^0) = 0\},\\
V_1 &= \{i \in M_1 \mid \phi_i(x^0) < 0\},\\
V_2 &= \{i \in M_2 \mid |\phi_i(x^0)| \neq 0\}.
\end{aligned}$$

Let η and γ be vectors of dimensions $|A_1|$ and $|A_2|$ respectively, whose components, in both cases, can attain *only* the values 1 or -1.

In the following theorem we show the equivalence between p and a class of nonlinear programming problems. Deducing the optimality conditions from this relationship is then straightforward.

Theorem 1. *Assuming that f, ϕ_i, $i \in M_1 \cup M_2$ are continuous, then*

x^0 is a local minimizer of p (for a given μ)

$$\Leftrightarrow$$

x^0 is a local minimizer to the problems:

$$\begin{aligned}
\underset{x}{\text{minimize}} \quad & \Big\{\mu f(x) - \sum_{i\in V_1} \phi_i(x) + \sum_{i\in V_2} (\operatorname{sgn} \phi_i(x^0)) \cdot \phi_i(x)\\
& \quad + \sum_{i\in A_1} \tfrac{1}{2}(\eta_i - 1)\phi_i(x) + \sum_{i\in A_2} \gamma_i\phi_i(x)\Big\}, \qquad (3)\\
\text{subject to} \quad & \eta_i\phi_i(x) \geq 0, \quad i \in A_1,\\
& \gamma_i\phi_i(x) \geq 0, \quad i \in A_2,
\end{aligned}$$

for all possible vectors η and γ satisfying the property that each component is either 1 *or* -1.

Proof. (i) Suppose x^0 solves $p(x)$ but not (3), for some $\bar{\eta}, \bar{\gamma}$. The non-empty feasible region for this problem is defined by

$$\bar{\eta}_i\phi_i(x) \geq 0, \quad i \in A_1,$$
$$\bar{\gamma}_i\phi_i(x) \geq 0, \quad i \in A_2.$$

But in this region p is equivalent to the objective function of (3) (if we are sufficiently close to x^0), which implies that x^0 is not a local minimizer of p, a contradiction.

(ii) Suppose x^0 solves (3), for all η, γ, but is not a local minimizer of p. It follows that there exists an infinite sequence $\{x^k\}$ converging to x^0, and satisfying $p(x^k) < p(x^0)$. But clearly a subsequence of $\{x^k\}$ is entirely contained in a region defined by

$$\bar{\eta}_i\phi_i(x) \geq 0, \quad i \in A_1,$$
$$\bar{\gamma}_i\phi_i(x) \geq 0, \quad i \in A_2$$

for some $\bar{\eta}, \bar{\gamma}$. Clearly then x^0 is not a local minimizer to (3), (for $\eta = \bar{\eta}, \gamma = \bar{\gamma}$) a contradiction.

Since the optimality conditions for problem (3) are well-known [7], we can now easily derive conditions for p. (To simplify notation, all function arguments are assumed to be x^0 when they are not explicitly written.)

Corollary 1 (first-order necessary conditions). *Assuming that the constraint and objective functions are continuously differentiable, and that $\{\nabla\phi_i(x^0) \mid i \in A_1 \cup A_2\}$ is linearly independent, then necessary conditions for x^0 to be a local minimizer of p are: There exist vectors λ, w satisfying*

(i) $$\mu\nabla f - \sum_{i\in V_1} \nabla\phi_i + \sum_{i\in V_2} \operatorname{sgn}(\phi_i)\nabla\phi_i = \sum_{i\in A_1} \lambda_i\nabla\phi_i - \sum_{i\in A_2} w_i\nabla\phi_i,$$

(ii) $$0 \leq \lambda_i \leq 1, \quad i \in A_1,$$
$$-1 \leq w_i \leq 1, \quad i \in A_2.$$

Proof. Let us consider two particular problems from the class represented by (3). Let P_0 refer to the problem where

$$\eta_i = 1, \quad i \in A_1$$
$$\gamma_i = 1, \quad i \in A_2,$$

and let P_j refer to the problem where

$$\eta_i = 1, \quad i \in A_1 \setminus \{j\},$$
$$\eta_j = -1,$$
$$\gamma_i = 1, \quad i \in A_2.$$

