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On the convergence of interior-reflective Newton methods for nonlinear minimization subject to bounds ☆

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Abstract

We consider a new algorithm, an interior-reflective Newton approach, for the problem of minimizing a smooth nonlinear function of many variables, subject to upper and/or lower bounds on some of the variables. This approach generates *strictly feasible* iterates by using a new affine scaling transformation and following piecewise linear paths (*reflection paths*). The interior-reflective approach does not require identification of an ''activity set''. In this paper we establish that the interior-reflective Newton approach is globally and quadratically convergent. Moreover, we develop a specific example of interior-reflective Newton methods which can be used for large-scale and sparse problems.

Keywords: Box constraints; Interior-point method; Nonlinear minimization

1. Introduction

This paper is concerned with minimizing a smooth nonlinear function subject to bounds on the variables:

$$\min_{x \in \mathbb{R}^n} f(x), \quad l \leqslant x \leqslant u, \tag{1.1}$$

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where $l \in \{\mathbb{R} \cup \{-\infty\}\}^n$, $u \in \{\mathbb{R} \cup \{\infty\}\}^n$, $l \leqslant u$, and $f: \mathbb{R}^n \to \mathbb{R}^1$. We denote the feasible set $\mathscr{F} = \{x: l \leqslant x \leqslant u\}$ and the strict interior $\text{int}(\mathscr{F}) = \{x: l < x < u\}$.

Minimization problems with upper and/or lower bounds on some of the variables form an important and common class of problems. There are many algorithms for this type of optimization problem, some of which are restricted to quadratic (in some cases convex quadratic) objective functions and some are more general (e.g., [2,5,11,12,14,15,18,20–23,29]). Compared to most of the existing approach for nonlinear minimization subject to bounds, the new interior-reflective Newton approach proposed in [10] has the following three distinctive features. Firstly, the iterates $\{x_k\}$ generated by the new approach are always in the strictly feasible region $\text{int}(\mathscr{F})$. This is done using a new affine scaling transformation, different from the affine scaling transformation used in the linear programming context [28]. Secondly, unlike the affine scaling method for linear programs, our new approach is able to achieve quadratic convergence. Finally, a novel reflective line search technique is used to accelerate convergence.

The main purpose of this paper is to consider the convergence properties of the new interior-reflective Newton approach. In particular, here we establish that interior-reflective Newton methods, applied to twice continuously-differentiable nonlinear functions f, are globally and quadratically convergent under reasonable assumptions.

An interior-reflective Newton method appears to have significant practical potential for large-scale problems. Consider, for example, the results quoted in [10] for the "obstacle problem" on a square m-by-m mesh – see Table 1. The column "its" refers to the number of iterations required to achieve an accurate solution. Full details are given in [10].

A remarkable feature of this type of algorithm, illustrated by this typical example, is the very slow growth in required number of iterations. Given a class of problems and a "natural" way to increase the problem dimension, interior-reflective Newton methods appear to be strikingly insensitive to problem size. Experiments reported in [10] are restricted to quadratic problems; we are currently experimenting on more general nonlinear problems and preliminary results continue to support this claim.

The presentation of this paper is organized as follows. In Section 2, we motivate our new affine scaling transformation. In Section 3, we discuss our unusual reflective path line search idea. The usual acceptance conditions for straight line search algorithms are generalized to our reflective path line search. In Section 4, we discuss important consequences of our new affine scaling transformation which allow us to establish convergence results for a broader

Table 1
Obstacle problem: Lower and upper bounds

m	n	its
30	900	11
40	1600	12
50	2500	14
60	3600	13
100	10000	14

class of methods. In Section 5, we establish first order convergence properties of our interior-reflective algorithms (second order information is not required). In Section 6, second order convergence properties of interior-reflective Newton methods are obtained. An example of interior-reflective Newton method with strong convergence properties, which can be used for large problems, is given in Section 7. Concluding remarks and a look ahead are given in Section 8.

Throughout the presentation, we denote $g = g(x) \stackrel{\text{def}}{=} \nabla f(x)$; $g_k \stackrel{\text{def}}{=} g(x_k)$; $g_* \stackrel{\text{def}}{=} g(x_*) = \nabla f(x_*)$, where x_* is a specified (usually optimal) point. Following Matlab notation, for any $s \in \mathbb{R}^n$, $\text{diag}(s)$ denotes an n-by-n diagonal matrix with the vector s defining the diagonal entries in their natural order. Moreover, as a general rule, we use a superscript to denote additional meaning of a quantity and a subscript to denote a component of a vector, except that a subscript k suggests the iteration k. For example, v_i and v_{k_i} denote the ith component of v and v_k respectively. More notations will be introduced when necessary.

The following assumptions are made throughout the presentation.

Compactness and smoothness assumption. Given an initial point $x_1 \in \mathcal{F}$, it is assumed that the level set $\mathcal{L} = \{x: x \in \mathcal{F} \text{ and } f(x) \leqslant f(x_1)\}$ is compact. Moreover, we assume $f(x)$ is twice continuously-differentiable on an open set $D \supseteq \mathcal{F}$.

2. Motivation of the new affine scaling transformation

Our approach [10] uses a new affine scaling transformation to maintain strict feasibility and a novel reflective path line search to achieve efficiency. We remark that many of the basic ideas behind the interior and reflective Newton approach originated in previous work on various convex optimization problems [6–9,19].

We first motivate the desirability of our new affine scaling transformation.

Currently, there are two distinct ways of handling linear constraints in linear and nonlinear programs. One approach, illustrated in Fig. 1, is to follow the boundary of the feasible region using an active set technique. The alternative philosophy, also illustrated in Fig. 1, is to approach a solution by going through the middle of the feasible region. The interior point approach is more recent and has primarily been applied to linear programs, e.g., [27]. By going through the middle, interior point methods can eliminate the combinatorial nature of many active set methods by handling the linear constraints in a simultaneous manner.

The simplest interior point method for linear programs is the affine scaling method [28,13]. To illustrate the affine scaling idea, assume that we have a strictly feasible point, $x_k \in \text{int}(\mathcal{F})$, i.e., $l < x_k < u$. As indicated in Fig. 2, the sides of the box can restrict the movement from the current point x_k. Hence it is reasonable to change units of each variable so that, in the new coordinates, the current feasible point x_k becomes equally distant to all the nearest sides of the box:

$$\min(\hat{x}_k - (D_k^{\text{affine}})^{-1} l_k,\ (D_k^{\text{affine}})^{-1} u_k - \hat{x}_k) = (D_k^{\text{affine}})^{-1} \min(x_k - l_k,\ u_k - x_k) = e$$

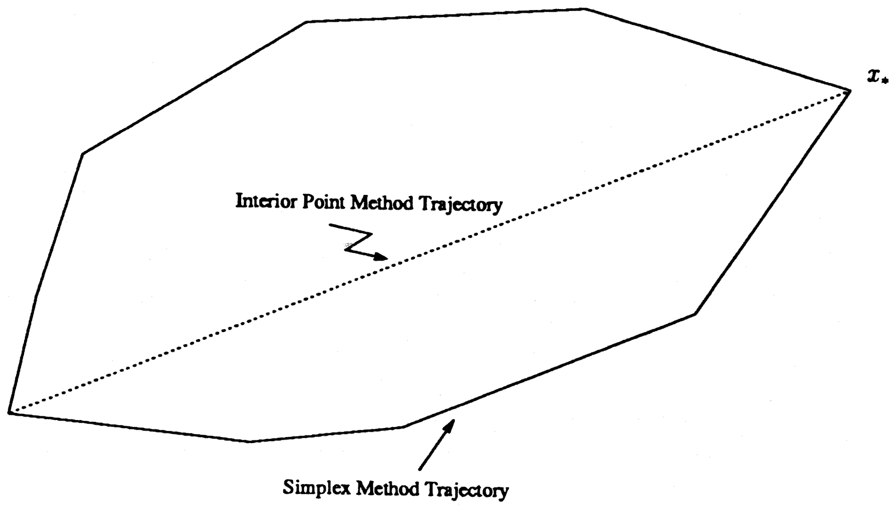

Fig. 1. Trajectories of the interior point method and simplex method.

where

$$D_k^{\text{affine}} + \text{diag}(\min(x_k - l_k,\, u_k - x_k)) \;.$$

The benefit of centering is that sufficient reduction of the objective function can be obtained before a variable bound is reached.

In particular, the best direction to take for linear programs (with bound constraints only), in the new coordinates, is steepest descent: $\hat{d}_k = -(D_k^{\text{affine}})g_k$. This corresponds to the scaled steepest descent direction $d_k = -(D_k^{\text{affine}})^2 g_k$ in the original variable space. This direction is angled away from the approaching bound, see Fig. 2. Moving along d_k, a step is determined from the current point to the nearest boundary and a large fraction (e.g., 0.9) is taken to stay strictly feasible.

The idea of using a local transformation as described above can also be applied to nonlinear problems, with bound constraints, in a straightforward manner. The notion of angling away from nearby constraints is an attractive one. However, the resulting algorithm will be linearly convergent at best. How can the affine scaling ideas be used for nonlinear problems to generate strictly feasible iterates converging globally and quadratically?

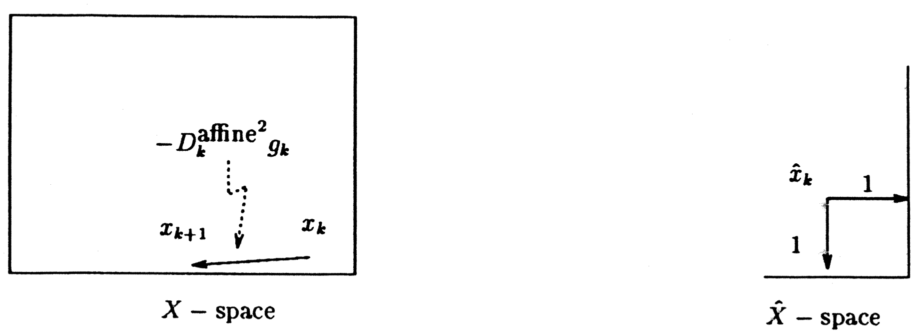

Fig. 2. Effect of affine scaling transformation $\hat{x} = D_k^{\text{affine}^{-1}} x$.

