Optimal Execution Under Jump Models For Uncertain Price Impact*†

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Abstract

In the execution cost problem, an investor wants to minimize the total expected cost and risk in the execution of a portfolio of risky assets to achieve desired positions. A major source of the execution cost comes from price impacts of both the investor's own trades and other concurrent institutional trades. Indeed price impact of large trades have been considered as one of the main reasons for fat tails of the short term return's probability distribution function. However, current models in the literature on the execution cost problem typically assume normal distributions. This assumption fails to capture the characteristics of tail distributions due to institutional trades. In this paper we provide arguments that compound jump diffusion processes naturally model uncertain price impact of other large trades. This jump diffusion model includes two compound Poisson processes where random jump amplitudes capture uncertain permanent price impact of other large buy and sell trades. Using stochastic dynamic programming, we derive analytical solutions for minimizing the expected execution cost under discrete jump diffusion models. Our results indicate that, when the expected market price change is nonzero, likely due to large trades, assumptions on the market price model, and values of mean and covariance of the market price change can have significant impact on the optimal execution strategy. Using simulations, we computationally illustrate minimum CVaR execution strategies under different models. Furthermore, we analyze qualitative and quantitative differences of the expected execution cost and risk between optimal execution strategies, determined under a multiplicative jump diffusion model and an additive jump diffusion model.

Keywords: uncertain price impact, execution cost problem, stochastic dynamic programming, jump diffusion models

1 Introduction

Investment performance is substantially related to the execution cost (Yang and Borkovec (2005)), which is the difference in the value between an ideal trade and the actual implementation (Almgren (2008)). The

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execution cost is comprised of *explicit costs*, such as commissions, and *implicit costs*, which are more difficult to characterize. Implicit costs are mainly due to price impact of trading by investors, and can be quite significant in large trades. As a result controlling execution costs becomes crucial for institutional traders, whose trades often comprise a large fraction of the average daily volume. To decrease price impact, trades are typically broken up into smaller packages (Chan and Lakonishok (1995)), which are executed over a short time horizon. Such a sequence of trades is called an *execution strategy*. However, the size of each package nonetheless remains large enough to induce a significant price change (Gabaix et al. (2006)).

Two distinctive types of price impact, the *permanent* price impact and the *temporary* price impact, are considered in the literature, see, e.g., (Holthausen et al., 1987; Barclay and Litzenberger, 1988; Holthausen et al., 1990; Barclay and Warner, 1993; Chan and Lakonishok, 1993, 1995; Almgren and Chriss, 2000/2001; Moazeni et al., 2010). The temporary price impact mainly comes from the liquidity cost, i.e., an additional price an investor pays for immediate execution of the trade (Focardi and Fabozzi (2004)); it affects the execution price at the moment of trading. In contrast, the permanent price impact moves the future market price, due to the imbalance between supply and demand and the information transmitted to the market.

Price impact is often modeled as a function of the trading rate, see, e.g., (Almgren and Chriss, 2000/2001). *temporary* and *permanent price impact functions*, along with the market price dynamics, determine the execution cost of each execution strategy. Given the market price model and price impact functions, an investor (decision maker) wants to determine an *execution strategy* to minimize the expected execution cost and possibly some measure of risk.

There is a large body of literature on the execution cost problem, see, e.g., (Huberman and Stanzl, 2004) and references therein, each of which deals with a particular model for price dynamics. In (Almgren and Chriss, 2000/2001), a static execution strategy is determined to minimize the mean and variance of the execution cost when market price evolves according to a Brownian diffusion process. Moazeni et al. (2010) study the sensitivity of this static optimal execution strategy to the change in the parameters of linear price impact functions under the same setting; an upper bound for the change of the execution strategy is established mathematically.

The typical assumption of a continuous or discrete Brownian motion market price model for the execution cost problem is questionable. In particular it fails to capture the impact of large trades from other institutions concurring during the course of the execution. Analogous to the fact that one's own large trade causes a discrete market price change, a large trade from others also induces a permanent (uncertain) price impact on the market price. These uncertain permanent price impact of other large trades should however be modeled appropriately in the market price dynamics when seeking an optimal execution strategy and evaluating the risk associated with an execution strategy. Unfortunately current quantitative analysis of the execution cost does not explicitly model this source of price depression; only the permanent price impact of the decision maker's own trade is explicitly considered. Indeed, the normal distribution assumption contradicts the well recognized empirical evidence that the short term (a day or less) asset return probability distribution function typically has fat tails, see, e.g., (Campbell et al., 1996; Pagan, 1996; Cont, 2001).

There are relatively few studies on the execution cost problem under a model which accounts for price impact of other concurrent large trades. Carlin et al. (2007) develop a repeated game of complete information to model repeated interaction of price impact of large investors who attempt to minimize the expected execution cost. This model however relies on the assumptions that participants are strategic, and their trading strategies and their overall trading target are common knowledge. In (Almgren and Lorenz, 2006), a Bayesian approach is proposed to introduce information on other large trades based on the observed price. This approach implicitly assumes that traders use VWAP-like strategies rather than the arrival price so that their trading is not front-loaded. In addition, the market price is still modeled through normal distributions. Thus risk assessment under this model, particularly the tail risk, is likely to be inaccurate. Note that, in both studies, no risk consideration is given in devising an optimal execution strategy. In Alfonsi et al. (2008) and Alfonsi et al. (2010), optimal execution strategies in order books are considered; the authors also mentioned, without any explicit discussion, that perhaps jump models for the market price should be considered.

In this paper, we make no assumption about the decision maker's knowledge of other institutions' trading targets or their execution strategies. Thus, arrivals and price impact of other large trades are uncertain. We investigate reasonable models for this uncertainty and their effect on the optimal execution strategy and execution risk. The main contributions of this paper include the following:

- Following the methodology in market microstructure theory in which uncertainty in order arrivals over time are modeled by Poisson processes, e.g., see (Garman, 1976), we explicitly model uncertain permanent price impact of other large trades using compound Poisson processes. Jump events, in this model, represent uncertain arrivals of other large trades and random jump amplitudes represent their uncertain permanent price impact. In the proposed model the market price evolution is defined by the summation of a continuous diffusion process (for "normal" trades) and two compound Poisson processes for permanent price impact of large buy and sell trades. Our proposed model accounts for discrete large changes in the market price to better capture the fat tails in the probability distribution of the price due to concurrent large trades by other institutions.

- Since the first concern in portfolio execution is the expected cost, we derive explicit formulae for optimal execution strategies to minimize the expected execution cost (*optimal risk neutral execution strategies*), under discrete additive jump diffusion models as well as multiplicative jump diffusion models with linear price impact functions. The additive diffusion model *without jumps* has been used previously in the literature (e.g., see Almgren and Chriss (2000/2001)). Since the stock price is typically modeled by a multiplicative model, we also consider multiplicative models with jumps. We analyze implications of model assumptions on the optimal execution strategies, execution cost, and execution risk. In addition we apply a computational method to determine the optimal execution strategy which minimizes the CVaR of the execution cost, assuming a strategy is deterministic.

- We compare the execution cost distribution and risk values for the optimal risk neutral execution strategy under a mean and volatility-adjusted diffusion model and the jump diffusion model. We illustrate that for quantitative assessment of risk, model assumption can make a significant difference, particularly with respect to the assessment of extreme risk. Therefore, using an appropriate model is crucial in evaluating the risk exposure associated with an execution strategy, even for a risk neutral investor who seeks a strategy which solely minimizes the expected execution cost. Furthermore, when a risk measure such as CVaR is minimized, the optimal solutions under the two models are different and the execution risk can be underestimated by a Brownian diffusion process with no jump.

Our theoretical and computational investigation also establishes the following result and observations. Firstly, under an additive diffusion market price model and with linear price impact functions, it has been noted that (e.g., see Bertsimas and Lo (1998)), when the expected market price change is zero, the optimal risk neutral execution strategy is the naive strategy of trading an equal amount in each period. We generalize this result by proving that, when the expected market price change aside from the permanent price impact of the decision maker's own trade is zero, the optimal risk neutral execution strategy derived from stochastic dynamic programming is always static, unrelated to the specification of the market price evolution. Moreover, for stationary linear price impact functions this static strategy is reduced to the naive strategy. *Unless otherwise stated explicitly, in this paper, we simply refer to the expected market price change aside from the permanent price impact of the decision maker's own trade as the expected market price change.*

Secondly, when the expected market price change is nonzero, specification of the market price evolution matters and the optimal execution strategy derived under each model can be significantly different from the naive strategy. The optimal risk neutral execution strategy obtained under the additive jump diffusion model is static and independent of the asset price volatility. In contrast, the optimal risk neutral execution strategy under the multiplicative jump diffusion model is dynamic and depends on the market price realization. Hence, this execution strategy adjusts the trading size according to the trading impact of other investors realized during the previous periods. In addition, the optimal risk neutral execution strategy under the multiplicative jump diffusion model depends on the covariance matrix.

Finally, we investigate the degree if suboptimality of both the naive strategy and the optimal risk neutral execution strategy under the additive jump diffusion model in terms of the expected execution cost. We observe that the expected execution cost associated with the optimal risk neutral execution strategy obtained under the multiplicative jump model can be significantly less than the expected execution cost of the naive strategy. Moreover, its expected execution cost can be notably smaller than that of the execution strategy optimal under the additive jump model with comparable expected market price change and volatility. This is particularly true as the asset return volatility or the trading horizon increases.

The paper is organized as follows. In §2, we motivate and describe the proposed jump process to capture uncertain permanent price impact of other large institutions. We present the mathematical formulation for the execution cost problem in §3. In this section, we also provide closed-form expressions for the optimal execution strategies under an additive jump diffusion model and a multiplicative jump diffusion model. The computational method to minimize the CVaR of the execution cost in described in §4. In §5, simulations are carried out to compare different execution strategies and model assumptions in terms of the expected execution cost and risk assessment. Concluding remarks are presented in §6.

2 Jump Processes for Uncertain Price Impact of Large Trades

In this paper, similar to (Bertsimas and Lo, 1998; Almgren and Chriss, 2000/2001; Huberman and Stanzl, 2004), our presentation mainly follows the discrete time framework since the analytic formula for optimal risk neutral execution strategy is presented under a discrete time model. We also analyze the execution risk in discrete time setting. We note that the continuous time optimal execution problem for the single asset has also been widely studied, see, e.g., (Forsyth, 2010). The rational for the jump process can also be appreciated in contrasted to a continuous time Brownian model.

Without loss of generality, we assume that an investor (decision maker) plans to liquidate his holdings in m assets during N periods in the time horizon T. Let $t_0 = 0 < t_1 < \cdots < t_N = T$, where $\tau \stackrel{\text{def}}{=} t_k - t_{k-1} = \frac{T}{N}$ for $k = 1, 2, \ldots, N$. The decision maker's position at time t_k is denoted by the m-vector $x_k = (x_{1k}, x_{2k}, \ldots, x_{mk})^T$, where x_{ik} is the decision maker's holding position in the number of units in the ith asset at time t_k, We assume that the decision maker's initial position is $x_0 = \bar{S}$ shares and final position is $x_N = 0$. The difference between positions at two consecutive times t_{k-1} and t_k is denoted by an m-vector n_k, where

$$n_k = x_{k-1} - x_k, \qquad k = 1, 2, \ldots, N. \tag{2.1}$$

Negative n_{ik} implies that the ith asset is bought between t_{k-1} and t_k. We refer to a sequence $\{n_k\}_{k=1}^N$ satisfying $\sum_{k=1}^N n_k = \bar{S}$ as an *execution strategy.*

Let the m-vector P_k denote the unit market price at time t_k. The deterministic initial market price is denoted by P_0. Similar to (Almgren and Chriss, 2000/2001), we assume that the permanent price impact of the decision maker's trade is a deterministic function $g(\cdot)$ of the trading rate:

$$P_k = \mathcal{F}_{k-1}(P_{k-1}) - \tau g\left(\frac{n_k}{\tau}\right), \quad k = 1, 2, \ldots, N-1, \tag{2.2}$$

where $\mathcal{F}_{k-1}(P_{k-1})$ denotes the market price at time t_k when the decision maker does not trade in $(t_{k-1}, t_k]$. Similar to (Bertsimas and Lo, 1998; Almgren and Chriss, 2000/2001; Huberman and Stanzl, 2004), the form of the permanent impact function suggests it as a function of trading amount in a period. Further discussion on properties of the permanent impact function $g(\cdot)$ can be found in Huberman and Stanzl (2004). For optimal execution in the continuous framework, the permanent impact function may need to be a function of trading rate.

For the execution cost problem, the random variable $\mathcal{F}_{k-1}(P_{k-1})$ is typically characterized by a normal random variable corresponding to an increment of a Brownian motion process. When the expected market price change is zero, the optimal risk neutral execution strategy is the naive strategy under many market price dynamics (e.g., see Moazeni et al. (2010), Bertsimas and Lo (1998), Huberman and Stanzl (2005)). This observation may suggest that one needs not be concerned with the specification of the market price dynamics (or price impact functions). However, secondary to the expected execution cost, the risk of the execution cost is another main concern for investors. Accurate assessment of the execution risk associated with an execution strategy needs an accurate model for the market price. In addition, based on 15 minutes returns of 1000 largest U.S. assets in several international indices, Gabaix et al. (2006) show that trades of large institutions cause nonzero expected short term market price changes.

Furthermore, empirical evidence indicates that the distribution of the short term asset return typically has fat tails, see, e.g., (Campbell et al., 1996; Pagan, 1996; Cont, 2001). One likely reason for the fat tail distribution is the price impact of trades from institutions. Gabaix et al. (2006) show that trades of large institutions generate excess asset price volatility.

There is an additional contradiction in modeling market price dynamics as a Brownian motion; this contradiction can be better seen in the context of a continuous time framework. When the market price is modeled by a continuous process, permanent price impact of the decision maker's own trade causes a discrete change in the market price while the impact of large trades from other institutions maintains price continuity.

In this paper we assume that the arrival time of large trades from other institutions as well as their impact are unknown to the decision maker. Following the approach proposed in Garman (1976), we model these uncertain arrivals using Poisson processes with constant arrival rates. The arrival of each trade induces an unknown permanent price impact and causes a jump in the market price. We use a random jump size to model the uncertain impact; the jump size is assumed to follow a known distribution. Combining this with uncertain arrivals, the uncertain price impact of uncertain trades from other institutions are modeled by *compound Poisson processes*. Including compound Poisson processes in the market price dynamics yields a price distribution with fatter tails than that of a normal distribution. The proposed model is likely to be a more accurate representation for the trading activities of institutional investors.

To further distinguish buys from sells, we assume that arrivals of buy and sell trades are independent Poisson processes with deterministic arrival rates. For simplicity, we first consider a single asset trading, and then generalize the model to trading of multiple assets. Let $\{X_t : t \in [0,T]\}$ be a Poisson process in the execution horizon $[0,T]$ with a constant arrival rate $\lambda_x \geq 0$. The process $\{X_t\}$ models uncertain arrivals of sell trades from other institutions. Similarly, a Poisson process $\{Y_t : t \in [0,T]\}$ with a constant arrival rate $\lambda_y \geq 0$ represents arrivals of buy trades. Processes $\{X_t\}$ and $\{Y_t\}$, respectively, count the number of sell and buy events during the time period $[0,t)$. Initially $X_0 = 0$ and $Y_0 = 0$. We assume that processes $\{X_t\}$ and $\{Y_t\}$ are independent of each other.