Define

$$\bar{\nabla} p = \mu \nabla f - \sum_{i \in V_1} \nabla \phi_i + \sum_{i \in V_2} \operatorname{sgn}(\phi_i) \nabla \phi_i.$$

Since x^0 is optimal to P_0, there exists λ, w, such that $\lambda \geq 0$, and

$$\bar{\nabla} p = \sum_{i \in A_1} \lambda_i \nabla \phi_i - \sum_{i \in A_2} w_i \nabla \phi_i. \tag{4}$$

Since x^0 is optimal to P_j, there exists λ^j, w^j such that $\lambda^j \geq 0$, and

$$\bar{\nabla} p - \nabla \phi_j = \sum_{i \in A_1 \sim \{j\}} \lambda_i^j \nabla \phi_i - \lambda_j^j \nabla \phi_j - \sum_{i \in A_2} w_i^j \nabla \phi_i. \tag{5}$$

Considering (4) and (5) along with linear independence gives

$$\lambda_i = \lambda_i^j, \quad i \in A_1 \sim \{j\},$$
$$w_i = w_i^j, \quad i \in A_2,$$

and $(1 - \lambda_j^j) = \lambda_j$. But $\lambda_j^j \geq 0 \Rightarrow \lambda_j \leq 1$.

In a similar fashion we can obtain the bounds on w.

Corollary 2 (second-order necessary conditions). *Assuming that f, ϕ_i, $i \in M_1 \cup M_2$ are twice continuously differentiable, and the set $\{\nabla \phi_i(x^0) \mid i \in A_1 \cup A_2\}$ is linearly independent, then necessary conditions for x^0 to be a local minimizer of p, are: There exist vectors λ, w satisfying*

(i) $$\mu \nabla f - \sum_{i \in V_1} \nabla \phi_i + \sum_{i \in V_2} \operatorname{sgn}(\phi_i) \nabla \phi_i = \sum_{i \in A_1} \lambda_i \nabla \phi_i - \sum_{i \in A_2} w_i \nabla \phi_i,$$

(ii) $$0 \leq \lambda_i \leq 1, \quad \forall i \in A_1,$$
$$-1 \leq w_i \leq 1, \quad \forall i \in A_2,$$

(iii) $\forall y$ *satisfying* $y^{\mathrm{T}} \nabla \phi_i = 0$, $\forall i \in A_1 \cup A_2$,

$$y^{\mathrm{T}} \left[\mu \nabla^2 f - \sum_{i \in V_1} \nabla^2 \phi_i + \sum_{i \in V_2} \operatorname{sgn}(\phi_i) \nabla^2 \phi_i - \sum_{i \in A_1} \lambda_i \nabla^2 \phi_i + \sum_{i \in A_2} w_i \nabla^2 \phi_i \right] y \geq 0.$$

Proof. The result follows directly from Theorem 1, the second-order necessary conditions for nonlinear programming [7], and Corollary 1.

Corollary 3 (second-order sufficiency). *Assuming that f, ϕ_i, $i \in M_1 \cup M_2$ are twice-differentiable functions, then sufficient conditions for x^0 to be an isolated local minimizer of p are: There exist vectors λ, w satisfying*

(i) $$\mu \nabla f - \sum_{i \in V_1} \nabla \phi_i + \sum_{i \in V_2} \operatorname{sgn}(\phi_i) \nabla \phi_i = \sum_{i \in A_1} \lambda_i \nabla \phi_i - \sum_{i \in A_2} w_i \nabla \phi_i,$$

(ii) $$0 \leq \lambda_i < 1, \quad \forall i \in A_1,$$
$$-1 \leq w_i \leq 1, \quad \forall_i \in A_2,$$