To answer this question we begin with a review of optimality conditions. First-order necessary optimality conditions for problem (1.1) are:

$$\text{first order:} \quad \begin{cases} (g_*)_i = 0 & \text{if } l_i < (x_*)_i < u_i, \\ (g_*)_i \leqslant 0 & \text{if } (x_*)_i = u_i, \\ (g_*)_i \geqslant 0 & \text{if } (x_*)_i = l_i. \end{cases} \tag{2.1}$$

Second-order conditions involve the Hessian matrix of f, $H = H(x) \stackrel{\text{def}}{=} \nabla^2 f(x)$. Let $Free_*$ denote the set of indices corresponding to "free" variables at point x_*:

$$Free_* = \{i\colon l_i < (x_*)_i < u_i\} .$$

Second-order necessary conditions can be written: If a feasible point x_* is a local minimizer of (1.1) then $D_*^2 g_* = 0$ and H_*^{Free*} is positive semi-definite where H_*^{Free*} is the submatrix of $H_* = H(x_*)$ corresponding to the index set $Free_*$.

These conditions are necessary but not sufficient. Sufficiency conditions that are achievable in practice often require a nondegeneracy assumption. This is the case here.

Definition 1. A point $x \in \mathbb{R}^n$ is *nondegenerate* if, for each index i:

$$g_i = 0 \;\Rightarrow\; l_i < x_i < u_i .$$

With this definition we can state **second-order sufficiency conditions:** if a nondegenerate feasible point x_* satisfies $D_*^2 g_* = 0$ and H_*^{Free*} is positive definite, then x_* is a local minimizer of (1.1).

The crucial observation, for our purposes, is that the first-order optimality conditions (2.1) can be written as a nonlinear (diagonal) system of equations. From this system comes a local Newton process, yielding local quadratic convergence, and from this process comes a natural connection with affine scaling ideas and, ultimately, a global method.

The diagonal system

$$D(x)^2 \nabla f(x) = 0 \tag{2.2}$$

is equivalent to the first-order optimality conditions (2.1), where v is defined below and D is given by [1],

$$D(x) = \text{diag}(|v(x)|^{1/2}) . \tag{2.3}$$

Definition 2. The vector $v(x) \in \mathbb{R}^n$ is defined:

(i) If $\nabla f(x)_i < 0$ and $u_i < \infty$ then $v_i \stackrel{\text{def}}{=} x_i - u_i$.

[1] Notation: If z is a vector then $|z|^{1/2}$ denotes a vector with the ith component equal to $|z_i|^{1/2}$.

(ii) If $\nabla f(x)_i \geqslant 0$ and $l_i > -\infty$ then $v_i \stackrel{\text{def}}{=} x_i - l_i$.

(iii) If $\nabla f(x)_i < 0$ and $u_i = \infty$ then $v_i \stackrel{\text{def}}{=} -1$.

(iv) *If* $\nabla f(x)_i \geqslant 0$ and $l_i = -\infty$ then $v_i \stackrel{\text{def}}{=} 1$.

System (2.2) is continuous but not everywhere differentiable. Nondifferentiability occurs when $v_i = 0$. Since only the strictly feasible points $x_k \in \text{int}(\mathcal{F})$ will be generated, this is not a concern. Assume that $x_k \in \text{int}(\mathcal{F})$. Let $J^v(x) \in \mathbb{R}^{n \times n}$ be the Jacobian matrix of $|v(x)|$ whenever $|v(x)|$ is differentiable. The function $|v(x)|$ is not differentiable at a point where there exists some $1 \leqslant i \leqslant n$ or $1 \leqslant j \leqslant n$ with $v_i(x) = 0$ or $g_j(x) = 0$. Since strict feasibility is maintained, $|v(x)_k| > 0$ always. If some $g_i = 0$, we define the ith row J_i^v of J^v to be zero, i.e., $J_i^v \stackrel{\text{def}}{=} 0$. Nondifferentiability of this type is not cause for concern because, for such a component, it is not significant which value v_i takes. Moreover, v_i is discontinuous at such a point but the product $v_i(x) g_i(x)$ is continuous.

A Newton step for (2.2) satisfies

$$\hat{B}_k \hat{d}_k^N = -\hat{g}_k \tag{2.4}$$

where $\hat{d}_k^N = D_k^{-1} d_k^N$ is a Newton step in the new coordinates under the new affine scaling transformation $\hat{x} \stackrel{\text{def}}{=} D_k^{-1} x$ and

$$\hat{g}_k \stackrel{\text{def}}{=} D_k g_k = \text{diag}(|v_k|^{1/2}) g_k,$$

$$\hat{B}_k \stackrel{\text{def}}{=} D_k H_k D_k + \text{diag}(g_k) J_k^v. \tag{2.5}$$

A local, quadratic, and feasible method can be based on (2.4). Feasibility requirements may prohibit a full (unit) step from being taken; however, as we indicate in Section 6, it is possible to set $x_{k+1} = x_k + \alpha_k d_k^N$ such that $\{x_k\}$ is strictly feasible and $\alpha_k \to 1$ sufficiently fast to ensure quadratic convergence. Beyond yielding a local Newton method, Eqs. (2.4) and (2.5) suggest a minimization process. To appreciate this consider Lemma 1.

Lemma 1. *Assume that* $x_* \in \mathcal{F}$.

(a) *If* x_* *is a local minimizer of* (1.1), *then* $\hat{g}_* = 0$.

(b) *If* x_* *is a local minimizer, then* $\hat{B}_*$ *is positive semi-definite and* $\hat{g}_* = 0$.

(c) *If* $\hat{B}_*$ *is positive definite and* $\hat{g}_* = 0$, *then* x_* *is a local minimizer of* (1.1).

This result is easily proved – it follows directly from the optimality conditions.

Lemma 1 indicates that computing a local minimizer of a bound-constrained problem (1.1) is equivalent to locating a point such that $\hat{g}_* = 0$ and $\hat{B}_*$ positive semidefinite. Therefore, loosely speaking, we have transformed a bound-constrained problem (1.1) to a problem of finding a local minimizer for some unconstrained problem.

Though $\hat{g}$ and $\hat{B}$ do not correspond to the gradient and Hessian of a specific nonlinear function, Lemma 1 suggests that, in the new coordinates, a solution of the following trust region subproblem is a reasonable step

$$\min_{\hat{s}}\{\hat{\psi}_k(\hat{s}): \|\hat{s}\|_2 \leqslant \Delta_k\} \tag{2.6}$$

where

$$\hat{\psi}_k(\hat{s}) \stackrel{\text{def}}{=} \hat{g}_k^T\hat{s}+\tfrac{1}{2}\hat{s}^T\hat{B}_k\hat{s}\,.$$

Let $s=D_k\hat{s}$ and $H_k=H(x_k)$. Subproblem (2.6) is equivalent to the following problem in the original variable space:

$$\min_{s}\ \{\psi_k(s): \|D_k^{-1}s\|_2 \leqslant \Delta_k,\} \tag{2.7}$$

where

$$\psi_k(s) \stackrel{\text{def}}{=} s^Tg_k+\tfrac{1}{2}s^TB_ks\,,$$

$$C_k \stackrel{\text{def}}{=} D_k^{-1}\,\mathrm{diag}(g_k)J_k^vD_k^{-1}\,, \tag{2.8}$$

$$B_k \stackrel{\text{def}}{=} H_k+C_k\,.$$

It is clear that $C(x)$ is a positive semi-definite diagonal matrix. This matrix contains the constraint information. Moreover, in the neighborhood of a local minimizer, the Newton step with respect to (2.2) is a solution to the trust region subproblem (2.7) if the trust region size Δ_k is sufficiently large.

Our affine scaling transformation $\hat{x}=D_k^{-1}x$ differs from the affine scaling transformation used for linear programming problems in two regards: D_k depends on the current gradient g_k and a diagonal component is the *squareroot* of the distance of the corresponding variable to its closest bound, if this bound is correct according to g_k. In this case, the scaled steepest descent direction, $-D_k^2g_k$, and the solution of trust region subproblem (2.7), are sufficiently angled away from the approaching correct bound. Our choice of D_k comes naturally from the Newton step with respect to the nonlinear systems characterizing the first order optimality conditions of (1.1). Note that, if $\hat{B}_k$ is positive definite and the ellipsoidal constraint is inactive, then the solution to the reduced trust region problem is $s_k^N=D_k\hat{s}_k^N$ where

$$\hat{s}_k^N=-\hat{B}_k^{-1}\hat{g}_k\,. \tag{2.9}$$

In a neighborhood of sufficiently nondegenerate point satisfying second-order sufficiency, s_k^N is a Newton step for system (2.2). Therefore a global step blends automatically into a Newton step locally and achieves fast local convergence.

At the current point x_k some variables may be approaching the wrong bounds according to the gradient, i.e., $|v_{k_i}|\neq\min(u_i-x_{k_i}, x_{k_i}-l_i)$, for some i. In this case there is no angle property and it is quite possible to reach this bound after only a short step along s_k. However, this is where the notion of a reflection, introduced in the next section, plays its part to ensure sufficient decrease.

The theory we develop in later sections allows for some latitude in the manner in which a descent direction is obtained; we often determine s_k, at x_k, by solving a subspace trust region subproblem. In addition, in order to be able to use the trust region subproblem when a problem is degenerate, we replace the second order matrix B_k in (2.8) by M_k which is a slight modification of B_k. More precisely, we consider a solution of the following trust region subproblem

$$\min_s \{ s^T g_k + \tfrac{1}{2} s^T M_k s \colon \|D_k^{-1} s\|_2 \leqslant \Delta_k,\ s \in \mathscr{S}_k \} \tag{2.10}$$

where $\mathscr{S}_k$ is a subspace of $\mathscr{R}^n$,

$$M_k \overset{\text{def}}{=} H_k + C_k^+ , \qquad C_k^+ \overset{\text{def}}{=} D_k^{-1} \operatorname{diag}(g_k^+) J_k^v D_k^{-1} , \tag{2.11}$$

and the vector $g^+(x)$ is an "extended gradient", extended to deal with possible degeneracy. In particular,

$$g_i^+ \overset{\text{def}}{=} \begin{cases} |g_i| + \tau_\epsilon & \text{if } |g_i| + |v_i|^{1/2} \leqslant \tau_\epsilon , \\ |g_i| & \text{otherwise} , \end{cases} \tag{2.12}$$

where τ_ϵ is a very small positive number, e.g., $\sqrt{\tau_\epsilon}$ and τ_ϵ is the machine precision. Throughout the remaining presentation, we will refer to a point satisfying $|v(x)| + |g(x)| > \tau_\epsilon$ as a sufficiently nondegenerate point. Clearly if x is a sufficiently nondegenerate point, then $g^+ = |g|$, $C^+ = C$ and $B(x) = M(x)$.