Using the Poisson processes $\{X_t\}$ and $\{Y_t\}$, we model uncertain permanent price impact of trades by other institutions in $[t_{k-1}, t_k)$ as below:

$$\mathcal{J}(k) \stackrel{\text{def}}{=} \sum_{\ell=1}^{Y_{t_k}-Y_{t_{k-1}}} \chi_\ell(k) - \sum_{\ell=1}^{X_{t_k}-X_{t_{k-1}}} \pi_\ell(k), \tag{2.3}$$

where $\chi_\ell(k)$ and $\pi_\ell(k)$ are random variables with known distributions. When the upper limit of a summation in (2.3) is zero, the summation itself is zero.

For every period k, the random variable $\pi_\ell(k)$ represents the permanent price impact of the ℓth sell trade in $[t_{k-1}, t_k)$. We assume that the random variables $\{\pi_\ell(k)\}$ are independently distributed with the mean $\mu_x(k)$ and standard deviation $\sigma_x(k)$. Similarly, the random variable $\chi_\ell(k)$ captures the permanent price impact of the ℓth buy trade in period k. The random variables $\{\chi_\ell(k)\}$ are assumed to be independently distributed with mean and standard deviation $\mu_y(k)$ and $\sigma_y(k)$, respectively.

Using two separate compound Poisson processes in equation (2.3) provides the flexibility to choose different arrival rates and distributional characteristics for permanent price impact of buys and sells from other institutions. Distinguishing permanent price impact of sell trades and buy trades by their arrival rates or distributions for the jump sizes is similar to the *double jump diffusion* process for modeling asset price dynamics (e.g., see Ramezani and Zeng (2007) and references therein). Furthermore, empirical studies on institutional trades indicate that market reacts differently to buy and sell orders: buys have larger permanent price impact than sells (e.g., see Saar (2001) and references therein). Employing two compound Poisson processes allows us to set $\mu_y(k) \geq \mu_x(k)$ to capture this market behavior.

The proposed jump diffusion model can be extended to a portfolio of m assets. For each asset $i = 1, 2, \ldots, m$, we similarly define two independent Poisson processes $\{X_t^{(i)}\}$ and $\{Y_t^{(i)}\}$ with constant arrival rates $\lambda_x^{(i)}$ and $\lambda_y^{(i)}$, respectively. In this case, $\mathcal{J}(k)$ is the m-vector

$$\mathcal{J}(k) \stackrel{\text{def}}{=} \left(\sum_{\ell=1}^{Y_{t_k}^{(1)} - Y_{t_{k-1}}^{(1)}} \chi_\ell^{(1)}(k) - \sum_{\ell=1}^{X_{t_k}^{(1)} - X_{t_{k-1}}^{(1)}} \pi_\ell^{(1)}(k) \;,\; \ldots \;,\; \sum_{\ell=1}^{Y_{t_k}^{(m)} - Y_{t_{k-1}}^{(m)}} \chi_\ell^{(m)}(k) - \sum_{\ell=1}^{X_{t_k}^{(m)} - X_{t_{k-1}}^{(m)}} \pi_\ell^{(m)}(k) \right)^T . \quad (2.4)$$

We further note that, if necessary, the compound Poisson processes of different assets can be allowed to include correlations to capture cross-asset relations observed.

For simplicity we assume subsequently that, for every period k, random jump sizes for sell trades at period k are independent of random jump sizes for buy trades at period k. In addition, we assume that the jump amplitudes are independent of the Poisson processes, and the compound Poisson processes are independent of the Brownian motion process used to model *normal* market price changes.

Below we incorporate jumps in two specifications for the market price dynamics, namely, additive model and multiplicative model. The additive diffusion process has been used frequently in the literature on the execution cost problem (e.g., see Almgren and Chriss (2000/2001)); this is mainly due to the simplicity of the additive model which leads to determination of the optimal execution strategy in the early literature. In practice, a multiplicative model is more accurate in modeling the stock price and it has been more widely adopted in the finance literature for asset price modeling.

Additive Jump Diffusion Models. Here we assume that the change in the market price comes from a Brownian increment and a jump $\mathcal{J}^{\mathbf{a}}(k)$, which represents permanent price impact of other large trades:

$$\mathcal{F}_{k-1}(P_{k-1}) = P_{k-1} + \tau^{1/2}\Sigma^{\mathbf{a}} Z_k + \tau\alpha_0^{\mathbf{a}} + \mathcal{J}^{\mathbf{a}}(k). \quad (2.5)$$

The m-vector $\tau\alpha_0^{\mathbf{a}}$ can be interpreted as the expected price change due to small trades, which is likely to be negligible. The random vector Z_k is an l-vector of independent standard normals, and $\Sigma^{\mathbf{a}}$ is an $m \times l$ volatility matrix of the asset price changes. Based on high frequency financial price data, it has been noted in McCulloch and Tsay (2001) that significant percentages of trades lead to no price change. Similarly, we decompose the market price change into random shocks which lead to no expected price change, and jump events that cause a nonzero expected price change. Notice that we have used the superscript $\mathbf{a}$ to distinguish the model parameters in the *additive* model (2.5) from those of *multiplicative* model subsequently presented. Throughout this paper bold superscripts of matrices and vectors should not be considered as exponents.

Together with the price impact of the decision maker's own trade, the market price dynamics is:

$$P_k = P_{k-1} + \tau^{1/2}\Sigma^{\mathbf{a}} Z_k + \tau\alpha_0^{\mathbf{a}} + \mathcal{J}^{\mathbf{a}}(k) - \tau g\left(\frac{n_k}{\tau}\right), \qquad \text{where} \quad (2.6)$$

$$\mathcal{J}^{\mathbf{a}}(k) = \sum_{j=1}^{Y_{t_k} - Y_{t_{k-1}}} \chi_j^{\mathbf{a}}(k) - \sum_{j=1}^{X_{t_k} - X_{t_{k-1}}} \pi_j^{\mathbf{a}}(k), \quad \text{for } k = 1, 2, \ldots, N.$$

We use $\mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k)$ and $\mathbf{Cov}_{\mathcal{J}}^{\mathbf{a}}(k)$ to refer to $\mathbf{E}(\mathcal{J}^{\mathbf{a}}(k))$ and $\mathbf{Cov}(\mathcal{J}^{\mathbf{a}}(k))$, respectively. In the additive market price dynamics (2.6), the total market price change is decomposed into two components, one due to small trades, captured by $\tau\alpha_0^{\mathbf{a}} + \tau^{1/2}\Sigma^{\mathbf{a}} Z_k$, and the other due to the permanent price impact of large trades, modeled by $\mathcal{J}^{\mathbf{a}}(k)$. Whence, the total expected market price change in each trading interval becomes $\tau\alpha_0^{\mathbf{a}} + \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k)$. Since Z_k and $\mathcal{J}^{\mathbf{a}}(k)$ are assumed to be independent, the covariance of the total market price change in the kth period equals $\Sigma^{\mathbf{a}}(\Sigma^{\mathbf{a}})^T + \mathbf{Cov}_{\mathcal{J}}^{\mathbf{a}}(k)$.

Multiplicative Jump Diffusion Models. In practice, one often explicitly models return rather than price change; here we incorporate jump in such a model. Let the market return, aside from the permanent price impact of the decision maker's trades, be characterized by a normal distribution plus uncertain permanent price impact of other large trades. In the single asset trading context, this corresponds to

$$\frac{\mathcal{F}_{k-1}(P_{k-1}) - P_{k-1}}{P_{k-1}} = \tau\alpha_0^{\mathbf{m}} + \tau^{1/2}\Sigma^{\mathbf{m}} Z_k + \mathcal{J}^{\mathbf{m}}(k),$$

or equivalently

$$\mathcal{F}_{k-1}(P_{k-1}) = P_{k-1}\left(1 + \tau\alpha_0^{\mathbf{m}} + \tau^{1/2}\Sigma^{\mathbf{m}} Z_k + \mathcal{J}^{\mathbf{m}}(k)\right). \tag{2.7}$$

Similarly, the multiplicative jump diffusion model for m assets, together with the price impact of the decision maker's own trade, can be described as below:

$$P_k = \mathrm{Diag}(P_{k-1})\left(e + \tau\alpha_0^{\mathbf{m}} + \tau^{1/2}\Sigma^{\mathbf{m}} Z_k + \mathcal{J}^{\mathbf{m}}(k)\right) - \tau g\left(\frac{n_k}{\tau}\right), \quad \text{where} \tag{2.8}$$

$$\mathcal{J}^{\mathbf{m}}(k) \stackrel{\text{def}}{=} \sum_{j=1}^{Y_{t_k} - Y_{t_{k-1}}} \left(\chi_j^{\mathbf{m}}(k) - e\right) - \sum_{j=1}^{X_{t_k} - X_{t_{k-1}}} \left(\pi_j^{\mathbf{m}}(k) - e\right). \tag{2.9}$$

Here, e is the m-vector of all ones and $\mathrm{Diag}(P_{k-1})$ is a diagonal matrix with the m-vector P_{k-1} as its diagonal. The components of the l-vector Z_k are independent standard normals and $\Sigma^{\mathbf{m}}$ is an $m \times l$ volatility matrix of the asset returns. The term $\tau\alpha_0^{\mathbf{m}}$ can be interpreted as the expected return due to small trades. Here, the superscript $\mathbf{m}$ emphasizes parameters in the multiplicative jump model. Jump amplitudes $\pi_j^{\mathbf{m}}(k)$ and $\chi_j^{\mathbf{m}}(k)$ represent uncertain permanent price impacts, and are assumed to be drawn from known distributions. We denote the expected value and covariance matrix of $\mathcal{J}^{\mathbf{m}}(k)$ with $\mathbf{E}_{\mathcal{J}}^{\mathbf{m}}(k)$ and $\mathbf{Cov}_{\mathcal{J}}^{\mathbf{m}}(k)$, respectively.

3 Optimal Execution Strategies

In addition to permanent impact, the decision maker's trade also induces a temporary price impact on the execution price. We assume that the m-vector unit execution price $\tilde{P}_k$ is given by

$$\tilde{P}_k = P_{k-1} - h\left(\frac{n_k}{\tau}\right), \quad k = 1, 2, \ldots, N, \tag{3.1}$$

where $h(\cdot)$ is the given temporary impact function.

Linear price impact functions have been well-studied in the market microstructure literature, e.g., see (Bertsimas and Lo, 1998; Bertsimas et al., 1999; Almgren and Chriss, 2000/2001; Huberman and Stanzl, 2004). In this paper, we mostly focus on linear price impact functions which are defined by the temporary impact matrix H and the permanent impact matrix G, as below:

$$g(v) = Gv, \qquad h(v) = Hv, \tag{3.2}$$

where $v = \frac{n}{\tau}$ is the trading rate. These impact matrices H and G are the expected price depressions caused by trading assets at a unit rate.

Given an execution strategy $\{n_k\}_{k=1}^N$, the total amount received at the end of the time horizon T is $\sum_{k=1}^N n_k^T \tilde{P}_k$. The difference between this quantity and the value of an ideal benchmark trade is the *execution* *cost* (Almgren (2008)). The benchmark is commonly taken as the value of the portfolio at the arrival price P_0. Hence, the *execution cost* associated with the strategy $\{n_k\}_{k=1}^N$ is defined as $P_0^T\bar{S} - \sum_{k=1}^N n_k^T \tilde{P}_k$. The main objective of the decision maker is to minimize the expected execution cost. In addition the decision maker is concerned with the uncertainty in the total amount that he receives from the trade implementation. Hence the execution cost problem in the generic form can be described as follows:

$$\min_{n_1,\cdots,n_N\in\mathbb{R}^m} \mathbf{E}\left(P_0^T\bar{S} - \sum_{k=1}^N n_k^T\tilde{P}_k\right) + c\cdot\rho\left(P_0^T\bar{S} - \sum_{k=1}^N n_k^T\tilde{P}_k\right) \quad \text{s.t.} \quad \sum_{k=1}^N n_k = \bar{S}, \tag{3.3}$$

where $\rho(\cdot)$ is a risk measure of the execution cost and $c \geq 0$ is a risk aversion parameter. The inequality constraints $n_k \geq 0$ can also be included in (3.3) to rule out buying in a sell execution.

We first consider here the optimal risk neutral execution strategy when purchasing is allowed, i.e.,

$$\min_{n_1,\cdots,n_N} \mathbf{E}\left(P_0^T\bar{S} - \sum_{k=1}^N n_k^T\tilde{P}_k\right) \quad \text{s.t.} \quad \sum_{k=1}^N n_k = \bar{S}. \tag{3.4}$$

We will also analyze properties of the optimal risk neutral execution strategy in terms of both the expected execution cost and execution risk.

Stochastic dynamic programming has been used to determine the optimal execution strategy when the market price evolves according to a Brownian motion (e.g., see Bertsimas and Lo (1998), Bertsimas et al. (1999), Huberman and Stanzl (2005)). The key ingredients of the stochastic dynamic programming for Problem (3.4) are described below.

Let the optimal-value function at t_{k-1} corresponding to Problem (3.4) be

$$V_k^*(P_{k-1}, x_{k-1}) = \min_{n_k,\cdots,n_N} \mathbf{E}\left(P_0^T\bar{S} - \sum_{j=k}^N n_j^T\tilde{P}_j \;\middle|\; P_{k-1},\ x_{k-1}\right), \quad \text{s.t.} \quad \sum_{j=k}^N n_j = x_{k-1}.$$

Here, $n_k, \cdots, n_N$ are over the set of $\mathbb{R}^m$-valued functions of the system state, namely current asset holdings x_{k-1} and current market price P_{k-1}.

For $k = N$, $n_N^* = x_{N-1}$ since there is no choice but to execute the entire remaining order x_{N-1}. Whence, for the model (3.1), the optimal-value function for the last period becomes

$$\begin{aligned} V_N^*(P_{N-1}, x_{N-1}) &= \min_{n_N} \mathbf{E}\left(P_0^T\bar{S} - n_N^T\tilde{P}_N \mid P_{N-1},\ x_{N-1}\right) \quad \text{s.t} \quad x_{N-1} - n_N = 0 \\ &= P_0^T\bar{S} - x_{N-1}^T\left(P_{N-1} - h\left(\frac{x_{N-1}}{\tau}\right)\right). \end{aligned} \tag{3.5}$$

For the linear temporary price impact function (3.2), we have

$$V_N^*(P_{N-1}, x_{N-1}) = P_0^T\bar{S} - x_{N-1}^T P_{N-1} + \frac{1}{2}x_{N-1}^T\frac{H+H^T}{\tau}x_{N-1}. \tag{3.6}$$

Now assume that n_{k+1}^* and $V_{k+1}^*(P_k, x_k)$ have been determined. The optimal execution n_k^* and the optimal-value function $V_k^*(P_{k-1}, x_{k-1})$ can be determined from the Bellman's principle of optimality which relates, recursively backwards in time, the optimal-value function in period k to the optimal-value function in period $k+1$:

$$V_k^*(P_{k-1}, x_{k-1}) = \min_{n_k} \mathbf{E}\left(-n_k^T\tilde{P}_k + V_{k+1}^*(P_k, x_k) \mid P_{k-1},\ x_{k-1}\right).$$

Next we present the optimal risk neutral execution strategies under three different model assumptions: when the expected market price change is zero, additive jump diffusion models, as well as multiplicative jump diffusion models. associated execution cost distribution. For given impact matrices H and G, we define the *combined impact matrix* Θ below:

$$\Theta \stackrel{\text{def}}{=} \frac{1}{\tau}\left(H + H^T\right) - G, \tag{3.7}$$

which will be used in the subsequent expressions for optimal execution strategies.

3.1 Effect of a Zero Expected Market Price Change

An optimal execution strategy in general depends on the market price dynamics, i.e., $\mathcal{F}_k(\cdot)$ in (2.2). For a single asset execution under an additive diffusion model with zero expected market price change, the optimal risk neutral execution strategy is the *naive strategy* $\bar{n}$ of liquidating an equal amount in each period, i.e.,

$$\bar{n}_k = \frac{\bar{S}}{N}, \qquad k = 1, 2, \ldots, N, \tag{3.8}$$

see, e.g., (Almgren and Chriss, 2000/2001; Bertsimas and Lo, 1998; Bertsimas et al., 1999; Moazeni et al., 2010). The assumption that the expected market price change is zero may be reasonable in the absence of large institutional trades.