(iii) $\forall y$ *satisfying* $(y^{\mathrm{T}}\nabla\phi_i = 0, i \in A_2)$ *and* $(y^{\mathrm{T}}\nabla\phi_i = 0, i \in A_1$ *and* $\lambda_i > 0)$ *and* $(y^{\mathrm{T}}\nabla\phi_i \geq 0, i \in A_1$ *and* $\lambda_i = 0)$

$$y^{\mathrm{T}}\left[\mu\nabla^2 f - \sum_{i\in V_1}\nabla^2\phi_i - \sum_{i\in V_2}\operatorname{sgn}(\phi_i)\nabla^2\phi_i - \sum_{i\in A_1}\lambda_i\nabla^2\phi_i + \sum_{i\in A_2} w_i\nabla^2\phi_i\right]y > 0.$$

Proof.
Follows immediately from Theorem 1 and the second-order sufficiency conditions for nonlinear programming [7].

Remarks. (1) We note that if x^0 is feasible to (1), then the preceding results characterize local minima of (1), with the additional provisos that w is bounded above and below by 1, and λ is bounded above by 1.

(2) Pietrzykowski [13] showed for all μ sufficiently small, a minimum of (1) is also a minimizer of p (under a linear independence assumption). Luenberger [11] demonstrated that, in the convex case, the threshold value of μ is

$$\frac{1}{\max\{\lambda_i^*, |w_i^*|\}}$$

where λ^* and w^* satisfy (6)

$$\nabla f(x^0) = \sum_{i\in A_1}\lambda_i^*\nabla\phi_i - \sum_{i\in A_2} w_i^*\nabla\phi_i.$$

Charalambous [4, 5] showed that this bound is valid without making convexity or linear independence assumptions, but assuming that x^0 satisfies the second-order sufficiency conditions for nonlinear programming [7] (see also Han and Mangasarian [10]). We note here that this latter result follows trivially from Corollary 3. That is, if μ satisfies (6), and x^0 satisfies second-order sufficiency for (1), then (i), (ii), and (iii) of Corollary 3, hold.

3. Non-minimal first-order points

Let us call x^0 a *stationary point* of p if x^0 satisfies (i) of Corollary 1. If, in addition, x^0 satisfies (ii), we term x^0 a *first-order point* of p.

It is entirely possible, of course, for x^0 to satisfy the first-order conditions whilst not satisfying the second-order requirements. Often, in nonlinear programming one is content to obtain a first-order point. However, if one bears in mind that our interest in minimizing p is to obtain solutions to (1) and that such a point may not be feasible to (1), it is clear that first-order points of p may be totally *unacceptable.*

Often, a reasonable strategy to leave such unacceptable points is to reduce the parameter μ. This reduction gives additional weight to the violated constraints and a descent direction is then usually available. That this is not always the case will be seen below.

Consider the problem

$$\begin{aligned} &\text{Minimize} \quad f(x, y) = x^2 + y^2, \\ &\text{subject to} \quad \phi_1(x, y) \equiv x^2 + y^2 - 2.25 \geq 0, \\ &\qquad\qquad\quad \phi_2(x, y) \equiv x + y - 2 = 0 \end{aligned} \tag{7}$$

with starting point $(x_0, y_0) = (\sqrt{1.125}, \sqrt{1.125})$.

Thus

$$\begin{aligned} &f(x_0, y_0) = 2.25, \\ &\phi_1(x_0, y_0) = 0, \\ &\phi_2(x_0, y_0) > 0, \\ &\nabla f(x_0, y_0) = \nabla\phi_1(x_0, y_0) = (2\sqrt{1.125}, 2\sqrt{1.125})^{\mathrm{T}}, \\ &\nabla\phi_2(x_0, y_0) = (1, 1)^{\mathrm{T}}. \end{aligned}$$

Our penalty function is

$$p(x, y, \mu) = \mu f(x, y) - \min(0, \phi_1(x, y)) + |\phi_2(x, y)|.$$

Let us define a continuously differentiable function

$$p_1(x, y, \mu) = \mu \cdot f(x, y) + \phi_2(x, y).$$

Thus

$$\nabla p_1(x_0, y_0, \mu) = (2\mu x_0 + 1, 2\mu y_0 + 1)^{\mathrm{T}},$$

$$G_{p_1}(x_0, y_0, \mu) = \begin{bmatrix} 2\mu & 0 \\ 0 & 2\mu \end{bmatrix}, \quad \text{and} \quad G_{\phi_1} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