Using definition (2.3), problem (2.10) can be written

$$\min_{\hat{s}} \{ \hat{s}^T \hat{g}_k + \tfrac{1}{2} \hat{s}^T \hat{M}_k \hat{s} \colon \|\hat{s}\|_2 \leqslant \Delta_k,\ D_k \hat{s} \in \mathscr{S}_k \} \tag{2.13}$$

where

$$\hat{M}_k = D_k M_k D_k = D_k H_k D_k + J_k^v D_k^{g^+} , \qquad \hat{g}_k = D_k g_k , \qquad \hat{s} = D_k^{-1} s , \tag{2.14}$$

and D^{g^+} is a diagonal matrix, $D^{g^+} \overset{\text{def}}{=} \operatorname{diag}(g^+)$. Similarly, if x is a sufficiently nondegenerate point, then $\hat{M} = \hat{B}$.

Typically subspace $\mathscr{S}_k$ is small, e.g., $|\mathscr{S}_k| = 2$; the issues concerning how to choose $\mathscr{S}_k$ appropriately are addressed in Section 7. A related reduced trust region idea has been explored in the unconstrained minimization setting [3,25]. The solution to (2.10) is of low cost, provided $|\mathscr{S}_k|$ is small.

3. Reflective line search

We employ an unconventional line search technique for our approach. The traditional line search follows a straight line path. We search for an improved point along a reflective path. A two dimensional reflective path is illustrated in Fig. 3.

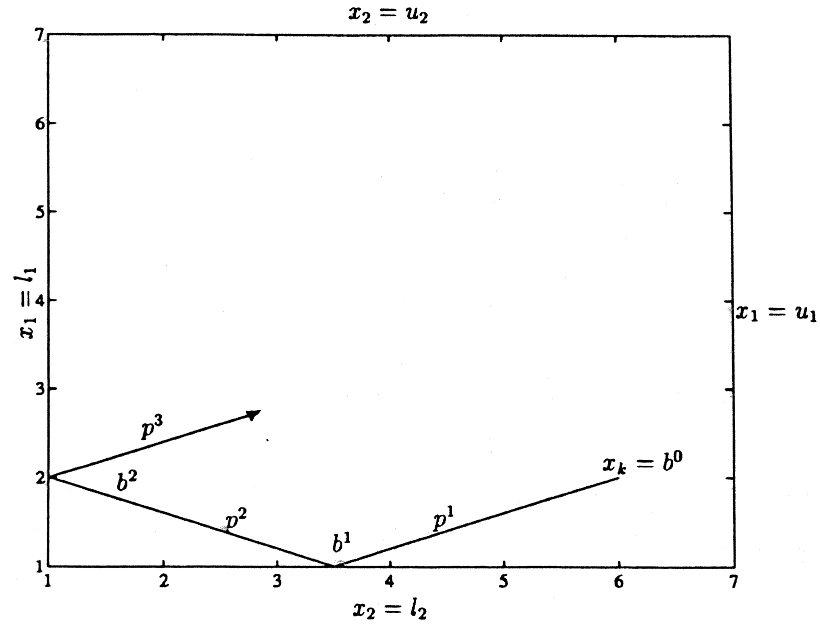

Fig. 3. A reflective path.

The piecewise reflective path can be described, recursively, as follows. Given any descent direction $s_k \in \mathbb{R}^n$, define the vector [2]

$$BR_k = \max[(l - x_k)./s_k, (u - x_k)./s_k)], \tag{3.1}$$

where the notation "./" indicates componentwise division. Component i of vector BR_k records the positive stepsize from x_k to the breakpoint corresponding to variable x_{k_i} in the direction s_k. The piecewise linear (reflective) path is defined by the reflective path described in Fig. 4. Since only a single outer iteration is considered, we do not include the subscript k with the variables in our description of the reflective path below – dependence on k is assumed.

Let $p_k(\alpha)$ denote the reflective path as defined in Fig. 4: For $\beta_k^{i-1} \leqslant \alpha < \beta_k^i$,

$$p_k(\alpha) = b_k^{i-1} + (\alpha - \beta_k^{i-1}) p_k^i. \tag{3.2}$$

Note that the reflective path $p_k(\alpha)$ is defined with respect to the current point and direction under consideration. This dependence is understood and will not be explicitly denoted in the presentation.

A model interior-reflective method is described in Fig. 5.

The notion of a reflective path line search may seem a bit odd at first glance. However, the reflective path line search supports our objective of staying relatively centered. More-

[2] For the purpose of computing BR we assume the following rules regarding arithmetic with infinities. If a is a finite scalar then $a + \infty = \infty$, $a - \infty = -\infty$, $\infty/a = \infty \cdot \text{sgn}(a)$, $-\infty/a = -\infty \cdot \text{sgn}(a)$, $a/0 = \text{sgn}(a) \cdot \infty$, $\infty/0 = \infty$, and $-\infty/0 = -\infty$, where $\text{sgn}(a) = +1$ if $a \geqslant 0$, $\text{sgn}(a) = -1$ if $a < 0$.

The reflective path: [Let $\beta^0=0$, $p^1=s$, set $b^0=x_k$.]
For $i=1:\infty$

1. Let β^i be the distance to the nearest breakpoint along p^i:

   $\beta^i=\min\{BR: BR>0\}$.

2. Define ith breakpoint: $b^i=b^{i-1}+(\beta^i-\beta^{i-1})p^i$.
3. Reflect to get new direction and update BR:
   (a) $p^{i+1}=p^i$
   (b) For each j such that $(b^i)_j=u_j$ (or $(b^i)_j=lj$)
   - $BR(j)=BR(j)+|u_j-l_j/(s)_j|$.
   - $(p^{i+1})_j=-(p^i)_j$.

Fig. 4. Determine the linear reflective path p.

over, as we indicate next, there is a familiar straight line interpretation of the reflective path line search. To see this we introduce the notion of a reflective mapping. For a problem with nonnegativity constraints only, $\mathscr{F}=\{x: x\geqslant 0\}$, a reflective mapping is merely the absolute value function, $R:\mathscr{R}^n \overset{\text{onto}}{\rightarrow} \mathscr{F}$, i.e., $x=R(y)=|y|$, where the absolute value notation is meant to apply to each component. More generally, a reflective mapping (or transformation) for problem (1.1) is an open mapping $R:\mathscr{R}^n \overset{\text{onto}}{\rightarrow} \mathscr{F}$ defined in Fig. 6. An illustration of a 1-dimensional reflective transformation is given in Fig. 7. Using this reflective transformation $R(y)$, (1.1) can be replaced with the unconstrained piecewise differentiable problem:

$$\min_{y\in R^n} \hat{f}(y) \tag{3.3}$$

where $\hat{f}(y)=f(R(y))$.

A model interior-reflective method

Choose $x_1\in \text{int}(\mathscr{F})$.
For $k=1, 2, \ldots$

1. Determine an initial descent direction s_k for f at $x_k\in\text{int}(\mathscr{F})$. Determine the reflective path $p_k(\alpha)$ as in Fig. 4.
2. Perform an approximate piecewise line minimization of $f(x_k+p_k(\alpha))$, with respect to α, to determine an acceptable stepsize α_k (such that α_k does not correspond to a breakpoint).
3. $x_{k+1}=x_k+p_k(\alpha_k)$.

Fig. 5. A model interior-reflective algorithm.

Case 1: $(l_i > -\infty, u_i < \infty)$

To evaluate $x_i = R(y)_i$:

$w_i = |y_i - l_i| \bmod [2(u_i - l_i)],\ x_i = \min(w_i, 2(u_i - l_i) - w_i) + l_i$

Case 2: $(l_i > -\infty, u_i = \infty)$

To evaluate $x_i = R(y)_i$: If $y_i \geqslant l_i$, $x_i = y_i$, else $x_i = 2l_i - y_i$.

Case 3: $(l_i = -\infty, u_i < \infty)$

To evaluate $x_i = R(y)_i$: If $y_i \leqslant u_i$, $x_i = y_i$, else $x_i = 2u_i - y_i$.

Case 4: $(l_i = -\infty, u_i = \infty)$.

In this case there are no constraints on x_i and so $x_i = y_i$.

Fig. 6. The reflective transformation R.

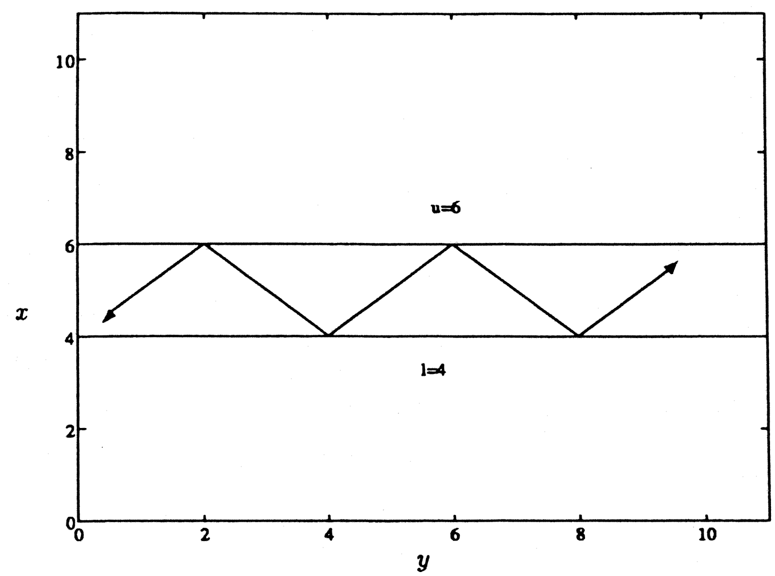

Fig. 7. A 1-dimensional reflective transformation example.