We now generalize this result to more general model assumptions for the portfolio case in Theorem 3.1.

Theorem 3.1. *Let the market price dynamics and the execution price model be given by equations* (2.2) *and* (3.1)*, respectively. In addition, assume that*

$$\mathbf{E}\left(\mathcal{F}_{k-1}\left(P_{k-1}\right) \mid P_{k-1}\right) = P_{k-1}, \; k = 1, 2, \cdots, N-1. \tag{3.9}$$

Assume further that the price impact functions $g(\cdot)$ and $h(\cdot)$ are deterministic functions of the trading rate $\frac{n_k}{\tau}$ and do not depend on the market prices. Then the unique optimal risk neutral execution strategy for the execution cost problem (3.4)*, when it exists, is static (state independent). Furthermore, for the linear price impact functions* (3.2) *with constant impact matrices, symmetric permanent impact matrix G, and positive definite combined impact matrix Θ defined in (3.7), the optimal risk neutral execution strategy $\{n_k^*\}_{k=1}^N$ is the naive strategy.*

This result highlights the important role of the expected market price change in the optimal execution strategy. Note that the results hold without specific assumption on the market price dynamics $\mathcal{F}_k(\cdot)$. The proof of Theorem 3.1 is provided in Appendix A.

In general, the expected market price change in each period is nonzero, likely due to institutional trades. We will show that, in this case, the model assumptions and the expected market price change can significantly affect the optimal execution strategy.

In §3.2 and §3.3, we focus on two specifications of the market price model (2.2) that include the jump process $\mathcal{J}(k)$.

3.2 Additive Jump Diffusion Market Price Models

We now present a closed-form expression for the optimal risk neutral execution strategy with respect to the additive jump diffusion model (2.6).

Theorem 3.2. *Assume that the $m \times m$ symmetric matrices $\{A_k\}_{k=1}^N$, specified by the following recursive equation:*

$$A_k = A_{k+1} - (A_{k+1} - \Theta^T) A_{k+1}^{-1} (A_{k+1} - \Theta^T)^T, \quad k = 1, 2, \ldots, N-1, \tag{3.10}$$

with $A_N = \Theta^T + \Theta$*, are positive definite. Moreover, let* m*-vectors* $\{b_k\}_{k=1}^N$ *and scalars* $\{c_k\}_{k=1}^N$ *be defined as follows:*

$$b_k = b_{k+1} + (\Theta^T - A_{k+1})A_{k+1}^{-1}\left(b_{k+1} - \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k+1) + \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k)\right) - \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k+1) + 2\mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k) + \tau\alpha_0^{\mathbf{a}}, \quad (3.11)$$

$$c_k = c_{k+1} + \frac{1}{2}\left(b_{k+1} - \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k+1) + \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k)\right)^T A_{k+1}^{-1}\left(b_{k+1} - \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k+1) + \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k)\right),$$

with $b_N = \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(N) + \tau\alpha_0^{\mathbf{a}}$ *and* $c_N = 0$*. Then the unique optimal risk neutral execution strategy* $n^* = \{n_k^*\}_{k=1}^N$ *of Problem (3.4) under the additive jump model (2.6) is:*

$$n_k^* = -A_{k+1}^{-1}\left(b_{k+1} - \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k+1) + \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k) + (\Theta^T - A_{k+1})^T x_{k-1}^*\right), \quad k = 1, 2, \ldots, N-1, \quad (3.12)$$

$$n_N^* = \bar{S} - \sum_{k=1}^{N-1} n_k^*,$$

where $x_0^* = \bar{S}$ *and* $x_k^* = x_{k-1}^* - n_k^*$ *for* $k = 1, 2, \ldots, N-2$*. Furthermore, the optimal expected execution cost equals:*

$$V_1^*(P_0, x_0) = P_0^T \bar{S} - \frac{1}{2}\bar{S}^T(\Theta^T - A_1 - G)\bar{S} - \left(P_0 + b_1 - \mathbf{E}_{\mathcal{J}}^{a}(1) - \tau\alpha_0^{\mathbf{a}}\right)^T \bar{S} - c_1.$$

A proof for Theorem 3.2 is given in Appendix B.

Theorem 3.2 indicates that the optimal risk neutral execution strategy under the additive model (2.6) does not depend on the market price realization. In addition, volatility $\Sigma^{\mathbf{a}}$ and covariance $\mathbf{Cov}_{\mathcal{J}}^{\mathbf{a}}(k)$ play no role in determining the optimal risk neutral execution strategy (3.12). However, the expected permanent price impact of other large trades, $\mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k)$, affects the optimal execution strategy. This can be seen more clearly from Proposition 3.1 below under an additional symmetry assumption.

Proposition 3.1. *Let the permanent impact matrix* G *be symmetric and the combined impact matrix* Θ *be positive definite. Moreover, assume for every* $k = 1, 2, \ldots, N$, $\mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k) = \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}$ *for some constant* $\mathbf{E}_{\mathcal{J}}^{\mathbf{a}}$*. Then the unique optimal risk neutral execution strategy is*

$$n_k^* = \frac{\bar{S}}{N} - \frac{(N+1-2k)}{2}\Theta^{-1}\left(\mathbf{E}_{\mathcal{J}}^{\mathbf{a}} + \tau\alpha_0^{\mathbf{a}}\right), \quad k = 1, 2, \ldots, N. \quad (3.13)$$

Note that the symmetry assumption holds when permanent impact matrix is a diagonal matrix; this is also assumed in the literature (e.g., see Almgren and Chriss (2000/2001)). We provide a proof for Proposition 3.1 in Appendix B.

In contrast to the naive strategy, the optimal execution strategy (3.13) now depends on the impact matrices and varies over time as a linear function of $\Theta^{-1}(\tau\alpha_0^{\mathbf{a}} + \mathbf{E}_{\mathcal{J}}^{\mathbf{a}})$. While the naive strategy never buys for a sell execution, the optimal risk neutral execution strategy (3.13) may include buying in some periods during liquidation. Note that the optimal solution (3.13) reduces to the naive strategy when the total expected market price change $\tau\alpha_0^{\mathbf{a}} + \mathbf{E}_{\mathcal{J}}^{\mathbf{a}} = 0$. This result is consistent with Theorem 3.1. When $\Theta^{-1}(\tau\alpha_0^{\mathbf{a}} + \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}) < 0$, the optimal risk neutral execution strategy is a strictly decreasing linear function of k. Specifically the decision maker trades more than $\frac{\bar{S}}{N}$ shares in the periods $1, 2, \ldots, \lceil \frac{N-1}{2} \rceil$, while, in the periods $\lfloor \frac{N+3}{2} \rfloor, \ldots, N$, he trades less than $\frac{\bar{S}}{N}$ shares per period. Similarly, when $\Theta^{-1}(\tau\alpha_0^{\mathbf{a}} + \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}) > 0$, the optimal risk neutral execution strategy is a strictly increasing function of the time period k.

We further examine what parameters $E_{\mathcal{J}}^{\mathbf{a}}$ depends on. Let the jump sizes $\pi_j^{\mathbf{a}}(k)$ and $\chi_j^{\mathbf{a}}(k)$ be normally distributed with means $\mu_x^{\mathbf{a}}(k)$ and $\mu_y^{\mathbf{a}}(k)$, and standard deviations $\sigma_x^{\mathbf{a}}(k)$ and $\sigma_y^{\mathbf{a}}(k)$, respectively. Hence, for the single asset execution, we have (e.g., see Theorem 9.1 in Karlin and Taylor (1981)):

$$\mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k) = \tau\lambda_y \mathbf{E}\left(\chi_j^{\mathbf{a}}(k)\right) - \tau\lambda_x \mathbf{E}\left(\pi_j^{\mathbf{a}}(k)\right) = \tau\left(\lambda_y \mu_y^{\mathbf{a}}(k) - \lambda_x \mu_x^{\mathbf{a}}(k)\right), \quad (3.14)$$

$$\begin{aligned}\mathbf{Cov}_{\mathcal{J}}^{\mathbf{a}}(k) &= \tau\lambda_x\left(\mathbf{Var}(\pi_j^{\mathbf{a}}) + \left(\mathbf{E}(\pi_j^{\mathbf{a}})\right)^2\right) + \tau\lambda_y\left(\mathbf{Var}(\chi_j^{\mathbf{a}}) + \left(\mathbf{E}(\chi_j^{\mathbf{a}})\right)^2\right) \\ &= \tau\lambda_x\left((\sigma_x^{\mathbf{a}}(k))^2 + (\mu_x^{\mathbf{a}}(k))^2\right) + \tau\lambda_y\left(\left(\sigma_y^{\mathbf{a}}(k)\right)^2 + \left(\mu_y^{\mathbf{a}}(k)\right)^2\right). \quad (3.15)\end{aligned}$$

Under the assumptions in Proposition 3.1, we observe that buy and sell arrival rates and the expected permanent price impacts directly affect the expected market price change and consequently the optimal risk neutral execution strategy. When $\lambda_x = \lambda_y$ and $\mu_x^{\mathbf{a}}(k) = \mu_y^{\mathbf{a}}(k)$, $\mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k) = 0$ while $\mathbf{Cov}_{\mathcal{J}}^{\mathbf{a}}(k)$ is strictly positive when either $\sigma_x^{\mathbf{a}}(k)\lambda_x$ is positive or $\sigma_y^{\mathbf{a}}(k)\lambda_y$ is positive. In this case, trades increase the volatility without causing a direction in the market price change.

3.3 Multiplicative Jump Diffusion Market Price Models

The simplicity of the additive jump diffusion model (2.5) leads to a static optimal risk neutral execution strategy. However, from a practical perspective, the additive model (2.5) has limitations. For example, its optimal strategy is static and therefore cannot adapt to the price information revealed during the course of trading.

Theorem 3.3 presents the optimal risk neutral execution strategy from problem (3.4) when market price dynamics and execution price model are given by the multiplicative jump diffusion model (2.8) and (3.1), respectively. Subsequently we denote the $m \times m$ identity matrix with I. Moreover, we use $A.* B$ to denote the componentwise (Hadamard) product of the matrices A and B.

Theorem 3.3. *Assume that the sequence of deterministic symmetric matrices $\{D_k\}_{k=1}^N$, defined by*

$$D_k = -2G^T A_k G + \frac{H + H^T}{\tau} - (G^T B_k + B_k^T G) - 2C_k, \tag{3.16}$$

are positive definite, where the deterministic matrix B_k and the symmetric matrices A_k and C_k are derived from

$$\begin{aligned} A_{k-1} &= A_k .* Q_{k-1} + L_{k-1} A_k L_{k-1} + \frac{1}{2}\left(I - L_{k-1}(2A_k G + B_k)\right) D_k^{-1} \left(I - L_{k-1}(2A_k G + B_k)\right)^T, \\ B_{k-1} &= L_{k-1} B_k - \left(I - L_{k-1}(B_k + 2A_k G)\right) D_k^{-1} \left(2C_k + G^T B_k\right), \\ C_{k-1} &= C_k + \frac{1}{2}\left(2C_k + B_k^T G\right) D_k^{-1} \left(2C_k + G^T B_k\right). \end{aligned} \tag{3.17}$$

Here $L_{k-1} = \mathrm{Diag}\left(e + \tau\alpha_0^{\mathbf{m}} + \mathbf{E}_{\mathcal{J}}^{\mathbf{m}}(k-1)\right)$, $Q_{k-1} = \tau\Sigma^{\mathbf{m}}(\Sigma^{\mathbf{m}})^T + \mathbf{Cov}_{\mathcal{J}}^{\mathbf{m}}(k-1)$, and $A_N = 0$, $B_N = I$, and $C_N = -\frac{H+H^T}{2\tau}$. Then the unique optimal risk neutral execution strategy $n^* = \{n_k^*\}_{k=1}^N$ is given by

$$\begin{aligned} n_k^* &= D_{k+1}^{-1}\left(I - \left(B_{k+1}^T + 2G^T A_{k+1}\right) L_k\right) P_{k-1} - D_{k+1}^{-1}\left(2C_{k+1} + G^T B_{k+1}\right) x_{k-1}, \quad k = 1, \ldots, N-1, \\ n_N^* &= \bar{S} - \sum_{k=1}^{N-1} n_k^*. \end{aligned} \tag{3.18}$$

Furthermore, the optimal expected execution cost becomes

$$V_1^*(P_0, x_0) = P_0^T \bar{S} - P_0^T A_1 P_0 - P_0^T B_1 x_0 - x_0^T C_1 x_0. \tag{3.19}$$

The proof of Theorem 3.3 is given in Appendix C.

The optimal risk neutral execution strategy (3.18), derived under the multiplicative jump diffusion model (2.8), is significantly different from the optimal execution strategy (3.12) under the additive jump diffusion model (2.6). Firstly, the optimal risk neutral execution strategy (3.18) does depend on the covariance matrices $\Sigma^{\mathbf{m}}(\Sigma^{\mathbf{m}})^T$ and $\mathbf{Cov}_{\mathcal{J}}^{\mathbf{m}}$. In addition, strategy (3.18), obtained under the multiplicative model (2.8), is stochastically dynamic and depends on the future market price realization P_{k-1}, when $\tau\alpha_0^{\mathbf{m}} + \mathbf{E}_{\mathcal{J}}^{\mathbf{m}}(k) \neq 0$. When $\tau\alpha_0^{\mathbf{m}} + \mathbf{E}_{\mathcal{J}}^{\mathbf{m}}(k)$ is zero for every period k, Theorem 3.1, applied to the price dynamics (2.8), implies that the solution (3.18) is static.

Assume that the jump amplitude is log-normally distributed, i.e., $\log \pi_j^{\mathbf{m}}(k)$ and $\log \chi_j^{\mathbf{m}}(k)$ have normal distributions with means $\mu_x^{\mathbf{m}}(k)$ and $\mu_y^{\mathbf{m}}(k)$, and standard deviations $\sigma_x^{\mathbf{m}}(k)$ and $\sigma_y^{\mathbf{m}}(k)$, respectively. For a single asset trading ($m = 1$), we have (e.g., see Karlin and Taylor (1975) page 268):

$$\mathbf{E}\left(\pi_j^{\mathbf{m}}(k)\right) = \exp\left(\mu_x^{\mathbf{m}}(k) + \tfrac{1}{2}\left(\sigma_x^{\mathbf{m}}(k)\right)^2\right), \quad \mathbf{Var}\left(\pi_j^{\mathbf{m}}(k)\right) = \left(\exp\left((\sigma_x^{\mathbf{m}}(k))^2\right) - 1\right)\exp\left(2\mu_x^{\mathbf{m}}(k) + \left(\sigma_x^{\mathbf{m}}(k)\right)^2\right),$$
$$\mathbf{E}\left(\chi_j^{\mathbf{m}}(k)\right) = \exp\left(\mu_y^{\mathbf{m}}(k) + \tfrac{1}{2}\left(\sigma_y^{\mathbf{m}}(k)\right)^2\right), \quad \mathbf{Var}\left(\chi_j^{\mathbf{m}}(k)\right) = \left(\exp\left((\sigma_y^{\mathbf{m}}(k))^2\right) - 1\right)\exp\left(2\mu_y^{\mathbf{m}}(k) + \left(\sigma_y^{\mathbf{m}}(k)\right)^2\right).$$

Therefore,

$$\mathbf{E}_{\mathcal{J}}^{\mathbf{m}}(k) = \tau\lambda_y\left(\exp\left(\mu_y^{\mathbf{m}}(k) + \frac{(\sigma_y^{\mathbf{m}}(k))^2}{2}\right) - 1\right) - \tau\lambda_x\left(\exp\left(\mu_x^{\mathbf{m}}(k) + \frac{(\sigma_x^{\mathbf{m}}(k))^2}{2}\right) - 1\right),$$
$$\mathbf{V}_{\mathcal{J}}^{\mathbf{m}}(k) = \tau\lambda_x\left(\mathbf{Var}\left(\pi_j^{\mathbf{m}}(k)\right) + \left(\mathbf{E}\left(\pi_j^{\mathbf{m}}(k)\right) - 1\right)^2\right) + \tau\lambda_y\left(\mathbf{Var}\left(\chi_j^{\mathbf{m}}(k)\right) + \left(\mathbf{E}\left(\chi_j^{\mathbf{m}}(k)\right) - 1\right)^2\right).$$

In contrast to $\mathbf{E}_{\mathcal{J}}^{\mathbf{m}}$, now the volatility of the permanent price impact affects the expected market price change. Notice that other distributions can also be considered for the jump amplitudes. For example, Pareto and Beta distributions have been considered for jump amplitudes in a double exponential jump diffusion process to model asset price evolution in the literature (e.g., see Kou (2002), Ramezani and Zeng (2007)).