We note that

$$\nabla p_1(x_0, y_0) = \nabla\phi_1(x_0, y_0)\left(\mu + \frac{1}{2x_0}\right)$$

and thus (x_0, y_0) is a stationary point for p. In fact if $0 < \mu \leq 1 - (1/2x_0)$, then (x_0, y_0) is a *first-order* point for p.

Claim 1. *For* $0 < \mu < 1 - (1/2x_0)$, $\exists$ *a descent direction for* p *at* (x_0, y_0).

Proof. Any direction d in $\mathbf{R}^2$ can be written as $d = d^1 + d^2$, where $(d^1)^{\mathrm{T}}\nabla\phi_1(x_0, y_0) = 0$, $d^2 = \beta\nabla\phi_1(x_0, y_0)$, $\beta \in \mathbf{R}$. Thus, if we define $\lambda = \mu + (1/2x_0)$,

$$\begin{aligned} p(x_0 + \alpha d_1, y_0 + \alpha d_2) = {} & p_1(x_0, y_0) + \alpha[\beta\lambda - \min(0, \beta)]\|\nabla\phi_1(x_0, y_0)\|^2 \\ & + \tfrac{1}{2}\alpha^2 d^{\mathrm{T}}\left\{\begin{bmatrix} 2\mu & 0 \\ 0 & 2\mu \end{bmatrix} - \min(0, \mathrm{sgn}(\beta))\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right\}d, \end{aligned}$$

for positive α sufficiently small.

Now, using the fact that $0 < \lambda < 1$, and considering the three cases $\beta > 0$, $\beta = 0$, $\beta < 0$, separately, it is easy to see that d is an ascent direction.

Claim 2. *For all positive μ, (x_0, y_0) is not a local minimizer of p.*

Proof. Clearly, along the curve $x^2 + y^2 - 2.25 = 0$, f and ϕ_1 remain constant. However, ϕ_2 decreases as we move from (x_0, y_0) along this curve and thus $p(x, y, \mu)$ decreases for any $\mu > 0$.

Therefore (x_0, y_0) is a point such that $p(x_0, y_0, \mu)$ has *no direction of descent* for $\mu < 1 - (1/2x_0)$ ($\simeq 0.52856$) and yet (x_0, y_0) is not a minimizer of p, for any positive μ. We emphasize that although (x_0, y_0) is a first-order point for p, it is not an acceptable solution to the original constrained problem since it is *not feasible.*

Therefore we have constructed a simple problem exhibiting a point from which p can be decreased only by following a curved path. Furthermore, it is not unlikely that a p-descent algorithm will converge to such a point x^0, due to the fact that x^0 is a *first-order point* for p. For example, the projection algorithm of Conn and Pietrzykowski [6] will *always* converge to $(\sqrt{1.125}, \sqrt{1.125})$ in the above example if the method is started at any point satisfying $y = x$.

4. Concluding remarks

Since it has become popular to use the exact penalty function (2) in nonlinear optimization procedures, the simple optimality characterizations given here should prove useful in future algorithmic development. In particular, the second-order conditions should aid in the design of second-order methods to minimize p (see [3]). In addition, the similarity between the optimality conditions for p and those for nonlinear programming reinforce the idea that the two problems are closely related. This is further justification for the design of nonlinear programming methods based on the minimization of p.

The example given in Section 3 demonstrates, however, that even if second-order information is available, it is not always possible to find a p-descent direction. This suggests that in some cases, either a nonlinear step, or a perturbation might be necessary.

The function p is a piece-wise differentiable function. As such, optimality characterizations are closely related to those for other piece-wise functions. In particular, these results relate to those given in [1, 2, 5, 8] concerning optimization in polyhedral norms.

Acknowledgment

We thank Jorge Moré for his careful reading and helpful remarks.

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