An interior-reflective algorithm for the original problem (1.1) is a descent direction algorithm with a (straight) line search [3] for $\hat{f}(y)$ – see Fig. 8. This straight line search descent algorithm generates the sequence $\{y_k\}$; the strictly feasible sequence $\{x_k\}$ can be obtained from the relation $x_k = R(y_k)$. (Note: strict feasibility is maintained because the line search does not accept breakpoints – breakpoints correspond to points on the boundary.)

The difference between the algorithm in Fig. 8 and the interior-reflective method in Fig. 5 is purely notational. The view presented by Fig. 5 has the advantage that it is in the original space – visualization of the reflective process is natural. The advantage of the second view, the method in Fig. 8, is that the algorithm is a straight line descent direction algorithm, a

[3] Direction s_k^y is a descent direction for $\hat{f}(y)$ at y_k if $\hat{f}(y_k + \alpha s_k^y) < \hat{f}(y_k)$ for all positive sufficiently small α.

A descent direction algorithm for $\hat{f}(y)$

Choose $y_1 \in \text{int}(\mathcal{F})$.

For $k=1, 2, \ldots$

1. Determine a descent direction s_k^y for $\hat{f}(y)$ at y_k
2. Perform an approximate line minimization of $\hat{f}(y_k+\alpha s_k^y)$, with respect to α, to determine an acceptable stepsize α_k (such that α_k does not correspond to a breakpoint)
3. $y_{k+1}=y_k+\alpha_k s_k^y$

Fig. 8. Descent direction algorithm for $\hat{f}(y)$.

familiar structure. It is probably useful for the reader to keep both views in mind. In this paper we will primarily work in the original space (x-space).

The Line Search Conditions: An exact line search is seldom adopted. Instead, an approximate line search satisfying conditions which guarantee convergence is used.

In the unconstrained setting, min $f(x)$, several such sufficiency conditions have been proposed. For example, Goldfarb [16] uses the modified Armijo [1] and Goldstein [17] conditions: Given $0<\sigma_l<\sigma_u<1$ and a descent direction s_k with $x_{k+1}=x_k+\alpha_k s_k$, α_k satisfies the modified Armijo/Goldstein conditions if

$$f(x_{k+1})<f(x_k)+\sigma_l(\alpha_k g_k^T s_k+\tfrac{1}{2}\alpha_k^2\min(s_k^T H_k s_k, 0)) \tag{3.4}$$

and

$$f(x_{k+1})>f(x_k)+\sigma_u(\alpha_k g_k^T s_k+\tfrac{1}{2}\alpha_k^2\min(s_k^T H_k s_k, 0)) . \tag{3.5}$$

Roughly speaking condition (3.4) can be interpreted as restricting the step length from being too large relative to the decrease in f; condition (3.5) can be interpreted as restricting the step length from being relatively too small. Both conditions can be combined to form a single expression: If we define

$$\phi_k(\alpha)=\frac{f(x_{k+1})-f(x_k)}{\alpha_k g_k^T s_k+\frac{1}{2}\alpha_k^2\min(s_k^T H_k s_k, 0)}, \tag{3.6}$$

conditions (3.4) and (3.5) can be expressed as

$$\sigma_l<\phi_k(\alpha_k)<\sigma_u . \tag{3.7}$$

We establish that conditions (3.4) and (3.5) can be satisfied for the relative path minimization process where $x_{k+1}=x_k+p_k(\alpha_k)$ and p_k is defined by (3.2). In particular, we prove that there is an interval (α_l, α_u), depending on k, such that for all $\alpha\in(\alpha_l, \alpha_u)$, (3.7) is satisfied.

Theorem 2. *Assume that $f(x)$ has two continuous derivatives and either $g_k^T s_k < 0$ or $g_k^T s_k = 0$ and $s_k^T H_k s_k < 0$ where $x_k \in \text{int}(\mathscr{F})$. Then either f is unbounded below along the reflective path $p_k(\alpha)$ or, for $0 < \sigma_l < \sigma_u < 1$, there exists an interval (α_l, α_u), depending on k, such that condition (3.7) is satisfied.*

Proof. First we denote that $\lim_{\alpha \to 0} \phi_k(\alpha) = 1$. To see this consider that from Taylor's theorem, for $\alpha < \beta_k^1$,

$$\phi_k(\alpha) = \frac{\alpha g_k^T s_k + \frac{1}{2}\alpha^2 s_k^T \bar{H}_k s_k}{\alpha g_k^T s_k + \frac{1}{2}\alpha^2 \min(s_k^T H_k s_k, 0)},$$

where

$$\bar{H}_k = H(x_k + \theta(\alpha)\alpha s_k), \quad 0 \leqslant \theta(\alpha) \leqslant 1 .$$

Therefore, if $g_k^T s_k \neq 0$, $\phi_k(0) \overset{\text{def}}{=} \lim_{\alpha \to 0} \phi_k(\alpha) = 1$ and so $\phi_k(0) > \sigma_u > \sigma_l$; if $g_k^T s_k = 0$ then $s_k^T H_k s_k < 0$ and clearly $\phi_k(0) \overset{\text{def}}{=} \lim_{\alpha \to 0} \phi_k(\alpha) = 1$ and so $\phi_k(0) \overset{\text{def}}{=} \lim_{\alpha \to 0} \phi_k(\alpha) = 1$.

Assume $\phi_k(\alpha) \leqslant \sigma_l$ for some $\alpha > 0$. Let α_u be the smallest positive α such that $\phi_k(\alpha) = \sigma_l$. Since $\phi_k(0) > \sigma_u > \sigma_l$ it follows that $\phi_k(\alpha) > \sigma_l$ for all $\alpha \in (0, \alpha_u)$. Therefore by continuity there exists a positive $\alpha_l < \alpha_u$ such that $\phi_k(\alpha) < \sigma_u$ for all $\alpha \in (\alpha_l, \alpha_u)$. Therefore (3.7) is satisfied on (α_l, α_u).

Now assume the contrary; i.e., $\phi_k(\alpha) > \alpha_l$ for all positive α. But since either $g_k^T s_k < 0$ or $s_k^T g_k = 0$ and $s_k^T H_k s_k < 0$, it follows that

$$\lim_{\alpha \to \infty} \alpha g_k^T s_k + \tfrac{1}{2}\alpha^2 \min(s_k^T H_k s_k, 0) = -\infty .$$

Therefore to achieve $\phi_k(\alpha) > \alpha_l$, for all positive α, it must be that

$$\lim_{\alpha \to \infty} f(x_k + p_k(\alpha)) - f(x_k) = -\infty .$$

Consequently f is unbounded below along the path $p_k(\alpha)$ as $\alpha \to \infty$. □

The interval (α_l, α_u) contains a finite number of breakpoints. Consequently, we can choose $\alpha_k \in (\alpha_l, \alpha_u)$ such that α_k is not a breakpoint.

A basic interior-reflective algorithm can now be stated. To allow for flexibility, especially with regard to the Newton step, we do not always require that both (3.4) and (3.5) be satisfied. Instead, we demand that either both conditions are satisfied or (3.4) is satisfied and α_k is guaranteed to be bounded away from zero, e.g., $\alpha_k > \rho > 0$. The latter conditions are used to allow for the liberal use of Newton steps and do not weaken the global convergence results.

An interior-reflective method is described in Fig. 9. Note that since $x_1 \in \text{int}(\mathscr{F})$, it follows that $x_k \in \text{int}(\mathscr{F})$.

We conclude this section with two comments on the economy of the line search. First we

An interior-reflective algorithm [ρ is a positive scalar.]

Choose $x_1 \in \text{int}(\mathscr{F})$.

For $k=1,2,\ldots$

1. Determine an initial descent direction s_k for f at x_k. Note that the piecewise linear path $p_k(\alpha)$ is defined by x_k, s_k.
2. Perform an approximate piecewise line minimization of $f(x_k+p_k(\alpha))$, with respect to α, to determine α_k such that:
   (a) α_k does not correspond to a breakpoint
   (b) condition (3.4) is satisfied
   (c) Either
      (i) α_k satisfies condition (3.5), or
      (ii) $\alpha_k > \rho > 0$
3. $x_{k+1} = x_k + p_k(\alpha_k)$.

Fig. 9. An interior-reflective algorithm satisfying line search conditions.

mention that it is not necessary to implement a line search in a left-to-right fashion; it is not necessary to predetermine the reflective path. For example, a simple bisection strategy can be quite effective in some cases [10]. Second, as indicated in Section 6, distances to the breakpoints that correspond to variables tight at the solution converge to unity (under nondegeneracy assumptions). This indicates that in a neighbourhood of a minimizer it is unnecessary to reflect.

4. Constraint compatibility and consistency

Satisfaction of the conditions (3.4) and (3.5) is not sufficient to ensure convergence for (1.1). In this section, we attempt to capture the effect of our new affine scaling transformation $\hat{x}=D_k x$ on the steepest descent direction $-g_k$ and the solution of the trust region subproblem (2.7). We indicate that the scaled steepest descent $\{-D_k^2 g_k\}$ and the trust region solution $\{p_k\}$ of (2.7) share two properties, *constraint compatibility* and *consistency*, which are enough to obtain first-order convergence, i.e., to guarantee that $\{D_k^2 g_k\} \to 0$.

We begin with a discussion of constraint-compatibility. Recall that the diagonal matrix D_k is defined by (2.3), i.e., $D_k^2 = D(x_k)^2 = \text{diag}(|v_k|)$.

Definition 3. A sequence of vectors $\{s_k\}$ is *constraint-compatible* if the sequence $\{D_k^{-2} s_k\}$ is bounded.

Constraint-compatibility of $\{s_k\}$ is important because it facilitates a sufficiently long step along s_k. In particular, if x_k is close to a boundary then a direction satisfying only $g_k^T s_k < 0$

may not guarantee that sufficient progress can be made to obtain a convergence result. In particular, s_k may point directly at a nearby constraint and descent beyond this first breakpoint, along p_k, is not guaranteed. (Conditions (3.4) and (3.5) can still be satisfied though.) For constraint-compatible directions, we prove the following.