4 Assessing and Controlling the Execution Risk

The optimal risk neutral execution strategy under the multiplicative jump diffusion model (2.8), given the values of the two state variables P_k and x_k, depends only on the expected market return $\mathbf{E}_{\mathcal{J}}^{\mathbf{m}}$ and the covariance $\mathbf{Cov}_{\mathcal{J}}^{\mathbf{m}}$ (see Theorem 3.3). Thus, the optimal risk neutral execution strategy is identical to the optimal strategy obtained under the following adjusted model *without jump* for the market price

$$P_k = P_{k-1} + \mathrm{Diag}(P_{k-1})\left(\tau\left(\alpha_0^{\mathbf{m}} + \tau^{-1}\mathbf{E}_{\mathcal{J}}^{\mathbf{m}}\right) + \tau^{1/2}\left(\Sigma^{\mathbf{m}} + \tau^{-1/2}\mathbf{Cov}_{\mathcal{J}}^{\mathbf{m}\,1/2}\right)Z_k\right) - Gn_k. \tag{4.1}$$

While the market price in model (4.1) is normally distributed and has no jump, the market price P_k of models (2.8) and (4.1) share the same first and second moments. Hence, for the purpose of determining the optimal risk neutral execution strategy, one does not need to differentiate between model (4.1) without jump and model (2.8) with jumps.

However, in addition to the expected execution cost, one needs to be concerned about the execution risk which can be assessed from the execution cost distribution. For risk management purposes, it is important to quantitatively measure and manage the execution risk. The multiplicative jump model (2.8) and the adjusted normal model (4.1) clearly leads to distinctively different execution cost distributions. We illustrate the difference computationally in §5.

If one also wants to control execution risk when choosing an execution strategy, then the stochastic programming problem (3.3) needs to be solved with an appropriate risk measure $\rho(\cdot)$ for the execution cost. Under the jump model, the distribution of the execution cost is asymmetric and the variance is not appropriate since it treats the cost and profit equally. Instead, CVaR or downside risk measure is more appropriate. In addition, CVaR is a coherent risk measure which can measure extreme events/execution costs and has attractive properties such as convexity, see, e.g., (Artzner et al., 1997; Rockafellar and Uryasev, 2000)).

Denote the execution cost by the random variable $L \stackrel{\text{def}}{=} \left(P_0^T \bar{S} - \sum_{k=1}^{N} n_k^T \tilde{P}_k\right)$. For a given confidence level $\beta \in (0,1)$, CVaR_β is given below

$$\mathrm{CVaR}_\beta(L) = \min_{\alpha\in\mathbb{R}}\left(\alpha + (1-\beta)^{-1}\mathbf{E}\left((L-\alpha)^+\right)\right), \tag{4.2}$$

where $(z)^+ = \max(z, 0)$, see, e.g., (Rockafellar and Uryasev, 2000). With the CVaR risk measure, the execution cost problem (3.3) becomes

$$\min_{n_1,\cdots,n_N \in \mathbb{R}^m, \alpha \in \mathbb{R}} \mathbf{E}\left(P_0^T \bar{S} - \sum_{k=1}^{N} n_k^T \tilde{P}_k\right) + c \cdot \left(\alpha + (1-\beta)^{-1} \mathbf{E}\left([P_0^T \bar{S} - \sum_{k=1}^{N} n_k^T \tilde{P}_k - \alpha]^+\right)\right)$$

$$\text{s.t.} \qquad \sum_{k=1}^{N} n_k = \bar{S}.$$

This is a multi-stage stochastic nonlinear programming problem. In particular, the execution cost depends nonlinearly on n_k due to the permanent price impact. Solving this problem is computationally challenging and we are currently developing methods to approximate the solution accurately.

Similar to (Almgren and Chriss, 2000/2001), here we assume that the strategy $\{n_1, \cdots, n_N\}$ is static. We use the following computational method to obtain the optimal static execution strategy under the CVaR risk measure. Since there is no analytic expression for the CVaR evaluation, Monte Carlo simulation is required to discretize a CVaR minimization problem. Unfortunately, under a discretization with M simulations, the objective function in (4.3) includes the sum of M piecewise nonlinear functions:

$$\min_{n_1,\cdots,n_N \in \mathbb{R}^m, \alpha \in \mathbb{R}} \frac{1}{M} \sum_{j=1}^{M} \left(P_0^T \bar{S} - \sum_{k=1}^{N} n_k^T \tilde{P}_k^{(j)}\right) + c \cdot \left(\alpha + \frac{1}{M(1-\beta)} \sum_{j=1}^{M} \left(P_0^T \bar{S} - \sum_{k=1}^{N} n_k^T \tilde{P}_k^{(j)} - \alpha\right)^+\right)$$

$$\text{s.t.} \qquad \sum_{k=1}^{N} n_k = \bar{S},$$

where the superscript (j) denotes the jth simulation.

The CVaR risk measure is typically continuously differentiable (Rockafellar and Uryasev, 2000). Since nondifferentiability here arises from simulation discretization, we apply a smoothing technique in (Alexander et al., 2006) for the single period CVaR optimization problem. The convergence property of this smoothing method is established in (Xu and Zhang, 2009). We approximate the nonsmooth piecewise linear function $[z]^+$ by a continuously differentiable piecewise quadratic function $\rho_\epsilon(z)$ for some small resolution parameter ϵ:

$$\rho_\epsilon(z) = \begin{cases} z & \text{if } z > \epsilon \\ \frac{z^2}{4\epsilon} + \frac{1}{2}z + \frac{1}{4}\epsilon & \text{if } -\epsilon \le z \le \epsilon \\ 0 & \text{if } z < -\epsilon \end{cases} \tag{4.3}$$

In particular, the execution strategy which minimizes the CVaR_β of the execution cost can be computed from the following minimization problem:

$$\min_{\alpha \in \mathbb{R}, n_1,\cdots,n_N \in \mathbb{R}^m} \alpha + \frac{1}{(1-\beta)M} \sum_{j}^{M} \left(\rho_\epsilon\left(P_0^T \bar{S} - \sum_{k=1}^{N} n_k^T \tilde{P}_k^{(j)} - \alpha\right)\right) \qquad \text{s.t.} \qquad \sum_{k=1}^{N} n_k = \bar{S}. \tag{4.4}$$

5 Performance Comparison

We now present our computational investigation of the potential effect of the model assumption on the optimal risk neutral execution strategy. We evaluate trading performance in terms of the expected execution cost, execution risk, and more generally execution cost distribution.

Because of a more accurate characterization for the short term asset return, the multiplicative jump diffusion model (2.8) with known model parameters is assumed for the future market price. Since trading impact of large institutions is likely to cause a nonzero change in the expected market price and return, we assume that the expected change in the market price is nonzero. Based on the assumed model, we then compare the following three strategies:

- Strategy$_M$: optimal risk neutral execution strategy under the assumed multiplicative jump model (2.8).
- Strategy$_A$: optimal risk neutral execution strategy under the additive jump diffusion model (2.6) with comparable means and covariances set as below

$$\mathbf{E}^{\mathbf{a}}_{\mathcal{J}} = P_0 \mathbf{E}^{\mathbf{m}}_{\mathcal{J}}, \qquad \tau \Sigma^{\mathbf{a}} (\Sigma^{\mathbf{a}})^T + \mathbf{Cov}^{\mathbf{a}}_{\mathcal{J}} = P_0^2 \left(\tau \Sigma^{\mathbf{m}} (\Sigma^{\mathbf{m}})^T + \mathbf{Cov}^{\mathbf{m}}_{\mathcal{J}} \right).$$

  We denote the total volatility $(\tau \Sigma^{\mathbf{m}} (\Sigma^{\mathbf{m}})^T + \mathbf{Cov}^{\mathbf{m}}_{\mathcal{J}})^{1/2}$ by σ_{tot}. Note that Strategy$_A$ does not depend on the covariance $\mathbf{Cov}^{\mathbf{a}}_{\mathcal{J}}$ and volatility $\Sigma^{\mathbf{a}}$.
- Strategy$_N$: the naive strategy which is optimal when the expected total market price change is zero, the permanent impact matrix G is symmetric, and the combined impact matrix Θ is positive definite. The naive strategy is used as the performance benchmark; the comparison illustrates the importance of accurate modeling of the market price dynamics in determining an optimal execution strategy.

We conduct computational investigations for a single asset trading example. The expected market price change due to small trades is assumed to be zero, i.e., $\alpha_0^{\mathbf{a}} = 0$ (\$/share)/day and $\alpha_0^{\mathbf{m}} = 0$ (1/day). We also assume that variance $\tau \Sigma^{\mathbf{m}} (\Sigma^{\mathbf{m}})^T$ (due to *normal* trading) constitutes 10% of the total standard deviation σ_{tot}. Specifically, we consider selling $\bar{S}$ shares over T days. Unless otherwise stated, the parameter values in Table 1 are used.

Parameters	Values
Number of Periods	$N = T$
Interval Length	$\tau = T/N = 1$ day
Temporary Impact Matrix	$H = 2.5 \times (10^{-6})$ (\$ $\cdot$ day)/share^2
Permanent Impact Matrix	$G = 2.5 \times (10^{-7})$ \$/share^2
Initial Asset Price	$P_0 = 50$ \$/share

Table 1: Parameter values for the single asset execution example.

In addition parameters λ_x and λ_y are trading arrival rates per day. We assume that the jump amplitudes $\pi^{\mathbf{m}}_j$ and $\chi^{\mathbf{m}}_j$ are log-normally distributed, and $\mathbf{E}^{\mathbf{a}}_{\mathcal{J}}(k) = \mathbf{E}^{\mathbf{a}}_{\mathcal{J}}$ and $\mathbf{E}^{\mathbf{m}}_{\mathcal{J}}(k) = \mathbf{E}^{\mathbf{m}}_{\mathcal{J}}$ for some constants $\mathbf{E}^{\mathbf{a}}_{\mathcal{J}}$ and $\mathbf{E}^{\mathbf{m}}_{\mathcal{J}}$, $\mathbf{Cov}^{\mathbf{m}}_{\mathcal{J}}(k) = \mathbf{Cov}^{\mathbf{m}}_{\mathcal{J}}$ and $\mathbf{Cov}^{\mathbf{a}}_{\mathcal{J}}(k) = \mathbf{Cov}^{\mathbf{a}}_{\mathcal{J}}$ for some constants $\mathbf{Cov}^{\mathbf{m}}_{\mathcal{J}}$ and $\mathbf{Cov}^{\mathbf{a}}_{\mathcal{J}}$.

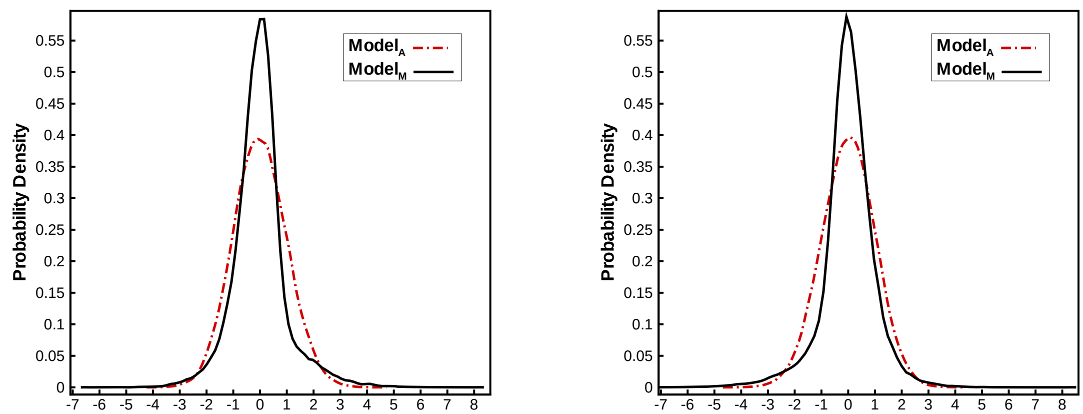
Furthermore, the market price dynamics is determined by the following parameters $\Sigma^{\mathbf{m}}$, $\mu^{\mathbf{m}}_x$, $\mu^{\mathbf{m}}_y$, $\sigma^{\mathbf{m}}_x$, $\sigma^{\mathbf{m}}_y$. λ_x, λ_y. In subsequent computational results, we have simply assigned reasonable parameter values for illustrative purposes; we also choose these parameter values so that the magnitudes of trading impact represented by $\mathbf{E}\left(\pi^{\mathbf{m}}_j(k)\right) - 1$ and $\mathbf{E}\left(\chi^{\mathbf{m}}_j(k)\right) - 1$ are reasonable. In addition, since in general the permanent price impact of buying is larger than selling, we choose larger values for means of jump amplitude for buys than for sells, i.e., $\mu^{\mathbf{m}}_y \geq \mu^{\mathbf{m}}_x$.

5.1 Comparison of the Execution Risk

We assess the difference in execution risk under the multiplicative jump diffusion model (2.8), denoted as Model$_M$, and the adjusted model (4.1) without jump, denoted as Model$_A$.

The market price model (4.1) leads to a normal distribution for the market price P_k which can underestimate the tail risk (likely due to large trades of other institutions). However, the multiplicative jump model (2.8), in which permanent price impact of other institutional trades are modeled by compound Poisson processes, is capable of better characterizing the short term asset returns and describing the fat tails.

In subplot (a) of Figure 1, the probability density function of the market price P_1 and the execution cost under the models (2.8) and (4.1) are compared. Subplot (b) compares the execution cost distribution associated with Strategy$_M$ and Strategy$_A$. Under the proposed jump model (2.8), compared with the normal model (4.1), the execution cost has larger probability of small costs and higher probability of extreme costs. Using the model (4.1), it is possible to significantly underestimate the execution risk.



(a) Standardized Market Price Distribution (with zero mean and unit variance)

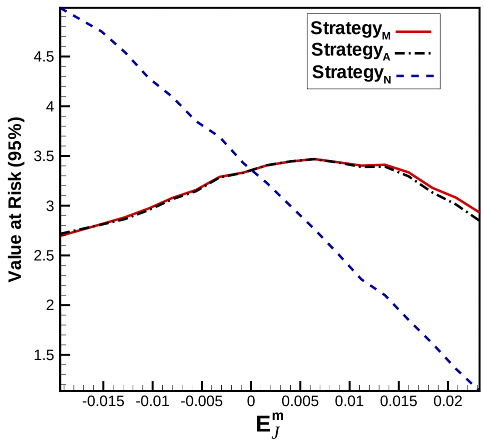
(b) Standardized Execution Cost Distribution (with zero mean and unit variance)

Figure 1: Probability density functions of Model$_M$ and Model$_A$ for $M = 50,000$ simulations. The kurtosis of P_1 for Model$_M$ is 7.03 while for Model$_A$ is 3.03. The kurtosis of the total execution cost per share for Model$_M$ is 7.50 while for Model$_A$ is 3.04. Initial holding is $\bar{S} = 10^6$ shares. The parameters are $\lambda_x = 1$, $\mu_x^{\mathbf{m}} = 9.901 \times 10^{-3}$, $(\sigma_x^{\mathbf{m}})^2 = 9.802 \times 10^{-5}$ $\lambda_y = 0.2$, $\mu_y^{\mathbf{m}} = 1.049 \times 10^{-2}$, $(\sigma_y^{\mathbf{m}})^2 = 2.873 \times 10^{-3}$. These values yield $\Sigma^{\mathbf{m}} = 9.045 \times 10^{-3}$, $\mathbf{E}_{\mathcal{J}}^{\mathbf{m}} = -0.0076$, and $\mathbf{Cov}_{\mathcal{J}}^{\mathbf{m}} = 8.182 \times 10^{-4}$.

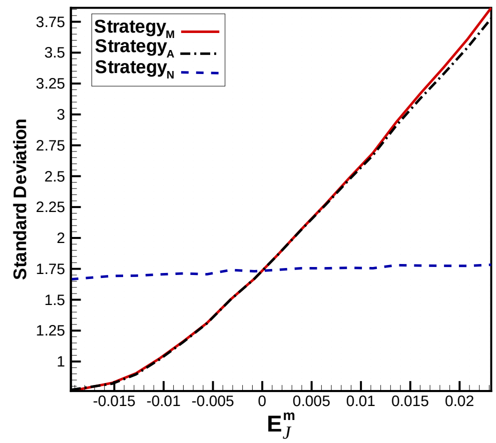
Figure 2 compares the risk measured in standard deviation and VaR for the execution strategies Strategy$_M$, Strategy$_A$, and Strategy$_N$, under the assumed multiplicative jump diffusion model (2.8). Figure 2 illustrates that the risk values are quite different between the naive strategy and Strategy$_M$ or Strategy$_A$. We note that at $\mathbf{E}_{\mathcal{J}}^{\mathbf{m}} = 0$ the risk measure values are identical since the three execution strategies Strategy$_N$, Strategy$_M$ and Strategy$_A$ coincide at this point.

Figures 2 also illustrates that including an appropriate risk measure in Problem (3.3) can be important in determining the optimal execution strategy. Under the proposed jump model, the coherent risk measure CVaR may be more appropriate.

Assume that a strategy $\{n_k\}$ is deterministic, we compute the minimum CVaR$_{95\%}$ strategies under Model$_M$ and Model$_A$. Table 2 illustrates the difference between the optimal static (price-independent) execution strategies to minimize the CVaR$_{95\%}$ of the execution cost computed under the two models Model$_M$ and Model$_A$. Table 2 also indicates that, although Model$_M$ and Model$_A$ share the same optimal risk neutral execution strategy, they yield different optimal execution strategies when CVaR$_{95\%}$ of the execution cost is



(a) VaR



(b) Standard Deviation

Figure 2: Risk measures of the execution costs for Strategy$_M$, Strategy$_A$ and Strategy$_N$ for $M = 40,000$ simulations. Initial holding is $\bar{S} = 10^6$ shares. Time horizon is $T = 5$. The parameter values are $\lambda_x = 2$, $\mu_x^{\mathbf{m}} = 9.9013 \times 10^{-3}$, $\sigma_x^{\mathbf{m}} = 9.9007 \times 10^{-3}$, and $\lambda_y \in [0.05, 3.6]$. These values yield $\Sigma^{\mathbf{m}} = 9.6484 \times 10^{-3}$, $\sigma_{tot} = 0.032$, and $\mathbf{Cov}_{\mathcal{J}}^{\mathbf{m}} = 9.3091 \times 10^{-4}$.

minimized. Here the difference in CVaR values is about 3.7%. While the strategy to minimize the variance liquidate completely immediately, the strategies for minimizing CVaR$_{95\%}$ under both models sell in the first couple of periods aggressively and purchases are made in the last couple of periods. Although here minimizing CVaR$_{95\%}$ strategies share a similar pattern under both Model$_M$ and Model$_A$, there is significant difference in the amount of trading in these two strategies.

5.2 Comparison of Optimal Execution Strategies

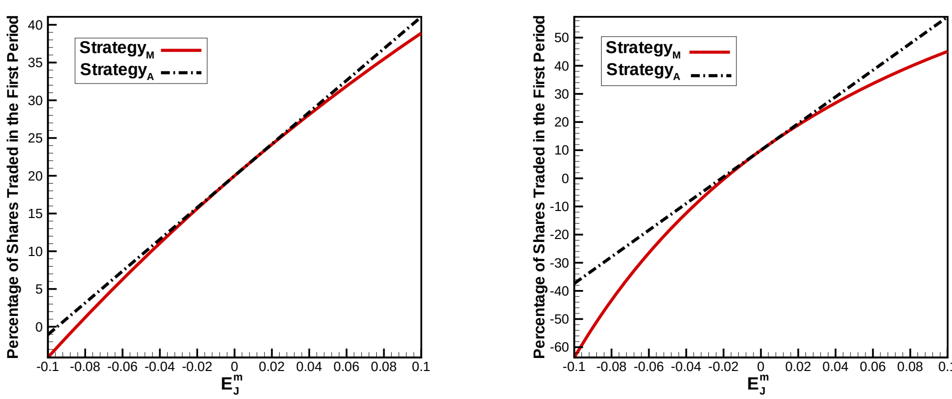
When the expected total market price change is nonzero, Strategy$_M$ is dynamic while the optimal Strategy$_A$ is static. However, since the initial price P_0 and the initial holding x_0 are known, the optimal execution n_1^* for both Strategy$_M$ and Strategy$_A$ are deterministic. Figure 3 compares Strategy$_M$ and Strategy$_A$ for the first period as a function of $\mathbf{E}_{\mathcal{J}}^{\mathbf{m}}$. As is illustrated in Figure 3, the difference in Strategy$_M$ and Strategy$_A$ increases as $\mathbf{E}_{\mathcal{J}}^{\mathbf{m}}$ moves away from zero.

Given a fixed $\mathbf{E}_{\mathcal{J}}^{\mathbf{m}}$, Figure 4 illustrates the optimal execution strategies Strategy$_M$ and Strategy$_A$ from period 2 to period 5 for $M = 1000$ simulations of the jump amplitudes and pricing shocks Z_k. These plots clearly illustrate the significant difference of these execution strategies. While the naive strategy Strategy$_N$ suggests to trade an equal amount in each period, Strategy$_A$ is time varying but independent of the market price realized at the beginning of each period. In contrast, Strategy$_M$ is stochastic and varies with the realized market prices. In comparison to the naive strategy, both execution strategies Strategy$_M$ and Strategy$_A$ suggest to sell more aggressively initially in order to take advantage of the expected impact of large trades $\mathbf{E}_{\mathcal{J}}^{m} < 0$.

According to Theorems 3.2 and 3.3, another main difference between Strategy$_M$ and Strategy$_A$ is that the execution strategy Strategy$_M$ depends on the covariance of the market return, while the execution strategy Strategy$_A$ does not depend on the covariance of market prices. We illustrate the implication of this property

	Model_M	Model_A
$\text{CVaR}_{95\%}$	2.50×10^6	2.59×10^6
n_1	9.83×10^5	9.61×10^5
n_2	1.91×10^4	1.12×10^5
n_3	7.11×10^3	-5.87×10^4
n_4	-1.13×10^3	-4.55×10^3
n_5	-8.59×10^3	-9.04×10^3

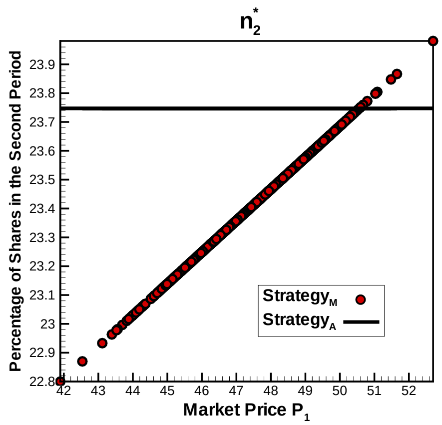
Table 2: Optimal strategies which minimize $\text{CVaR}_{95\%}$ under Model_M and Model_A, and the corresponding optimal values using $M = 50,000$ simulation in executing a single asset. Parameters are as in Table 1, and $T = 5$ days, and $\bar{S} = 10^6$ shares. The parameter values are $\lambda_x = 3$, $\mu_x^{\mathbf{m}} = 9.5 \times 10^{-3}$, $\sigma_x^{\mathbf{m}} = 10^{-2}$, $\lambda_y = 0.5$, $\mu_y^{\mathbf{m}} = 6.9 \times 10^{-3}$, and $\sigma_y^{\mathbf{m}} = 3.2 \times 10^{-2}$. These values yield $\Sigma^{\mathbf{m}} = 0.009$, $\mathbf{E}_{\mathcal{J}}^{\mathbf{m}} = -0.025050$, and $\mathbf{Cov}_{\mathcal{J}}^{\mathbf{m}} = 1.106555 \times 10^{-3}$.



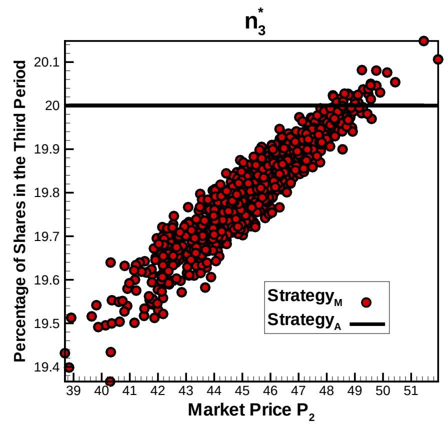
(a) $T = 5$ days

(b) $T = 10$ days

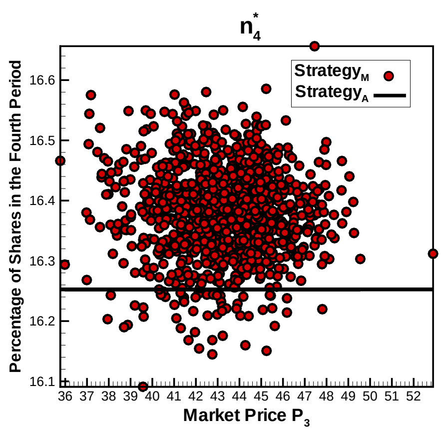
Figure 3: Comparison of the optimal execution n_1^* under the multiplicative jump model and under the additive jump model. Initial holding is $\bar{S} = 10^7$ shares. The total volatility is $\sigma_{tot} = 0.03$.



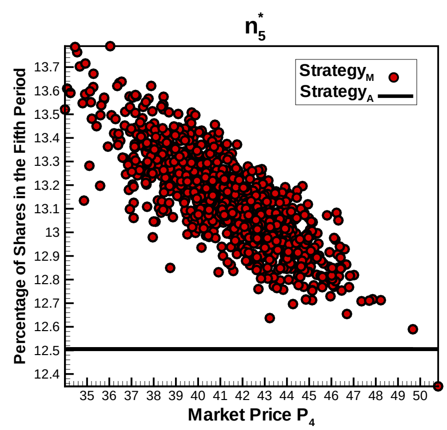
(a) Strategy at the second period



(b) Strategy at the third period



(c) Strategy at the fourth period



(d) Strategy at the fifth period

Figure 4: Optimal strategies n_2^*, n_3^*, n_4^* and n_5^* under the multiplicative jump model and the additive jump model for $M = 1000$ simulations. The trading horizon is $T = 5$ days. Initial holding is $\bar{S} = 10^7$ shares. The parameter values are $\lambda_x = 3.8$, $\mu_x^{\mathbf{m}} = 9.901 \times 10^{-3}$, $\sigma_x^{\mathbf{m}} = 9.901 \times 10^{-3}$, $\lambda_y = 0.2$, $\mu_y^{\mathbf{m}} = 1.186 \times 10^{-2}$, and $\sigma_y^{\mathbf{m}} = 1.198 \times 10^{-2}$. These values yield $\Sigma^{\mathbf{m}} = 9.045 \times 10^{-3}$, $\mathbf{E}_{\mathcal{J}}^{\mathbf{m}} = -3.560 \times 10^{-2}$, and $\mathbf{Cov}_{\mathcal{J}}^{\mathbf{m}} = 8.182 \times 10^{-4}$.

σ_{tot}	$\mu_y^{\mathbf{m}}$	$\sigma_y^{\mathbf{m}}$	$\Sigma^{\mathbf{m}}$	Strategy$_M$	Strategy$_N$	Strategy$_A$	$\mathbf{D^{am}}(\%)$
0.020	5.90×10^{-3}	1.25×10^{-2}	6.03×10^{-3}	21.40	292.29	22.73	86.26
0.025	5.40×10^{-3}	3.41×10^{-2}	7.54×10^{-3}	21.19	293.08	22.66	89.10
0.030	4.78×10^{-3}	4.90×10^{-2}	9.05×10^{-3}	21.04	293.90	22.60	92.48
0.035	4.06×10^{-3}	6.21×10^{-2}	10.55×10^{-3}	20.97	293.31	22.65	95.77
0.040	3.22×10^{-3}	7.44×10^{-2}	12.06×10^{-3}	20.58	293.43	22.62	100.41
0.045	2.27×10^{-3}	8.62×10^{-2}	13.57×10^{-3}	20.32	292.01	22.59	104.61
0.050	1.21×10^{-3}	9.77×10^{-2}	15.08×10^{-3}	20.13	292.78	22.59	108.71

Table 3: Average expected cost (cents per share) and $\mathbf{D^{am}}$ (percentage) for $M = 100,000$ simulations. Trading horizon is $T = 10$ days. Initial holding is $\bar{S} = 10^6$ shares. Jump parameters are set $\lambda_x = 2.6$, $\lambda_y = 0.2$, $\mu_x^{\mathbf{m}} = 4.938 \times 10^{-3}$ and $\sigma_x^{\mathbf{m}} = 9.950 \times 10^{-3}$. Thus, $\mathbf{E}_{\mathcal{J}}^{\mathbf{m}} = -1.180 \times 10^{-2}$.

next.

5.3 Comparison of Expected Execution Costs

We now compare expected execution costs associated with Strategy$_N$, Strategy$_M$, and Strategy$_A$, presented in cents per share. We quantify the average execution cost difference per period for a single asset trading using the following measure,

$$\mathbf{D^{am}} \stackrel{\text{def}}{=} \frac{1}{NM\bar{S}} \sum_{k=1}^{N} \sum_{i=1}^{M} \left| n_k^{(\mathbf{m})}(i) \tilde{P}_k^{(\mathbf{m})}(i) - n_k^{(\mathbf{a})} \tilde{P}_k^{(\mathbf{a})}(i) \right|, \tag{5.1}$$

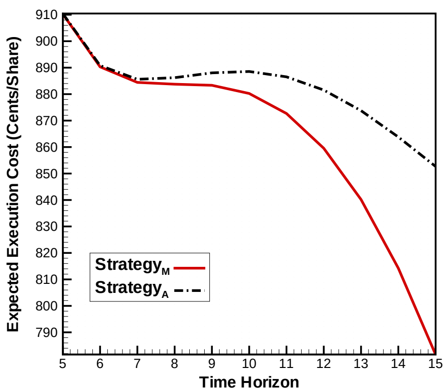
where the number M is the total number of simulations, $n_k^{(\mathbf{m})}(i)$ is the optimal risk neutral execution under the multiplicative jump diffusion model for the kth period in the ith simulation, $n_k^{(\mathbf{a})}$ is the optimal execution for the kth period derived under the additive jump diffusion model. The prices $\tilde{P}_k^{(\mathbf{m})}(i)$ and $\tilde{P}_k^{(\mathbf{a})}(i)$ are the execution prices at period k in the ith simulation, corresponding to the execution strategies $n^{(\mathbf{m})}(i)$ and $n^{(\mathbf{a})}$, respectively. The market price is assumed to follow a multiplicative jump diffusion model.