Theorem 3. *If* $\{s_k\}$ *is a constraint-compatible sequence then* $\{BR_k(j)\colon BR_k(j) = |v_{kj}| / |s_{kj}|\}$ *is bounded away from zero.*

Proof. By constraint compatibility there exists $\chi > 0$ such that, for all iterations k and all indices j,

$$\frac{|s_{kj}|}{|v_{kj}|} \leqslant \chi .$$

Clearly if $BR_k(j) = |v_{kj}| / |s_{kj}|$, then $BR_k(j) \geqslant 1/\chi$. □

In other words, constraint-compatibility helps avoid the problem of running directly into a bound by ensuring that the stepsizes to breakpoints, corresponding to "correct sign conditions", remain bounded away from zero. Specifically, if $\{s_k\}$ is constraint-compatible then the positive distance to constraint j along s_k, $BR_k(j) = \max\{(l_j - x_{kj})/s_{kj},\ (u_j - x_{kj})/s_{kj}\}$, is bounded away from zero for any j with the correct "sign condition". The "sign condition" refers to a consistency between v_j and $\max\{(l_j - x_{kj})/s_{kj},\ (u_j - x_{kj})/s_{kj}\}$. The "sign condition" holds when $s_{kj} g_{kj} < 0$, and so $BR_k(j) = |v_{kj}| / |s_{kj}|$.

When the "sign condition" is violated, i.e., when $s_{kj} g_{kj} > 0$, then it is possible to hit a bound after only a short step along s_k. However, the reflective line search guarantees that the new direction passing this breakpoint will maintain descent if the bound is encountered soon enough (since $-s_{kj} g_{kj} < 0$). This reflection is essential for convergence: see Lemma 7.

Summarizing the above discussion, constraint-compatibility together with the reflective line search guarantee sufficient progress from the current point.

A technical lemma is required to establish constraint-compatibility of some useful directions.

Lemma 4. *Let* $\{s_k\}$ *be a sequence of vectors and assume* $\{s_k\}$ *is bounded. Assume that for each iteration k and each index i such that* $0 < |v_{ki}| < 1$,

$$e_{k_i} s_{k_i} = |v_{k_i}| z_{k_i} , \tag{4.1}$$

where e_{k_i} *satisfies* $|e_{k_i}| \geqslant g^+_{k_i}$. *Assume* $\{z_k\}$ *is bounded. Then* $\{s_k\}$ *is constraint-compatible.*

Proof. Consider any subsequence, denoted by indices $\bar{k}$. If $\{v_{\bar{k}_i}\}$ is bounded away from zero then $\{s_{\bar{k}_i} / |v_{\bar{k}_i}|\}$ is bounded since, by assumption, $\{s_k\}$ is bounded. On the other hand, if $\{v_{\bar{k}_i}\} \to 0$ then by (2.12), $|e_{\bar{k}_i}| \geqslant \tau_\epsilon > 0$. But $\{z_{k_i}\} = \{e_{k_i} s_{k_i} / |v_{k_i}|\}$ is bounded by assumption; therefore, $\{s_{\bar{k}_i} / |v_{\bar{k}_i}|\}$ is bounded. Since every subsequence of $\{s_{k_i} / |v_{k_i}|\}$ is bounded, the sequence itself is bounded. □

Theorem 5 below establishes that several useful directions satisfy the constraint compatibility requirement. The sequences $\{D_k u_k\colon \hat{M}_k u_k = \mu_k u_k \text{ for some } \mu_k \leqslant 0\}$ and $\{-D_k^2 \operatorname{sgn}(g_k)\}$ play an important role in forming subspaces of a reduced trust region subproblem (2.10) (see Section 7).

Theorem 5. *Assume* $0 < \Delta_l \leqslant \Delta_k \leqslant \Delta_u < \infty$, *where* Δ_l *and* Δ_u *are positive scalars satisfying* $\Delta_l < \Delta_u$. *Under the compactness and smoothness assumption, the following definitions yield constraint-compatible sequences* $\{s_k\}$:

(1) $s_k = -D_k^2 g_k$,

(2) $s_k = -D_k^2 \operatorname{sgn}(g_k)$, [4]

(3) $s_k = D_k u_k$, *where* u_k *with* $\|u_k\|_2 = 1$ *is an eigenvector of* $\hat{M}_k$ *corresponding to a non-positive eigenvalue*,

(4) $s_k = D_k \hat{s}_k^N$ *where* $\hat{s}_k^N$ *is the Newton step in the scaled space*, $\hat{s}_k^N = -\hat{M}_k^{-1} \hat{g}_k$, $\hat{g}_k = D_k g_k$, $\|\hat{s}_k^N\| \leqslant \Delta_k \leqslant \Delta_u$ *and* $\hat{M}_k$ *positive definite*,

(5) $s_k = D_k \hat{s}_k^N / \|\hat{s}_k^N\|$, $\hat{s}_k^N\| \geqslant \Delta_k \geqslant \Delta_l$ *and* $\hat{M}_k$ *positive definite*,

(6) s_k *is the solution to* (2.10) *with* $\mathcal{S}_k = \mathcal{R}^n$.

Proof. Constraint-compatibility of the first two choices for s_k follows directly from the definition and boundedness of $\{g_k\}$.

For case 3, let $\mu_k \leqslant 0$ be a nonpositive eigenvalue of $\hat{M}_k$ and $\hat{M}_k u_k = \mu_k u_k$. Then

$$(\mu_k I - J_k^v D_k^{g+}) s_k = D_k^2 H_k D_k u_k , \quad \mu_k \leqslant 0 ,$$

where $D_k^{g+} = \operatorname{diag}(g_k^+)$. For each index i with $|v_{k_i}| < 1$, $J_{k_i}^v = 1$ and $|\mu_{k_i} I - J_{k_i}^v D_k^{g+}| \geqslant g_{k_i}^+$. Using compactness, $\{H_k D_k u_k\}$ and $\{s_k\} = \{D_k u_k\}$ are bounded. Therefore, by Lemma 4, $\{s_k\}$ is constraint-compatible.

For case 4, note that s_k satisfies

$$J_k^v D_k^{g+} s_k = -D_k^2 (g_k + H_k D_k \hat{s}_k^N) .$$

But if $\|\hat{s}_k^N\| \leqslant \Delta_k \leqslant \Delta_u$ then, using compactness, both $\{g_k + H_k D_k \hat{s}^{N_k}\}$ and $\{s_k\}$ are bounded. Constraint-compatibility then follows from Lemma 4.

In case 5,

$$J_k^v D_k^{g+} s_k = -D_k^2 \left(\frac{g_k}{\|\hat{s}_k^N\|} + \frac{H_k D_k \hat{s}_k^N}{\|\hat{s}_k^N\|} \right).$$

But $\|\hat{s}_k^N\| \geqslant \Delta_k \geqslant \Delta_l > 0$; therefore, using compactness, $\{g_k / \|\hat{s}_k^N\| + H_k D_k \hat{s}_k^N / \|\hat{s}_k^N\|\}$ is bounded. The sequence $\{s_k\}$ is bounded since $s_k = D_k \hat{s}_k^N / \|\hat{s}_k^N\|$; constraint-compatibility follows from Lemma 4.

Finally in case 6 note that s_k satisfies

[4] If z is a vector, then $w = \operatorname{sgn}(z)$ is a vector; $w_i = 1$ if $z_i \geqslant 0$, $w_i = -1$ if $z_i < 0$.

$$(J_k^v D_k^{g+} + \mu_k I) s_k = -D_k^2(g_k + H_k D_k \hat{s}_k) \tag{4.2}$$

for some $\mu_k \geqslant 0$ and $\hat{s}_k = D_k^{-1} s_k$. But $\|\hat{s}_k\| \leqslant \Delta_k \leqslant \Delta_u$ and so, using compactness, both $\{g_k + H_k D_k \hat{s}_k\}$ and $\{s_k\}$ are bounded. Therefore, Lemma 4 can be applied to yield constraint-compatibility. □

Note that a constraint-compatible sequence $\{s_k\}$ can be obtained by mixing the various steps s_k given in Theorem 5. Constraint-compatibility is not sufficient to guarantee convergence: It is also important that first-order descent, represented by $g_k^T s_k$, be *consistent* with first-order optimality, represented by $D_k^2 g_k$. The following definition captures this concept.

Definition 4. A sequence $\{s_k\}$ satisfies the *consistency condition* if $\{s_k^T g_k\} \to 0$ implies $\{D_k g_k\} \to 0$.

In Theorem 6 we give five useful examples of sequences that satisfy consistency.

Theorem 6. *Under the compactness and smoothness assumption, the following definitions yield sequences $\{s_k\}$ satisfying the consistency condition.*

(1) $s_k = -D_k^2 g_k$,

(2) $s_k = -D_k^2 \mathrm{sgn}(g_k)$,

(3) $s_k = D_k \hat{s}_k^N$ *where* $\hat{s}_k^N = -\hat{M}_k^{-1} \hat{g}_k$, *assuming* $\hat{M}_k$ *is symmetric positive definite,*

(4) s_k *is a solution to* (2.10) *where* $\mathscr{S}_k$ *has the property that* $w_k = D_k \hat{w}_k \in \mathscr{S}_k$ *for somevector* $\hat{w}_k$ *such that* $\{\|\hat{w}_k\|\}$ *is bounded away from zero and* $\{w_k\}$ *is consistent, i.e.,* $\{w_k^T g_k\} \to 0$ *implies* $\{D_k g_k\} \to 0$,

(5) s_k *is a solution to* (2.10) *with* $\mathscr{S}_k = \mathscr{R}^n$.

Proof.

(1) The first case is clear since $s_k^T g_k = \|D_k g_k\|_2^2$.

(2) In this case $s_k^T g_k = \mathrm{sgn}(g_k)^T D_k^2 g_k = \|D_k |g_k|^{1/2}\|$, and so the result follows.

(3) If s_k is the Newton step, then

$$-g_k^T s_k = (D_k g_k)^T \hat{M}_k^{-1} (D_k g_k) .$$

But by compactness $\hat{M}_k$ is bounded, i.e., there exists a finite bound χ_M such that $\|\hat{M}_k\|_2 \leqslant \chi_M$. Therefore, $-g_k^T s_k \geqslant (1/\chi_M) \|D_k g_k\|^2$. The result follows.