Using simulation, we compute $\mathbf{D^{am}}$ measure for $T = 10$ days and various values of σ_{tot}. These quantities are reported in Table 3 which also includes the averaged execution costs of the three execution strategies Strategy$_M$, Strategy$_A$, and Strategy$_N$. As Table 3 indicates, the average relative difference $\mathbf{D^{am}}(\%)$ can be quite significant. Moreover, the value of $\mathbf{D^{am}}(\%)$ increases as σ_{tot} increases. Notice that, as Strategy$_A$ and Strategy$_N$ do not depend on the asset price volatility, their corresponding expected execution costs are constant as σ_{tot} changes; the slight variations for Strategy$_A$ and Strategy$_N$ are due to Monte Carlo simulations.

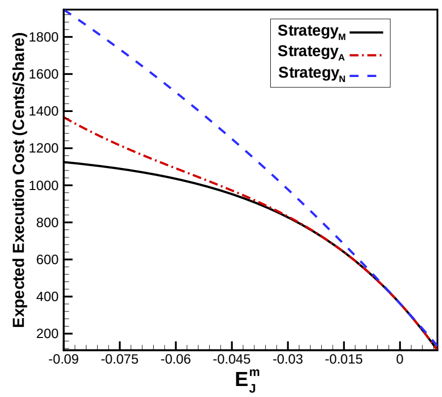
Proposition D.1 provides an analytical formula for the expected execution cost of the optimal strategy obtained under the additive jump diffusion model. For Strategy$_M$, which is optimal under the multiplicative jump diffusion model, the expected execution cost of Strategy$_M$ decreases as σ_{tot} increases. This is due to the fact that, under the multiplicative model, the optimal solution is truly stochastically dynamic; thus it is capable of capturing price variations. In contrast, Strategy$_A$ is static and its execution cost do not depend on σ_{tot}.

Subplot (a) in Figure 5 depicts the dependence of the expected execution cost on the trading horizon T. As Figure 5 demonstrates, when the time horizon increases, the expected execution cost of the execution strategy Strategy$_A$ becomes much higher than the expected execution cost of Strategy$_M$. Subplot (b) in

Figure 5 illustrates how the expected execution cost associated with each of these three execution strategies varies as $\mathbf{E}^{\mathbf{m}}_{\mathcal{J}}$ changes, focusing when $\mathbf{E}^{\mathbf{m}}_{\mathcal{J}} < 0$. This figure clearly illustrates that in the depicted range, as $\mathbf{E}^{\mathbf{m}}_{\mathcal{J}}$ deviates from zero, the expected execution cost of the naive strategy, Strategy$_N$, becomes significantly higher than the expected execution costs of the strategies Strategy$_M$ and Strategy$_A$. Moreover, as it is expected, the expected execution cost corresponding to the execution strategy Strategy$_A$ is also greater than that of the optimal strategy Strategy$_M$. This difference becomes more prominent as $\mathbf{E}^{\mathbf{m}}_{\mathcal{J}}$ moves away from zero.



(a) Execution cost as time horizon varies



(b) Execution cost as $\mathbf{E}^{\mathbf{m}}_{\mathcal{J}}$ varies

Figure 5: Comparison in the expected execution cost (cents per share). The expected costs in the plot were computed using Theorem 3.3 for Strategy$_M$ and Proposition D.1 for Strategy$_A$. Initial holding is $\bar{S} = 10^7$ shares. The values specified for the model parameters yield $\Sigma^{\mathbf{m}} = 9.045 \times 10^{-3}$. For Subplot (a), $\lambda_x = 3.8$,$\mu^{\mathbf{m}}_x = 9.901 \times 10^{-3}$, $\sigma^{\mathbf{m}}_x = 9.901 \times 10^{-3}$, $\lambda_y = 0.2$,$\mu^{\mathbf{m}}_y = 1.186 \times 10^{-2}$,$\sigma^{\mathbf{m}}_y = 1.198 \times 10^{-2}$, $\mathbf{E}^{\mathbf{m}}_{\mathcal{J}} = -3.560 \times 10^{-2}$, $\mathbf{Cov}^{\mathbf{m}}_{\mathcal{J}} = 8.182 \times 10^{-4}$

6 Concluding Remarks

Current literature on the execution cost problem typically assumes that the market return (or price change) has a normal distribution. There are two main problems with this assumption. Firstly, the empirical study indicates that the short term return distribution often has fat tail, possibly due to permanent price impact of institutional trades. Such fat tails cannot be described by normal distributions. Secondly, while the permanent price impact of a decision maker's own trade causes a discrete price depression, it is not reasonable to model permanent price impact of other concurrent large trades by a continuous Brownian motion.

In this paper, we suggest using jump processes to capture uncertain permanent price impact of trades by other institutions. The proposed model includes two compound Poisson processes corresponding to buy and sell trades, respectively. Using stochastic dynamic programming, we provide an analytical solution to the risk neutral execution strategy which minimizes the expected execution cost under the proposed jump diffusion models for the evolution of market prices. This solution is static (state independent) when the expected market price change is zero. However, when the expected market price change is nonzero, the optimal execution strategy derived under the multiplicative jump diffusion model is stochastic and dynamic. In addition the optimal execution strategy depends on the asset return volatility. In contrast, under an additive jump diffusion model, the optimal execution strategy does not depend on the asset price realization or the volatility, even when the expected market price change is nonzero.

Under the proposed jump diffusion model, more accurate assessment of the execution risk can be made. When the market price change is modeled by normal distributions, the tail execution risk can be significantly underestimated. Using simulations, we illustrate that the execution cost distribution associated with the naive strategy, optimal risk neutral strategy under the additive jump diffusion model, and the optimal risk neutral strategy under the multiplicative jump diffusion model are qualitatively different. This highlights the importance of using an appropriate model to determine an optimal execution strategy. In addition, we assess differences in the optimal execution strategies derived under different model assumptions. We illustrate that, when the expected market return deviates from zero, the naive strategy can perform poorly. Assuming that the market price dynamics is characterized by a multiplicative jump diffusion model, we show that the execution strategy optimal under an additive jump diffusion model (with comparable mean and standard deviation) can perform notably sub-optimally when the asset return volatility or the trading horizon increases.

Our main focus is on investigating model assumptions and the resulting optimal execution strategies for the execution cost problem. We consider the optimal execution strategy for minimizing the expected execution cost. In addition, we compute the minimum CVaR risk execution strategies assuming that the strategy is deterministic and leave extending the results to more general cases for future work. There are many possible objective functions which are of interest to institutional investors. In particular, a natural additional criterion to include is some measure of risk, e.g., variance, VaR or CVaR of the execution cost. While the inclusion of a risk measure into the objective function is conceptually straightforward and probably desirable, an analytical expression for the optimal execution strategy is not available except in very special cases (Bertsimas and Lo (1998)). Including risk measures in the objective function might make the optimal-value function non-separable in the sense of stochastic dynamic programming (e.g., see Yao et al. (2003)). Therefore, the presence of some nonlinear risk measure makes solving the stochastic dynamic programming very challenging.

In this paper, we do not address how to estimate the parameters of the proposed jump diffusion model. It will be interesting to investigate techniques to estimate the parameters using high frequency trading data and techniques such as maximum likelihood estimation. Empirical performance assessment of specifications of the proposed model using real world data suggests another future research direction. In addition, we have assumed here that impact matrices are constant and independent of the arrivals of other trades in previous periods. It might be interesting to appropriately model the effect of trading from other institutions on the impact matrices and to investigate its implications on the optimal execution strategy. Investigating the stability of the jump diffusion model comparing to a diffusion model is another direction for future work.

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A Optimal Execution Strategy with Zero Expected Price Change

In this appendix, we present the proof of Theorem 3.1.

Proof. We first prove by induction on k that, when $\mathbf{E}\left(\mathcal{F}_{k-1}\left(P_{k-1}\right)|P_{k-1}\right) = P_{k-1}$ holds and (deterministic) price impact functions are independent of the market prices, optimal execution n_k^* does not depend on P_{k-1}, and for $k = 1, 2, \ldots, N$, the optimal-value function is given by

$$V_k^*(P_{k-1}, x_{k-1}) = P_0^T \bar{S} - x_{k-1}^T P_{k-1} + R_{k-1}(x_{k-1}), \tag{A.1}$$

where $R_{k-1}(\cdot)$ is a deterministic function independent of P_{k-1}.

For $k = N$, optimal execution n_N^* equals x_{N-1} and from equation (3.5) the optimal-value function in the last period becomes

$$V_N^*(P_{N-1}, x_{N-1}) = P_0^T \bar{S} - x_{N-1}^T P_{N-1} + x_{N-1}^T h\left(\frac{x_{N-1}}{\tau}\right).$$

This confirms the correctness of (A.1) for $k = N$ with $R_{N-1}(x_{N-1}) = x_{N-1}^T h\left(\frac{x_{N-1}}{\tau}\right)$. Assume that in the period $(k+1)$, optimal execution n_{k+1}^* only depends on x_k and the optimal-value function at time period $k+1$ is

$$V_{k+1}^*(P_k, x_k) = P_0^T \bar{S} - x_k^T P_k + R_k(x_k),$$

where $R_k(x_k)$ does not depend on P_k. The Bellman's principle of optimality in the kth step yields:

$$\begin{aligned} V_k^*(P_{k-1}, x_{k-1}) &= \min_{n_k \in \mathbb{R}^m} \mathbf{E}\left[-n_k^T \tilde{P}_k + V_{k+1}^*(P_k, x_k) \mid P_{k-1},\ x_{k-1}\right] \\ &= \min_{n_k \in \mathbb{R}^m} \mathbf{E}\left[-n_k^T \left(P_{k-1} - h\left(\frac{n_k}{\tau}\right)\right) + P_0^T \bar{S} - x_k^T P_k + R_k(x_k) \mid P_{k-1},\ x_{k-1}\right]. \end{aligned}$$

Applying the market price dynamics (2.2), equation (2.1) and assumption (3.9), the optimal-value function $V_k^*(P_{k-1}, x_{k-1})$ becomes

$$\begin{aligned} &\min_{n_k \in \mathbb{R}^m} \mathbf{E}\left[-n_k^T \left(P_{k-1} - h\left(\frac{n_k}{\tau}\right)\right) + P_0^T \bar{S} - (x_{k-1} - n_k)^T \left(\mathcal{F}_{k-1}(P_{k-1}) - \tau g\left(\frac{n_k}{\tau}\right)\right) + R_k(x_{k-1} - n_k) \mid P_{k-1}, x_{k-1}\right] \\ &= \min_{n_k \in \mathbb{R}^m} \left(-n_k^T \left(P_{k-1} - h\left(\frac{n_k}{\tau}\right)\right) + P_0^T \bar{S} - (x_{k-1} - n_k)^T \left(P_{k-1} - \tau g\left(\frac{n_k}{\tau}\right)\right) + R_k(x_{k-1} - n_k)\right) \\ &= P_0^T \bar{S} - x_{k-1}^T P_{k-1} + \min_{n_k \in \mathbb{R}^m} \left(n_k^T h\left(\frac{n_k}{\tau}\right) + (x_{k-1} - n_k)^T \tau g\left(\frac{n_k}{\tau}\right) + R_k(x_{k-1} - n_k)\right). \end{aligned} \tag{A.2}$$

The objective function of the minimization problem in (A.2) does not depend on P_{k-1} and is only in terms of x_{k-1} and specifications of the price impact functions $h(\cdot)$ and $g(\cdot)$. Hence, optimal execution n_k^* does not depend on P_{k-1} and consequently is static. Moreover, the optimal objective value of the minimization problem in (A.2) becomes

$$V_k^*(P_{k-1}, x_{k-1}) = P_0^T \bar{S} - x_{k-1}^T P_{k-1} + R_{k-1}(x_{k-1}),$$

where

$$R_{k-1}(x_{k-1}) = \min_{n_k \in \mathbb{R}^m} \left(n_k^T h\left(\frac{n_k}{\tau}\right) + (x_{k-1} - n_k)^T \tau g\left(\frac{n_k}{\tau}\right) + R_k(x_{k-1} - n_k)\right).$$

This proves the correctness of equation (A.1) for k. Thus, for $k = 1, 2, \ldots, N$, the optimal-value function is as in equation (A.1), and the optimal execution n_k^* is independent of P_{k-1} and consequently is static.

Now, let the price impact functions be given by (3.2) where the permanent impact matrix G is symmetric and the matrix Θ is positive definite. By induction on k, we prove that for $k = 1, 2, \ldots, N$,

$$n_k^* = \frac{1}{N-k+1} x_{k-1}, \qquad V_k^*(P_{k-1}, x_{k-1}) = P_0^T \bar{S} - P_{k-1}^T x_{k-1} + \frac{1}{2} x_{k-1}^T \left(\frac{\Theta}{N-k+1} + G\right) x_{k-1}. \tag{A.3}$$

¿From (3.6), the optimal execution n_N^* equals x_{N-1}, and the optimal-value function $V_N^*(P_{N-1}, x_{N-1})$ becomes

$$V_N^*(P_{N-1}, x_{N-1}) = P_0^T \bar{S} - x_{N-1}^T P_{N-1} + \frac{1}{2} x_{N-1}^T \frac{H + H^T}{\tau} x_{N-1} = P_0^T \bar{S} - x_{N-1}^T P_{N-1} + \frac{1}{2} x_{N-1}^T (\Theta + G) x_{N-1}.$$

This confirms the correctness of (A.3) for $k = N$. Now assume (A.3) is true for $k+1$. Therefore,

$$V_{k+1}^*(P_k, x_k) = P_0^T \bar{S} - P_k^T x_k + \frac{1}{2} x_k^T \left(\frac{\Theta}{N-k} + G \right) x_k.$$

Using this assumption, we show that (A.3) is true for k. The Bellman's principle of optimality in the kth step becomes:

$$\begin{aligned} V_k^*(P_{k-1}, x_{k-1}) &= \min_{n_k \in \mathbb{R}^m} \mathbf{E} \left[-n_k^T \tilde{P}_k + V_{k+1}^*(P_k, x_k) \mid P_{k-1},\ x_{k-1} \right] \\ &= \min_{n_k \in \mathbb{R}^m} \mathbf{E} \left[-n_k^T P_{k-1} + n_k^T \frac{H}{\tau} n_k + \left(P_0^T \bar{S} - P_k^T x_k + \frac{1}{2} x_k^T \left(\frac{\Theta}{N-k} + G \right) x_k \right) \mid P_{k-1},\ x_{k-1} \right] \\ &= \min_{n_k \in \mathbb{R}^m} \left(-n_k^T P_{k-1} + n_k^T \frac{H}{\tau} n_k + P_0^T \bar{S} - \mathbf{E} \left[P_k^T x_k | P_{k-1}, x_{k-1} \right] + \frac{1}{2} (x_{k-1} - n_k)^T A (x_{k-1} - n_k) \right), \end{aligned}$$

where $A = \frac{\Theta}{N-k} + G$. Applying the market price dynamics (2.2) and equation (2.1), the expected value in the above statement can be stated in terms of P_{k-1} and x_{k-1}:

$$\mathbf{E} \left[P_k^T x_k | P_{k-1}, x_{k-1} \right] = \mathbf{E} \left[(\mathcal{F}_{k-1}(P_{k-1}) - G n_k)^T (x_{k-1} - n_k) \mid P_{k-1},\ x_{k-1} \right] = (P_{k-1} - G n_k)^T (x_{k-1} - n_k),$$

where the last equality comes from the assumption $\mathbf{E} [\mathcal{F}_{k-1}(P_{k-1}) \mid P_{k-1},\ x_{k-1}] = P_{k-1}$. Hence, after some algebraic manipulation, the optimal-value function $V_k^*(P_{k-1}, x_{k-1})$ equals

$$P_0^T \bar{S} - P_{k-1}^T x_{k-1} + \frac{1}{2} x_{k-1}^T A x_{k-1} + \min_{n_k \in \mathbb{R}^m} \left(\frac{1}{2} n_k^T \left(\frac{N-k+1}{N-k} \Theta \right) n_k - \left(\frac{\Theta}{N-k} x_{k-1} \right)^T n_k \right).$$

When the matrix $\Theta + \Theta^T = 2\Theta$ is positive definite, the unique optimal solution of the above minimization problem becomes

$$n_k^* = \frac{1}{(N-k+1)} x_{k-1}. \tag{A.4}$$

Therefore, the optimal value function $V_k^*(P_{k-1}, x_{k-1})$ equals

$$V_k^*(P_{k-1}, x_{k-1}) = P_0^T \bar{S} - P_{k-1}^T x_{k-1} + \frac{1}{2} x_{k-1}^T \left(\frac{\Theta}{N-k+1} + G \right) x_{k-1}.$$

This completes the induction. Using equation (2.1) and $x_0 = \bar{S}$, it can be shown that n_k^* obtained in (A.4) equals $\frac{\bar{S}}{N}$, which is the naive strategy.