(4) Let $\mathscr{S}_k = \langle V_k \rangle$ for some full-column rank matrix V_k; let Y_k be an orthonormalization of the columns of $D_k^{-1} V_k$. Since $w_k \in \mathscr{S}_k$ we can assume, without loss of generality, that one of the columns of Y_k is $\hat{w}_k / \|\hat{w}_k\|$. We can write the solution to (2.10) as $s_k = D_k Y_k s_{Y_k}$, where

$$s_{Y_k} = -(Y_k^T \hat{M}_k Y_k + \mu_k I)^+ Y_k^T \hat{g}_k + \omega_k u_k^1$$

where u_k^1 is a unit eigenvector corresponding to the most negative eigenvalue of $Y_k^T \hat{M}_k Y_k$ and $(Y_k^T \hat{g}_k)^T u_k^1 = 0$. Using a trust region solution characterization, e.g., [26], the matrix $Y_k^T \hat{M}_k Y_k + \mu_k I$ is positive semi-definite and $(Y_k^T \hat{g}_k) \in \text{range}(Y_k^T \hat{M}_k Y_k + \mu_k I)$. Since $\Delta_k \geqslant \Delta_1 > 0$, it follows that $\{\mu_k\}$ is bounded above. Therefore, using compactness, $\{Y_k^T \hat{M}_k Y_k + \mu_k I\}$ is bounded and so there exists a positive scalar χ_M such that

$$\|Y_k^T \hat{M}_k Y_k + \mu_k I\|_2 \leqslant \chi_M .$$

Therefore,

$$-g_k^T s_k = (Y_k D_k g_k)^T (Y_k^T \hat{M}_k Y_k + \mu_k I)^+ Y_k^T (D_k g_k) \geqslant \frac{1}{\chi M} \|Y_k^T D_k g_k\|^2 .$$

Therefore $\{s_k^T g_k\} \to 0$ implies $\{\|Y_k^T D_k g_k\|\} \to 0$. However, $\hat{w}_k / \|\hat{w}_k\|$ is a column of Y_k and $\{\|\hat{w}_k\|\}$ is bounded from zero. Therefore, $\{\|Y_k^T D_k g_k\|\} \to 0$ implies $\{w_k^T g_k\} \to 0$ which implies $\{D_k g_k\} \to 0$ since $\{w_k\}$ is consistent (by assumption).

(5) The proof follows from the above case 4 by letting $\mathscr{S}_k = \mathbb{R}^n$ and $Y_k = I$. □

5. First-order convergence of the interior-reflective algorithm

In this section we establish that constraint-compatibility and consistency allow the interior-reflective algorithm in Fig. 9 to achieve first-order convergence. Recall that a feasible point x is a first-order point if and only if $D^2(x) g(x) = 0$ where D is defined by (2.3)

The main result of this section is that, assuming that $\{s_k\}$ satisfies the constraint-compatible and consistency conditions, every limit point generated by an interior-reflective method in Fig. 9 is a first-order point. We first state a technical result which says that the change in f along the reflective path $p_k(\alpha)$ is primarily represented by linear term $g_k^T s_k$ as $\alpha_k \to 0$.

Lemma 7. *Assume that $\{x_k\}$ is generated by the interior-reflective algorithm in Fig. 9. Let $\{s_k\}$ be a sequence satisfying the constraint-compatibility conditions. Assume $\{\alpha_k\} \to 0$. Then,*

$$f(x_{k+1}) - f(x_k) = \alpha_k g_k^T s_k + O(\alpha_k^2) .$$

Proof. From Theorem 3 and $\{\alpha_k\} \to 0$, if $0 < \beta_k^i < \alpha_k$ corresponding to variable x_j, then $s_{k_j} g_{k_j} \geqslant 0$ where β_k^i is defined by the reflective path in Fig. 4. Moreover, the compactness assumption implies that a sequence of constraint-compatible direction $\{s_k\}$ is bounded.

Without loss of generality, and for notational simplicity, suppose that the ordering of the breakpoints along s_k corresponds to the natural variable ordering. Note that since $\{\alpha_k\} \to 0$ we can assume that the indices corresponding to $0 < \beta_k^i < \alpha_k$ are distinct and so $\beta_k^i = BR_k(i)$ where BR is defined by (3.1). Assume that

$$0 \leqslant \beta_k^j < \alpha_k < \beta_k^{t_k+1}, \quad j = 1{:}\ t_k . \tag{5.1}$$

Therefore,

$$s_{k_j} g_{k_j} \geqslant 0, \quad j=1:t_k. \tag{5.2}$$

By definition of the reflective path p_k (see Fig. 4) and using (5.2),

$$g_k^T s_k \geqslant g_k^T p_k^j, \quad j=1:t_k+1. \tag{5.3}$$

Now using the definition of the breakpoints b_j^k (see Fig. 4) and applying Taylor's theorem (repeatedly),

$$\begin{aligned}
&f(x_{k+1}) - f(x_k) \\
&= f(x_{k+1}) - f(b_k^{t_k}) + \sum_{j=2}^{t_k} [f(b_k^j) - f(b_k^{j-1})] + f(b_k^1) - f(x_k) \\
&= (\alpha_k - \beta_k^{t_k}) \nabla f(b_k^{t_k})^T p_k^{t_k+1} + \sum_{j=2}^{t_k} [\beta_k^j - \beta_k^{j-1}] \nabla f(b_k^{j-1})^T p_k^j \\
&+ \beta_k^1 \nabla f(x_k)^T p_k^1 + O(\alpha_k^2) \\
&= (\alpha_k - \beta_k^{t_k}) g_k^T p_k^{t_k+1} + \sum_{j=2}^{t_k} [\beta_k^j - \beta_k^{j-1}] g_k^T p_k^j + \beta_k^1 g_k^T p_k^1 + O(\alpha_k^2).
\end{aligned}$$

Now apply (5.3) to get

$$\begin{aligned}
f(x_{k+1}) - f(x_k) &\leqslant (\alpha_k - \beta_k^{t_k}) g_k^T s_k + \sum_{j=2}^{t_k} [\beta_k^j - \beta_k^{j-1}] g_k^T s_k + \beta_k^1 g_k^T s_k + O(\alpha_k^2) \\
&= \alpha_k g_k^T s_k + O(\alpha_k^2). \quad \square
\end{aligned}$$

The main result in this section is first-order convergence, i.e., $\{D_k^2 g_k\} \to 0$. This result is established in Theorem 8 where we also show that $\{\alpha_k^2 \min(s_k^T H_k s_k, 0)\} \to 0$; the latter is not part of the first-order conditions but is useful subsequently.

Theorem 8. *Assume that $\{x_k\}$ is a sequence generated by the interior-reflective path algorithm in Fig. 9 and that $\{s_k\}$ is the corresponding sequence satisfying both the consistency and constraint-compatibility conditions. Then the corresponding sequences $\{D_k^2 g_k\}$ and $\{\alpha_k^2 \min(s_k^T H_k s_k, 0)\}$ converge to zero.*

Proof. Since condition (3.4) is satisfied,

$$\begin{aligned}
f(x_m) - f(x_1) &= \sum_{k=1}^{m-1} (f(x_{k+1}) - f(x_k)) \\
&< \sum_{k=1}^{m-1} (\sigma_l \alpha_k g_k^T s_k + \tfrac{1}{2} \sigma_l \alpha_k^2 \min(s_k^T H_k s_k, 0)) \\
&\leqslant 0.
\end{aligned}$$

By the compactness and smoothness assumption, $\{f(x)\}$ is bounded on $\mathscr{L}$; therefore,

$$\lim_{k\to\infty} (\sigma_l\alpha_k g_k^T s_k + \tfrac{1}{2}\sigma_l\alpha_k^2 \min(s_k^T H_k s_k, 0)) = 0 .$$

But

$$\sigma_l\alpha_k g_k^T s_k \leqslant 0 \quad \text{and} \quad \sigma_l\alpha_k^2 \min(s_k^T H_k s_k, 0) \leqslant 0 ,$$

hence

$$\lim_{k\to\infty} \alpha_k g_k^T s_k = 0 \quad \text{and} \quad \lim_{k\to\infty} \alpha_k^2 \min(s_k^T H_k s_k, 0) = 0 .$$

Now we establish that $\{D_k^2 g_k\}$ converges to zero by contradiction. Suppose this is not true. Since $\{s_k\}$ satisfies the consistency condition, $\{g_k^T s_k\}$ does not converge to zero. Hence $g_k^T s_k < -\chi$ for some $\chi > 0$. Therefore, $\{\alpha_k\}$ converges to zero. Using Lemma 7,

$$\begin{aligned}\lim_{k\to\infty} \phi_k(\alpha_k) &= \lim_{k\to\infty} \frac{f(x_{k+1}) - f(x_k)}{\alpha_k g_k^T s_k + \frac{1}{2}\alpha_k^2 \min(s_k^T H_k s_k, 0)} \\ &\geqslant \lim_{k\to\infty} \frac{\alpha_k g_k^T s_k + O(\alpha_k^2)}{\alpha_k g_k^T s_k + \frac{1}{2}\alpha_k^2 \min(s_k^T H_k s_k, 0)} \\ &= 1 .\end{aligned}$$

This contradicts (3.5); hence, $\{D_k^2 g_k\}$ converges to zero. □

Theorems 5 and 6 provide several examples of directions satisfying consistency and constraint-compatibility; therefore, by Theorem 8, first-order convergence is achieved by the interior-reflective path algorithm in Fig. 9 with these choices. As an example, if we let $s_k = -D_k^2 g_k$ in the interior-reflective approach described in Fig. 5, the resulting method generates iterations $\{x_k\}$ with the property that, at every limit point, the first order necessary conditions are satisfied. In order to achieve better convergence properties, we need to use more sophisticated directions such as solutions of trust region subproblems (2.10).