□

B Optimal Execution Strategy Under Additive Jump Market Price Models

Below, we provide a proof for Theorem 3.2.

Proof. We prove by induction on k that the optimal execution and the optimal-value function are given by:

$$V_k^*(P_{k-1}, x_{k-1}) = P_0^T \bar{S} - \frac{1}{2} x_{k-1}^T \left(\Theta^T - A_k - G\right) x_{k-1} - \left(P_{k-1} + b_k - \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k) - \tau \alpha_0^{\mathbf{a}}\right)^T x_{k-1} - c_k, \quad \text{(B.1)}$$
$$n_{k-1}^* = A_k^{-1} \left(b_k - \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k) + \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k-1) + \Theta^T - A_k^T x_{k-2}\right), \qquad k = 2, 3, \ldots, N,$$

where A_k, b_k and c_k are defined as in equations (3.10) and (3.11), and the matrix A_k is symmetric.

For $k = N$, optimal execution n_N^* equals x_{N-1}. From equation (3.6), the optimal-value function in the last period becomes

$$V_N^*(P_{N-1}, x_{N-1}) = P_0^T \bar{S} - P_{N-1}^T x_{N-1} + \frac{1}{2} x_{N-1}^T \frac{H + H^T}{\tau} x_{N-1}, \quad \text{(B.2)}$$

Hence, equation (B.1) holds for $k = N$ with $A_N = \Theta^T + \Theta$, $b_N = \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(N) + \tau \alpha_0^{\mathbf{a}}$, and $c_N = 0$. Notice that the matrix A_N is symmetric. Assume that the statement (B.1) holds for $k + 1$, particularly:

$$V_{k+1}^*(P_k, x_k) = P_0^T \bar{S} - \frac{1}{2} x_k^T \left(\Theta^T - A_{k+1} - G\right) x_k - \left(P_k + b_{k+1} - \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k+1) - \tau \alpha_0^{\mathbf{a}}\right)^T x_k - c_{k+1}, \quad \text{(B.3)}$$

where A_{k+1} is symmetric. We will prove the correctness of (B.1) for k. Applying Bellman's principle of optimality in the kth step yields

$$V_k^*(P_{k-1}, x_{k-1}) = \min_{n_k \in \mathbb{R}^m} \mathbf{E}\left[-n_k^T \tilde{P}_k + V_{k+1}^*(P_k, x_k) \mid P_{k-1},\ x_{k-1}\right]. \quad \text{(B.4)}$$

Substituting (B.3) into equation (B.4), the objective function in (B.4) becomes

$$\begin{aligned}
&\mathbf{E}\left[-n_k^T \tilde{P}_k + P_0^T \bar{S} - \frac{1}{2} x_k^T (\Theta^T - A_{k+1} - G) x_k - (P_k + b_{k+1} - \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k+1) - \tau \alpha_0^{\mathbf{a}})^T x_k - c_{k+1} \mid P_{k-1},\ x_{k-1}\right] \\
&= P_0^T \bar{S} - \frac{1}{2} x_{k-1}^T (\Theta^T - A_{k+1} - G) x_{k-1} - \left(P_{k-1} + b_{k+1} - \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k+1) + \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k)\right)^T x_{k-1} + \frac{1}{2} n_k^T A_{k+1} n_k \\
&+ \left(b_{k+1} - \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k+1) + \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k) + (\Theta^T - A_{k+1})^T x_{k-1}\right)^T n_k - c_{k+1}.
\end{aligned}$$

Note that this function to be minimized is quadratic in n_k. Moreover, from the induction hypothesis the matrix A_{k+1} is symmetric. In addition, the matrix A_{k+1} is positive definite by assumption, and consequently the objective function is convex. It is straightforward to verify that the optimal solution is attained at

$$n_k^* = -A_{k+1}^{-1} \left(b_{k+1} - \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k+1) + \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k) + (\Theta^T - A_{k+1})^T x_{k-1}\right).$$

Whence the optimal-value function $V_k^*(P_{k-1}, x_{k-1})$ equals

$$\begin{aligned}
&P_0^T \bar{S} - \frac{1}{2} x_{k-1}^T \left((\Theta^T - A_{k+1}) A_{k+1}^{-1} (\Theta^T - A_{k+1})^T + \Theta^T - A_{k+1} - G\right) x_{k-1} \quad \text{(B.5)} \\
&- \left(P_{k-1} + (\Theta^T - A_{k+1}) A_{k+1}^{-1} \left(b_{k+1} - \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k+1) + \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k)\right) + b_{k+1} - \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k+1) + \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k)\right)^T x_{k-1} \\
&- \frac{1}{2} \left(b_{k+1} - \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k+1) + \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k)\right)^T A_{k+1}^{-1} \left(b_{k+1} - \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k+1) + \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}(k)\right) - c_{k+1}.
\end{aligned}$$

Substituting equations (3.10) and (3.11) in (B.5) yields the correctness of equation (B.1) for k. Furthermore, equation (3.10) and the symmetry assumption of A_{k+1} yield the matrix A_k is symmetric. This completes the induction. □

We now prove the statement in Proposition 3.1.

Proof. By a simple induction we can prove that, when $G = G^T$ and $\mathbf{E}^{\mathbf{a}}_{\mathcal{J}}(k) = \mathbf{E}^{\mathbf{a}}_{\mathcal{J}}$ for $k = 1, 2, \ldots, N$, equations (3.10) and (3.11) yield

$$A_k = \left(\frac{N+2-k}{N+1-k}\right)\Theta, \quad b_k = \frac{N-k+2}{2}\left(\mathbf{E}^{\mathbf{a}}_{\mathcal{J}} + \tau\alpha^{\mathbf{a}}_0\right). \tag{B.6}$$

Positive definiteness of Θ implies that the matrix A_k is positive definite, for every $k = 1, 2, \ldots, N$. Hence, the assumption in Theorem 3.2 is satisfied and stochastic dynamic programming offers a unique optimal solution. Substituting equations (B.6) in (3.12), we get:

$$n^*_k = \frac{1}{N-k+1}x^*_{k-1} - \frac{(N-k)}{2}\Theta^{-1}\left(\mathbf{E}^{\mathbf{a}}_{\mathcal{J}} + \tau\alpha^{\mathbf{a}}_0\right), \quad k = 1, 2, \ldots, N-1, \tag{B.7}$$

or equivalently

$$x^*_{k-2} = (N-k+2)\left(n^*_{k-1} + \frac{(N-k+1)}{2}\Theta^{-1}\left(\mathbf{E}^{\mathbf{a}}_{\mathcal{J}} + \tau\alpha^{\mathbf{a}}_0\right)\right), \quad k = 2, 3, \ldots, N. \tag{B.8}$$

Applying equation (B.7), equation (B.8) and equation (2.1), we get:

$$\begin{aligned} n^*_k &= \frac{1}{N-k+1}(x^*_{k-2} - n^*_{k-1}) - \frac{(N-k)}{2}\Theta^{-1}\left(\mathbf{E}^{\mathbf{a}}_{\mathcal{J}} + \tau\alpha^{\mathbf{a}}_0\right) \\ &= \frac{1}{N-k+1}\left((N-k+2)\left(n^*_{k-1} + \frac{(N-k+1)}{2}\Theta^{-1}(\mathbf{E}^{\mathbf{a}}_{\mathcal{J}} + \tau\alpha^{\mathbf{a}}_0)\right) - n^*_{k-1}\right) - \frac{(N-k)}{2}\Theta^{-1}\left(\mathbf{E}^{\mathbf{a}}_{\mathcal{J}} + \tau\alpha^{\mathbf{a}}_0\right) \\ &= n^*_{k-1} + \Theta^{-1}(\mathbf{E}^{\mathbf{a}}_{\mathcal{J}} + \tau\alpha^{\mathbf{a}}_0), \qquad k = 2, 3, \ldots, N-1. \end{aligned} \tag{B.9}$$

Now, we use equations (B.7) and (B.9) to prove (3.13) by induction on $k \leq N-1$. Equation (B.7) for $k = 1$ directly implies the correctness of (3.13) for $k = 1$. Assuming that equation (3.13) holds for $k-1$, we will prove it for k. Using equation (B.9), we have

$$n^*_k = \frac{\bar{S}}{N} - \frac{(N+3-2k)}{2}\Theta^{-1}\left(\mathbf{E}^{\mathbf{a}}_{\mathcal{J}} + \tau\alpha^{\mathbf{a}}_0\right) + \Theta^{-1}\left(\mathbf{E}^{\mathbf{a}}_{\mathcal{J}} + \tau\alpha^{\mathbf{a}}_0\right) = \frac{\bar{S}}{N} - \frac{N+1-2k}{2}\Theta^{-1}\left(\mathbf{E}^{\mathbf{a}}_{\mathcal{J}} + \tau\alpha^{\mathbf{a}}_0\right),$$

which proves the correctness of equation (3.13) for $k = 2, 3, \ldots, N-1$, and the induction is complete.

Since $\sum_{k=1}^{N} n^*_k = \bar{S}$, for $k = N$ we must have

$$n^*_N = \bar{S} - \sum_{k=1}^{N-1} n^*_k = \bar{S} - (N-1)\frac{\bar{S}}{N} + \frac{1}{2}\Theta^{-1}\left(\mathbf{E}^{\mathbf{a}}_{\mathcal{J}} + \tau\alpha^{\mathbf{a}}_0\right)\sum_{k=1}^{N-1}(N+1-2k) = \frac{\bar{S}}{N} + \frac{(N-1)}{2}\Theta^{-1}\left(\mathbf{E}^{\mathbf{a}}_{\mathcal{J}} + \tau\alpha^{\mathbf{a}}_0\right),$$

which shows the correctness of (3.13) for $k = N$. $\square$

C Optimal Execution Strategy Under Multiplicative Jump Price Models

In this Appendix, we prove Theorem 3.3. Recall that $\mathbf{E}^{\mathbf{m}}_{\mathcal{J}}(k)$ and $\mathbf{Cov}^{\mathbf{m}}_{\mathcal{J}}(k)$ denote the expected value and covariance matrix of $\mathcal{J}^{\mathbf{m}}(k)$. Moreover, we refer to the $m \times m$ identity matrix, and the $m \times m$ zero matrix as I and 0, respectively. Moreover, we denote the m-vector of all ones with e.

Proof. By backward induction on k, we prove that optimal execution is given by (3.18), matrices A_k and C_k are symmetric, and the optimal-value function is given by equation (3.19).

For $k = N$, the constraint $x_N = 0$ yields the optimal execution n_N^* must equal x_{N-1}. Using equation (3.6), the optimal-value function in the last period becomes

$$V_N^*(P_{N-1}, x_{N-1}) = P_0^T \bar{S} - P_{N-1}^T x_{N-1} + \frac{1}{2} x_{N-1}^T \frac{H + H^T}{\tau} x_{N-1}, \tag{C.1}$$

which is obtained from substitution $A_N = 0$, $B_N = I$ and $C_N = -\frac{H+H^T}{2\tau}$ in equation (3.19). Note that matrices A_N and C_N are symmetric.

Assume that statement (3.19) holds for $k+1$, i.e., the optimal-value function $V_{k+1}^*(P_k, x_k)$ is given by

$$V_{k+1}^*(P_k, x_k) = P_0^T \bar{S} - P_k^T A_{k+1} P_k - P_k^T B_{k+1} x_k - x_k^T C_{k+1} x_k, \tag{C.2}$$

with A_{k+1} and C_{k+1} are symmetric. We now prove the correctness of equation (3.19) for k. Bellman's principle of optimality implies

$$V_k^*(P_{k-1}, x_{k-1}) = \min_{n_k \in \mathbb{R}^m} \quad \mathbf{E}\left[-n_k^T \tilde{P}_k + V_{k+1}^*(P_k, x_k) \mid P_{k-1},\ x_{k-1} \right]. \tag{C.3}$$

Substituting equation (C.2) into equation (C.3), we obtain:

$$V_k^*(P_{k-1}, x_{k-1}) = \min_{n_k \in \mathbb{R}^m} \ \mathbf{E}\left[P_0^T \bar{S} - n_k^T \tilde{P}_k - P_k^T A_{k+1} P_k - P_k^T B_{k+1} x_k - x_k^T C_{k+1} x_k \mid P_{k-1},\ x_{k-1} \right]. \tag{C.4}$$

Given P_{k-1} and x_{k-1}, equation $x_k = x_{k-1} - n_k$ and the execution price model (2.8), the terms $n_k^T \tilde{P}_k$ and $x_k^T C_{k+1} x_k$ in the objective function of the minimization problem in (C.4) are deterministic. Hence:

$$\mathbf{E}\left[n_k^T \tilde{P}_k \mid P_{k-1},\ x_{k-1} \right] = n_k^T \tilde{P}_k = n_k^T \left(P_{k-1} - \frac{H}{\tau} n_k \right) = n_k^T P_{k-1} - n_k^T \frac{H}{\tau} n_k, \tag{C.5}$$

$$\mathbf{E}\left[x_k^T C_{k+1} x_k \mid P_{k-1},\ x_{k-1} \right] = (x_{k-1} - n_k)^T C_{k+1} (x_{k-1} - n_k). \tag{C.6}$$

Define $\mathcal{L}_k = e + \tau \alpha_0^{\mathbf{m}} + \mathcal{J}^{\mathbf{m}}(k) + \tau^{1/2} \Sigma^{\mathbf{m}} Z_k$. Using market price dynamics (2.8), we get:

$$\mathbf{E}\left[P_k^T B_{k+1} x_k \mid P_{k-1},\ x_{k-1} \right] = P_{k-1}^T L_k B_{k+1} (x_{k-1} - n_k) - n_k^T G^T B_{k+1} (x_{k-1} - n_k), \tag{C.7}$$

where $L_k = \text{Diag}\left(\mathbf{E}(\mathcal{L}_k)\right)$. Similarly, using the market price dynamics (2.8), the term $P_k^T A_{k+1} P_k$ is stated as:

$$P_k^T A_{k+1} P_k = P_{k-1}^T \text{Diag}\left(\mathcal{L}_k\right) A_{k+1} \text{Diag}\left(\mathcal{L}_k\right) P_{k-1} - 2 P_{k-1}^T \text{Diag}\left(\mathcal{L}_k\right) A_{k+1} G n_k + n_k^T G^T A_{k+1} G n_k. \tag{C.8}$$

In addition

$$\mathbf{E}\left[P_{k-1}^T \text{Diag}\left(\mathcal{L}_k\right) A_{k+1} \text{Diag}\left(\mathcal{L}_k\right) P_{k-1} \mid P_{k-1},\ x_{k-1} \right] = P_{k-1}^T \left(A_{k+1} .* \mathbf{E}\left[\mathcal{L}_k \mathcal{L}_k^T \mid P_{k-1},\ x_{k-1} \right] \right) P_{k-1},$$

where $.*$ denotes the Hadamard product.