6. Second-order convergence

In order to achieve a second-order algorithm (i.e., guarantee convergence to a second-order point; obtain quadratic convergence) we further restrict the descent direction s_k in the interior-reflective algorithm in Fig. 9. In particular, we now assume that when $\hat{M}_k$ is positive definite and $\|\hat{s}_k^N\| \leqslant \Delta_k$ then the Newton direction $s_k = D_k \hat{s}_k^N$ is taken; if $\hat{M}_k$ is not positive definite, the direction s_k is defined by a reduced trust region problem [5]: s_k solves

$$\min_s \{s^T g_k + \tfrac{1}{2} s^T M_k s \colon \|D_k^{-1} s\|_2 \leqslant \Delta_k,\ s \in \mathscr{S}_k\} . \tag{6.1}$$

[5] We do not (yet) specify how s_k might be determined when $\hat{M}_k$ is positive definite and $\|\hat{s}_k^N\| > \Delta_k$.

An interior-reflective Newton algorithm: Given $\Delta_l < \Delta_u$.

Choose $x_1 \in \mathrm{int}(\mathcal{F})$.

For $k = 1, 2, \ldots,$

1. Determine an initial descent direction s_k for f at x_k: If $\hat{M}_k$ is positive definite and $\|\hat{s}_k^N\| \leqslant \Delta_k$, choose $s_k = D_k \hat{s}_k^N$. If $\hat{M}_k$ is not positive definite, choose $\Delta_k \in [\Delta_l, \Delta_u]$, choose subspace $\mathcal{S}_k$, and solve (6.1) to get s_k.
2. Determine α_k: If $s_k = s_k^N$ and $x_k + p_k(1)$ satisfies (3.4), then set $\alpha_k = 1$; otherwise, perform an approximate piecewise line minimization of $f(x_k + p_k(\alpha))$, with respect to α, to determine α_k such that
   (a) α_k is not a breakpoint;
   (b) α_k satisfies (3.4) and (3.5).
3. $x_{k+1} = x_k + p_k(\alpha_k)$.

Fig. 10. A second-order interior-reflective Newton algorithm.

Fig. 10 describes a (second-order) interior-reflective Newton algorithm.

Note. If $\alpha_k = 1$ is accepted by the line search but corresponds to a breakpoint, then modify α_k: $\alpha_k = \tilde{\alpha}_k \overset{\mathrm{def}}{=} 1 - \epsilon_k$ where $\tilde{\alpha}_k$ is not a breakpoint, $\tilde{\alpha}_k$ satisfies (3.4), and $\epsilon_k < \chi_\alpha \|D_k g_k\|$ for some $\chi_\alpha > 0$.

The first important result of this section (Theorem 10) is that the method in Fig. 10 generates points $\{x_k\}$ such that the second-order necessary conditions are satisfied at every sufficiently nondegenerate limit point of $\{x_k\}$, provided $\{s_k\}$ is constraint-compatible, satisfies the consistency conditions, and $\mathcal{S}_k$ is chosen so that negative curvature of $\hat{M}_k$ is "well-represented".

A preliminary technical result is required. We denote the smallest eigenvalue of a real symmetric matrix A by $\lambda_{\min}(A)$. So if $\lambda(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, with $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$, then $\lambda_{\min}(A) = \lambda_1$.

Lemma 9. *Assume that $\{x_k\}$ is generated by the interior-reflective Newton algorithm in Fig. 10 where the initial point is strictly feasible. Let $\{s_k\}$ satisfy the consistency and constraint-compatibility conditions. Let $\mathcal{S}_k = \langle Y_k \rangle$, for some orthonormal matrix Y_k, be chosen such that when $\lambda_{\min}(\hat{M}_k) \leqslant 0$,*

$$\lambda_{\min}(Y_k^T \hat{M}_k Y_k) \leqslant \max(-\epsilon_{nc}, \tau \lambda_{\min}(\hat{M}_k)), \tag{6.2}$$

for some $\epsilon_{nc} > 0$, $\tau > 0$. Then for any subsequence satisfying $\{\min(s_k^T H_k s_k, 0)\} \to 0$, the corresponding subsequence satisfies $\lim_{k\to\infty}\{\min(\lambda_{\min}(\hat{M}_k), 0)\} = 0$.

Proof. In this proof the subscript k is identified with the subsequence under consideration. By definition s_k satisfies

$$s_k^T H_k s_k + s_{Y_k}^T Y_k^T D_k^{g+} Y_k s_{Y_k} + \mu_k \|s_{Y_k}\|^2 = -s_{Y_k}^T Y_k^T D_k g_k .$$

But by Theorem 8 $\lim_{k\to\infty} D_k g_k = 0$. By assumption $\lim_{k\to\infty}\{\min(s_k^T H_k s_k, 0)\} = 0$, and D_k^{g+} is positive semidefinite by definition (2.12). Moreover, since s_k solves (6.1), $\|S_{Y_k}\| = \Delta_k \geqslant \Delta_1 > 0$; therefore,

$$\lim_{k\to\infty} \{\mu_k\} = 0 .$$

However,

$$0 \leqslant -\min(\lambda_{\min}(Y_k^T \hat{M}_k Y_k), 0) \leqslant \mu_k ,$$

hence

$$\lim_{k\to\infty} \{\min(\lambda_{\min}(Y_k^T \hat{M}_k Y_k), 0)\} = 0 ,$$

and applying assumption (6.2),

$$\lim_{k\to\infty} \{\min(\max(-\epsilon_{nc}, \tau\lambda_{\min}(\hat{M}_k)), 0)\} = 0 .$$

Therefore

$$\lim_{k\to\infty} \{\min(\lambda_{\min}(\hat{M}_k), 0)\} = 0 . \qquad \square$$

Theorem 10. *Assume that x_* is a nondegenerate limit point of $\{x_k\}$. If the assumptions of Lemma 9 hold, then $\lambda_{\min}(\hat{M}_*)$ is positive semi-definite.*

Proof. Our proof is by contradiction. Assume $\lambda_{\min}(\hat{M}_*) < 0$. Applying Lemma 9, this means that there exists a subsequence with

$$\lim_{k\to\infty} \min(s_k^T H_k s_k, 0) < 0 .$$

Using Theorem 8, $\lim_{k\to\infty} \alpha_k \min(s_k^T H_k s_k, 0) = 0$; hence, $\lim_{k\to\infty} \alpha_k = 0$.

By Theorem 8, $D_* g_* = 0$, and by assumption, x_* is a nondegenerate point; therefore, for k sufficiently large, $\operatorname{sgn}(g_{k_j}) = \operatorname{sgn}(g_{*j})$ if $j \notin Free_*$. Hence, for any $j \notin Free_*$, $BR_k(j) = |v_{k_j}| / |s_{k_j}|$. Alternatively, if $j \notin Free_*$, then $|BR_k(j)| \to \infty$. By Theorem 3, $\{BR_k(j): BR_k(j) = |v_{k_j}| / |s_{k_j}|\}$ is bounded away from zero. It follows, since $\alpha_k \to 0$, that $0 \leqslant \alpha_k < \beta_k^1$ for sufficiently large k, where β is defined in Fig. 4. Therefore, due to the absence of breakpoints on $(0, \alpha_k)$, Taylor's Theorem can be applied straightforwardly to yield, for some subsequence:

$$\lim_{k\to\infty} \phi_k(\alpha_k) = \lim_{k\to\infty} \frac{f(x_k+\alpha_k s_k) - f(x_k)}{\alpha_k g_k^T s_k + \frac{1}{2}\alpha_k^2 \min(s_k^T H_k s_k, 0)}$$

$$= \lim_{k\to\infty} \frac{\alpha_k g_k^T s_k + \frac{1}{2}\alpha_k^2 s_k^T H(x_k + \theta(\alpha_k)) s_k}{\alpha_k g_k^T s_k + \frac{1}{2}\alpha_k^2 s_k^T H_k s_k}, \quad 0 \leqslant \theta(\alpha_k) \leqslant \alpha_k$$

$$= 1 .$$

This contradicts condition (3.5). Hence we conclude that every nondegenerate limit point is a second order point. □

Next we work toward establishing convergence of the entire sequence $\{x_k\}$.

First we establish that there is a natural (local) Newton process for problem (1.1). This view is similar to the development given in [6] for the convex quadratic problem. Let x_* be a specified nondegenerate point satisfying the second-order sufficiency conditions.

Consider a finite set $\mathscr{V}$ of functions defined by x_*:

$$F_\nu(x) = D_\nu(x) g(x) \tag{6.3}$$

where $D_\nu(x) = \mathrm{diag}(\nu(x))$ and $\nu(x)$ is a vector defined

$$\nu_i = \begin{cases} +1 \text{ or} -1 \text{ or } u_i - x_i \text{ or } x_i - l_i & \text{if } g_i^* = 0 \\ u_i - x_i & \text{if } g_i^* < 0 \\ x_i - l_i & \text{if } g_i^* > 0 . \end{cases} \tag{6.4}$$

Note. When $g_i^* = 0$ the choice $\nu_i = u_i - x_i$ is valid only when u_i is finite; the choice $\nu_i = x_i - l_i$ is valid only when l_i is finite.

Each function F_ν is continuously differentiable; furthermore, $F_\nu(x_*) = 0$ for every possible ν. Of course, F_ν cannot be used computationally since x_* is not known a priori. However, locally each step of our proposed algorithms is an approximate Newton step for exactly one set of equations based on the definition of $v(x)$, i.e., $\nu(x) = |v(x)|$. Therefore, $\mathscr{V}$ and F_ν are useful in a theoretical sense to help establish asymptotic convergence results of our proposed algorithm.

The next result formalizes the simple observation that any member of $\mathscr{V}$ can be used interchangeably with any other, at any iteration, and there remains a neighborhood around α_* retaining quadratic convergence properties of a Newton process.