Since $\mathbf{E}(Z_k) = 0$ and the random vectors Z_k and $\mathcal{J}^{\mathbf{m}}(k)$ are independent, we obtain

$$\mathbf{E}\left[\mathcal{L}_k \mathcal{L}_k^T \right] = \mathbf{E}[\mathcal{L}_k] \mathbf{E}[\mathcal{L}_k]^T + \tau \Sigma^{\mathbf{m}} (\Sigma^{\mathbf{m}})^T + \mathbf{Cov}^{\mathbf{m}}_{\mathcal{J}}(k).$$

Hence,

$$\begin{aligned}&\mathbf{E}\left[P_{k-1}^T \mathrm{Diag}\left(\mathcal{L}_k\right) A_{k+1} \mathrm{Diag}\left(\mathcal{L}_k\right) P_{k-1} \mid P_{k-1},\ x_{k-1}\right] \\ &= P_{k-1}^T \left(L_k A_{k+1} L_k + \left(\tau \Sigma^{\boldsymbol{m}} (\Sigma^{\boldsymbol{m}})^T + \mathbf{Cov}_{\mathcal{J}}^{\boldsymbol{m}}(k)\right) .* A_{k+1}\right) P_{k-1}.\end{aligned} \tag{C.9}$$

Taking expectation from (C.8) and substituting equation (C.9), the term $\mathbf{E}\left[P_k^T A_{k+1} P_k \mid P_{k-1},\ x_{k-1}\right]$ equals

$$P_{k-1}^T \left(A_{k+1} .* \left(\tau \Sigma^{\boldsymbol{m}} (\Sigma^{\boldsymbol{m}})^T + \mathbf{Cov}_{\mathcal{J}}^{\boldsymbol{m}}(k)\right) + L_k A_{k+1} L_k\right) P_{k-1} - 2 P_{k-1}^T L_k A_{k+1} G n_k + n_k^T G^T A_{k+1} G n_k. \tag{C.10}$$

Substituting equations (C.5), (C.6), (C.7) and (C.10) into equation (C.4), the objective function of the minimization problem in (C.4) is reduced to:

$$\begin{aligned}&P_0^T \bar{S} - x_{k-1}^T C_{k+1} x_{k-1} - P_{k-1}^T L_k B_{k+1} x_{k-1} - P_{k-1}^T \left(A_{k+1} .* \left(\tau \Sigma^{\boldsymbol{m}} (\Sigma^{\boldsymbol{m}})^T + \mathbf{Cov}_{\mathcal{J}}^{\boldsymbol{m}}(k)\right) + L_k A_{k+1} L_k\right) P_{k-1} \\ &+ \left(x_{k-1}^T \left(2 C_{k+1} + B_{k+1}^T G\right) + P_{k-1}^T \left(-I + L_k B_{k+1} + 2 L_k A_{k+1} G\right)\right) n_k + \frac{1}{2} n_k^T D_{k+1} n_k,\end{aligned} \tag{C.11}$$

where D_{k+1} is as in equation (3.16). Hence, the minimization problem in (C.4) is quadratic in n_k. Since D_{k+1} is assumed to be positive definite, the unique optimal solution is attained at

$$n_k^* = \quad -D_{k+1}^{-1} \left(x_{k-1}^T \left(2 C_{k+1} + B_{k+1}^T G\right) + P_{k-1}^T \left(-I + L_k B_{k+1} + 2 L_k A_{k+1} G\right)\right)^T. \tag{C.12}$$

Substituting n_k^* into (C.11) and after some algebraic manipulation, the optimal-value function $V_k^*(P_{k-1}, x_{k-1})$ becomes

$$V_k^*(P_{k-1}, x_{k-1}) = P_0^T \bar{S} - P_{k-1}^T A_k P_{k-1} - P_{k-1}^T B_k x_{k-1} - x_{k-1}^T C_k x_{k-1},$$

where the matrices A_k, B_k and C_k are given by equations (3.17). Notice that when A_{k+1} and C_{k+1} are symmetric, equations (3.17) indicate that A_k and C_k are also symmetric. This completes the induction. ☐

D Analytical Formulae for Expected Execution Costs

Assuming that the true model for the market price is the multiplicative model (2.8), closed-form expressions for the expected execution costs of the naive strategy and the execution strategy (3.13) can be easily derived. The following proposition presents these formulae.

Proposition D.1. *Let the true model for the market price be the multiplicative model* (2.8), *and for every* k, $\mathbf{E}^{\mathbf{m}}_{\mathcal{J}}(k) = \mathbf{E}^{\mathbf{m}}_{\mathcal{J}}$, *for some constant* $\mathbf{E}^{\mathbf{m}}_{\mathcal{J}}$. *Assume that the permanent impact matrix is symmetric and the matrix* Θ *is positive definite. Then the expected (true) execution cost of the execution strategy* n^*, *optimal under the additive jump diffusion model, given in equation* (3.13), *equals*

$$\mathbf{E}\left[P_0^T\bar{S} - \sum_{k=1}^{N}\tilde{P}_k^T n_k^*\right] = \bar{S}^T P_0 + \bar{S}^T\frac{H}{N\tau}\bar{S} + \frac{\bar{S}^T}{N}\sum_{k=1}^{N}L^{k-1}\left(\frac{N-k}{N}G\bar{S} - P_0\right) \tag{D.1}$$
$$+\sum_{k=0}^{N-1}\left(\mathbf{E}^{\mathrm{a}}_{\mathcal{J}} + \tau\alpha_0^{\mathrm{a}}\right)^T\Theta^{-1}\left(\frac{k(N-k)}{2N}\left(L^{k-1}G - GL^{k-1}\right)\bar{S} + \frac{(N-2k-1)}{2}L^kP_0\right)$$
$$+\left(\mathbf{E}^{\mathrm{a}}_{\mathcal{J}} + \tau\alpha_0^{\mathrm{a}}\right)^T\Theta^{-1}\left(\frac{N(N^2-1)}{12\tau}H + \frac{1}{12}\sum_{k=1}^{N-1}(N-k)\left(N^2-1-2k(k+N)\right)L^{k-1}G\right)\Theta^{-1}\left(\mathbf{E}^{\mathrm{a}}_{\mathcal{J}} + \tau\alpha_0^{\mathrm{a}}\right),$$

where $L \stackrel{\mathrm{def}}{=} \mathrm{Diag}\left(e + \tau\alpha_0^{\mathbf{m}} + \mathbf{E}^{\mathbf{m}}_{\mathcal{J}}\right)$. *Here the superscript* k *in the term* L^k *is the exponent of* L.

Proof. Let the market price evolves according to the multiplicative jump diffusion model in (2.8). Whence, following the optimal execution strategy n^* given in equation (3.13), the total amount received at the end of the time horizon equals

$$\sum_{k=1}^{N}\tilde{P}_k^T n_k^* = \sum_{k=1}^{N}\left(P_{k-1} - \frac{H}{\tau}n_k^*\right)^T n_k^* = \sum_{k=1}^{N}P_{k-1}^T\left(\frac{\bar{S}}{N} - \frac{(N+1-2k)}{2}\Theta^{-1}(\tau\alpha_0^{\mathrm{a}} + \mathbf{E}^{\mathrm{a}}_{\mathcal{J}})\right)$$
$$-\sum_{k=1}^{N}\left(\frac{\bar{S}}{N} - \frac{(N+1-2k)}{2}\Theta^{-1}(\tau\alpha_0^{\mathrm{a}} + \mathbf{E}^{\mathrm{a}}_{\mathcal{J}})\right)^T\frac{H}{\tau}\left(\frac{\bar{S}}{N} - \frac{(N+1-2k)}{2}\Theta^{-1}(\tau\alpha_0^{\mathrm{a}} + \mathbf{E}^{\mathrm{a}}_{\mathcal{J}})\right).$$

After simplifying the expression in the right-hand-side of the above equation and using the equalities $\sum_{k=1}^{N}(N+1-2k)^2 = \frac{N(N^2-1)}{3}$ and $\sum_{k=1}^{N}(N+1-2k) = 0$, we arrive at

$$\sum_{k=1}^{N}\tilde{P}_k^T n_k^* = \sum_{k=1}^{N}\frac{\bar{S}^T}{N}P_{k-1} - \sum_{k=1}^{N}\frac{(N+1-2k)}{2}\left(\mathbf{E}^{\mathrm{a}}_{\mathcal{J}} + \tau\alpha_0^{\mathrm{a}}\right)^T\Theta^{-1}P_{k-1}$$
$$-\bar{S}^T\frac{H}{N\tau}\bar{S} - \frac{N(N^2-1)}{12\tau}\left(\mathbf{E}^{\mathrm{a}}_{\mathcal{J}} + \tau\alpha_0^{\mathrm{a}}\right)^T\Theta^{-1}H\Theta^{-1}\left(\mathbf{E}^{\mathrm{a}}_{\mathcal{J}} + \tau\alpha_0^{\mathrm{a}}\right).$$

Therefore, the expected value of the execution cost becomes

$$\mathbf{E}\left[P_0^T\bar{S} - \sum_{k=1}^{N}\tilde{P}_k^T n_k^*\right] = P_0^T\bar{S} - \frac{\bar{S}^T}{N}\sum_{k=1}^{N}\mathbf{E}[P_{k-1}] + \left(\mathbf{E}^{\mathrm{a}}_{\mathcal{J}} + \tau\alpha_0^{\mathrm{a}}\right)^T\Theta^{-1}\sum_{k=1}^{N}\frac{(N+1-2k)}{2}\mathbf{E}[P_{k-1}] \tag{D.2}$$
$$+\bar{S}^T\frac{H}{N\tau}\bar{S} + \frac{N(N^2-1)}{12\tau}\left(\mathbf{E}^{\mathrm{a}}_{\mathcal{J}} + \tau\alpha_0^{\mathrm{a}}\right)^T\Theta^{-1}H\Theta^{-1}\left(\mathbf{E}^{\mathrm{a}}_{\mathcal{J}} + \tau\alpha_0^{\mathrm{a}}\right).$$

Since the random variables Z_k and $\mathcal{J}^{\mathbf{m}}(k)$ are independent of P_{k-1}, using the conditional expectation theorem (e.g., see Varadhan (2001)), we get

$$\mathbf{E}[P_k] = \mathbf{E}\left[\mathbf{E}[P_k|P_{k-1}]\right] = \mathbf{E}\left[\mathrm{Diag}(P_{k-1})\left(e + \tau\alpha_0^{m} + \tau^{1/2}\Sigma^{m}Z_k - \mathcal{J}^{m}(k)\right)\right] - Gn_k^* = L\mathbf{E}[P_{k-1}] - Gn_k^*, \tag{D.3}$$

where $L = \mathrm{Diag}(e + \tau\alpha_0^{\mathbf{m}} + \mathbf{E}^{\mathbf{m}}_{\mathcal{J}})$. Therefore,

$$\sum_{k=1}^{N}\mathbf{E}[P_{k-1}] = P_0 + \sum_{k=2}^{N}\left(L\mathbf{E}[P_{k-2}] - Gn_{k-1}^*\right) = P_0 + LP_0 + \sum_{k=1}^{N-2}L\mathbf{E}[P_k] - \sum_{k=1}^{N-1}Gn_k^*.$$

Using equation (D.3), the summation $\sum_{k=1}^{N-2} \mathbf{E}[P_k]$ can be further simplified and we get

$$\sum_{k=1}^{N} \mathbf{E}\left[P_{k-1}\right] = \left(\sum_{k=1}^{N} L^{k-1}\right) P_0 - \sum_{k=1}^{N-1} \sum_{i=1}^{N-k} L^{k-1} G n_i^*. \tag{D.4}$$

Applying the expression for n^* in equation (3.13), and the equation $\sum_{i=1}^{N-k}(N+1-2i) = k(N-k)$, we get

$$\sum_{k=1}^{N} \mathbf{E}[P_{k-1}] = \sum_{k=1}^{N} L^{k-1}\left(P_0 - \frac{N-k}{N} G\bar{S}\right) + \sum_{k=1}^{N-1} \frac{k(N-k)}{2} L^{k-1} G\Theta^{-1}\left(\mathbf{E}_{\mathcal{J}}^{\boldsymbol{m}} + \tau\alpha_0^{\mathbf{a}}\right).$$

Similarly by using equation (D.3), we may show that

$$\sum_{k=1}^{N}(N+1-2k)\mathbf{E}[P_{k-1}] = \sum_{k=0}^{N-1} (N-2k-1)\, L^k P_0 - \sum_{k=1}^{N-1} \sum_{i=1}^{N-k} (N-2(k+i)+1)\, L^{k-1} G n_i^*.$$

Applying the expression of n^* in equation (3.13) and the equations

$$\sum_{i=1}^{N-k} (N-2\,(k+i)+1) = -k(N-k),$$
$$\sum_{i=1}^{N-k} (N-2(k+i)+1)(N+1-2i) = \frac{(N-k)}{3}(N^2-1-2k(k+N)),$$

the value of $\sum_{k=1}^{N}(N+1-2k)\mathbf{E}[P_{k-1}]$ equals

$$\sum_{k=0}^{N-1} (N-2k-1)\, L^k P_0 - \sum_{k=1}^{N-1} \sum_{i=1}^{N-k} (N-2\,(k+i)+1)\, L^{k-1} G \left(\frac{\bar{S}}{N} - \frac{(N+1-2i)}{2}\Theta^{-1}(\tau\alpha_0^{\mathbf{a}} + \mathbf{E}_{\mathcal{J}}^{\mathbf{a}})\right)$$
$$= \sum_{k=0}^{N-1} (N-2k-1)\, L^k P_0 + \sum_{k=1}^{N-1} \frac{k(N-k)}{N} L^{k-1} G\bar{S} + \sum_{k=1}^{N-1} \frac{(N-k)}{6}(N^2-1-2k(k+N)) L^{k-1} G\Theta^{-1}(\tau\alpha_0^{\mathbf{a}} + \mathbf{E}_{\mathcal{J}}^{\mathbf{a}}).$$

We use this quantity to compute the third term in the right-hand-side of equation (D.2). Therefore, the expected execution cost in (D.2) is reduced to (D.1). □

When $(\mathbf{E}_{\mathcal{J}}^{\mathbf{a}} + \tau\alpha_0^{\mathbf{a}}) = 0$, the execution strategy (3.13) is reduced to the naive strategy. Therefore, equation (D.1) with $(\mathbf{E}_{\mathcal{J}}^{\mathbf{a}} + \tau\alpha_0^{\mathbf{a}}) = 0$ yields the expected execution cost of the naive strategy.

Corollary D.1. *The (true) expected execution cost of the naive strategy $\bar{n}$ equals*

$$\mathbf{E}\left[P_0^T \bar{S} - \sum_{k=1}^{N} \tilde{P}_k^T \bar{n}_k\right] = \bar{S}^T P_0 + \bar{S}^T \frac{H}{N\tau}\bar{S} + \frac{\bar{S}^T}{N} \sum_{k=1}^{N} L^{k-1}\left(\frac{N-k}{N} G\bar{S} - P_0\right). \tag{D.5}$$