Theorem 11. *Let $\mathscr{V} = \{F_\nu: R^n \to R^n\}$ be a finite set of functions satisfying the following assumptions*:

- *Each F_ν is continuously differentiable in an open convex set $\mathscr{C}$.*
- *There is a x_* in $\mathscr{C}$ such that $F_\nu(x_*) = 0$ and $\nabla F_\nu(x_*)$ is nonsingular for all $F_\nu \in \mathscr{S}$.*
- *There is a constant κ_0 such that for all $F_\nu \in \mathscr{S}$,*

$$\| \nabla F_\nu(x) - \nabla F_\nu(x_*) \| \leqslant \kappa_0 \| x - x_* \| , \tag{6.5}$$

for $x \in \mathscr{C}$.
Let $\{x_k\}$ *and* $\{s_k\}$ *be sequences such that* $x_{k+1} = x_k + s_k$ *and suppose*

$$\|s_k + s_k^{N_{\nu_k}}\| = O(\|x_k - x_*\|)^2 ,$$

where $s_k^{N_{\nu_k}}$ *is the Newton step for one of the functions* $F_{\nu_k} \in \nu$ *at* x_k, *i.e.*,

$$s_k^{N_{\nu_k}} = -(\nabla F_{\nu_k}(x_k))^{-1} F_{\nu_k}(x_k) .$$

Then, for $\mathscr{C}$ *sufficiently small,* $\{x_k\}$ *converges quadratically to* x^*.

Proof. The argument is straightforward and uses a standard result in the last step, e.g., [24]:

$$\begin{aligned} \|x_{k+1} - x_*\| &= \|x_k + s_k - x_*\| \\ &= \|x_k + s_k^{N_{\nu_k}} - x_* + s_k - s_k^{N_{\nu_k}}\| \\ &\leqslant \|x_k + s_k^{N_{\nu_k}} - x_*\| + \|s_k - s_k^{N_{\nu_k}}\| \\ &= O(\|x_k - x_*\|)^2 . \qquad \square \end{aligned}$$

Our next main result is that the local interior-reflective Newton method is locally and quadratically convergent. The local reflective Newton method, given in Fig. 11, is merely the interior-reflective algorithm in Fig. 9 with $\hat{M}_k$ replaced by $\hat{B}_k$ (compare (2.14) and (2.5)), direction s_k specified as the Newton step and α_k chosen so that $|\alpha_k - 1| = O(\|D_k g_k\|)$. We assume that $x_1 \in \text{int}(\mathscr{F})$.

Note that the kth iteration is computable provided x_k is sufficiently close to x_* and $x_k \neq x_*$. To see this note that the Newton direction and the step size α_k are always computable in a neighborhood of x_*. In particular, $\hat{B}_k$ is positive definite in a neighborhood of x_*, assuming x_* is nondegenerate and satisfies second-order sufficiency, and $\hat{g}_k \neq 0$ unless $x_k = x_*$. Stepsize $\alpha_k = 1$ satisfies the stepsize condition (step 2 in Fig. 10) unless $x_k + p_k(1)$ is on the boundary, i.e., $(x_k + p_k(1))_j$ is tight for some index j. In this case α_k can be chosen slightly smaller than unity, satisfying $|\alpha_k - 1| = O(\|D_k g_k\|)$, and strict feasibility will be maintained. It is clear that $\|D_k g_k\| = O(\|x_k - x_*\|)$. Hence $|\alpha_k - 1| = O(\|x_k - x_*\|)$.

A key observation is that, provided x_* satisfies nondegeneracy and second-order sufficiency and x_1 is sufficiently close to x_*, the search direction d_k generated by the local

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A local interior-reflective Newton algorithm

Choose $x_1 \in \text{int}(\mathscr{F})$.
For $k = 1, 2, \ldots,$
1. Solve $\hat{B}_k \hat{d}_k^N = -\hat{g}_k = -D_k g_k$, set $d_k = D_k \hat{d}_k^N$.
2. Determine α_k s.t. $|\alpha_k - 1| = O(\|D_k g_k\|)$ and $x_k + p_k(\alpha_k) \in \text{int}(\mathscr{F})$.
3. $x_{k+1} = x_k + p_k(\alpha_k)$.

Fig. 11. A local interior-reflective Newton method.

Newton algorithm in Fig. 11 is a Newton step for one of the functions in $\mathscr{Y}$. Therefore, to establish quadratic convergence we focus on the relationship between $p_k(\alpha_k)$ and d_k. The following result provides the necessary connection.

Lemma 12. *Let x_* be a nondegenerate point satisfying second-order sufficiency conditions. Assume that $\nu(x)$ is chosen such that $\nu(x)=|v(x)|$. Let $d^N(x)$ be the corresponding Newton direction, i.e.,*

$$d^N(x) = -(D^2H+J^vD^g)^{-1}D^2g \tag{6.6}$$

where $D^g=D^g(x)=\text{diag}(|g|)$, $D^2=D(x)^2=\text{diag}(|v(x)|)$, $J^v=J^v(x)$ is the diagonal Jacobian matrix of $|v(x)|$. There exists an open neighborhood $\mathscr{C}$ containing x_* such that for all $x\in \text{int}(\mathscr{F})\cap\mathscr{C}$, $d^N(x)$ is well defined and for each $j\notin Free_*$,

$$|1-\beta_j^N(x)| = O(\|x_*-x\|) \tag{6.7}$$

where $\beta_j^N=|v_j(x)|/|d_j^N(x)|$.

Proof. Since x_* satisfies nondegeneracy and second-order sufficiency, it follows that the matrix $D^2H+J^vD^g$ is nonsingular in a neighborhood of x_* and so $d^N(x)$ is well-defined. From the definition of the Newton step (6.6) it follows that if $j\notin Free_*$,

$$d_j^N = -|v_j|\cdot\text{sgn}(g_j) - \frac{|v_j|}{|g_j|}(Hd^N)_j$$

which implies

$$|v_j| - \frac{|v_j|}{g_j}\cdot|(Hd^N)_j| \leqslant |d_j^N| \leqslant |v_j| + \frac{|v_j|}{|g_j|}|(Hd^N)_j|. \tag{6.8}$$

The first inequality in (6.8) uses the fact that $g_j^* \neq 0$ (by nondegeneracy), and $Hd^N\to 0$ as $x\to x^*$. Therefore,

$$1-\frac{|(Hd^N)_j|}{|g_j|} \leqslant \frac{|d_j^N|}{|v_j|} \leqslant 1+\frac{|(Hd^N)_j|}{|g_j|}. \tag{6.9}$$

But, by nondegeneracy and continuity, $|g_j|$ is bounded away from zero in a neighborhood of x_*; H is bounded; $\|d^N\|=O(\|x-x_*\|)$; therefore, from (6.9) it is easy to show that $|1-\beta_j^N|=O(\|x-x_*\|)$. □

Theorem 13. *Let x_* be a nondegenerate point satisfying the second-order sufficiency conditions. Assume that $\{x_k\}$ is generated by the local Newton algorithm in Fig. 11. Then, for $x_1\in\text{int}(\mathscr{F})$ and sufficiently close to x_*, $\{x_k\}\in\text{int}(\mathscr{F})$ and $\{x_k\}$ converges quadratically to x_*.*

Proof. Let β_k^1 be the steplength to the first breakpoint along direction d_k. If $\alpha_k<\beta_k^1$ then $p_k(\alpha_k)=\alpha_k d_k$ where d_k is the Newton step. However, $|\alpha_k-1|=O(\|x_k-x_*\|)$ and since

d_k is the Newton step for some function in $\mathcal{F}_\nu$, $\|d_k\|=O(\|x_k-x_*\|)$; therefore, $\|p_k(\alpha_k)-d_k\|=O(\|x_k-x_*\|^2)$ and so Theorem 11 applies and the result follows.

Assume that $\beta_k^{t_k}<\alpha_k<\beta_k^{t_k+1}$. From the definition of the reflective process, we can write

$$p_k(\alpha_k)-d_k=\sum_{i=2}^{t_k}(\beta_k^i-\beta_k^{i-1})p_k^i+(\alpha_k-\beta_k^{t_k})p_k^{t_k+1}+\beta_k^1 d_k-d_k\,.$$

But applying Lemma 12,

$$\|p_k(\alpha_k)-d_k\|=O(\|d_k\|\cdot\|x_k-x_*\|)\,.$$

But d_k is the Newton step for some function in $\mathcal{F}_\nu$; hence, $\|d_k\|=O(\|x_k-x_*\|)$. It follows that $\|p_k(\alpha_k)-d_k\|=O(\|x_k-x_*\|^2)$; applying Lemma 11 the result follows. □

We have established global convergence results for the interior-reflective method in Fig. 9 (and therefore the second order method in Fig. 10) and we have established that the local interior-reflective Newton method described in Fig. 11 yields quadratic convergence. We now show that the second-order method in Fig. 10 reduces to the local Newton algorithm (Fig. 11) in a neighborhood of a nondegenerate second-order point: global and quadratic convergence properties follow. In particular, we show that in a neighborhood of a nondegenerate point satisfying second-order sufficiency conditions, a Newton step will satisfy line search condition (3.4).

Theorem 14. *Assume x_* is a nondegenerate point satisfying second-order sufficiency conditions. Let $0<\sigma_1<\frac{1}{2}$. Suppose $\{x_k\}$ is generated by the local Newton Algorithm in Fig. 11. Then for x_1 sufficiently close to x_* and k sufficiently large,*

$$f(x_k+p_k(\alpha_k))<f(x_k)+\sigma_1(g_k^Td_k+\tfrac{1}{2}\min(d_k^TH_kd_k,0))\,. \tag{6.10}$$

Proof. Suppose there are t_k-1 breakpoints $b_1, b_2, \ldots, b_{t_k-1}$, to the left of α_k, corresponding to step lengths $\beta_k^1, \beta_k^2, \ldots, \beta_k^{t_k-1}$. For notational simplicity let us label $x_k+p_k(\alpha_k)$ with $b_k^{t_k}$. Clearly,

$$f(x_k+p_k(\alpha_k))-f(x_k)=f(b_k^1)-f(x_k)+\sum_{i=1}^{t_k-1}[f(b_k^{i+1})-f(b_k^i)]\,. \tag{6.11}$$

Note that $p_k^{i+1}=D_k^{\sigma_i+1}d_k$ where $D_k^{\sigma_i+1}$ is a diagonal matrix with each diagonal entry equal to ±1; therefore, $\|p_k^{i+1}\|=O(\|d_k\|)$. Consequently, applying Lemma 12, for any $1\leqslant i\leqslant t_k-1$,

$$\begin{aligned}&f(b_k^{i+1})-f(b_k^i)\\&\quad=(\beta_k^{i+1}-\beta_k^i)g(b_k^i)^Tp_k^{i+1}+\tfrac{1}{2}(\beta_k^{i+1}-\beta_k^i)^2(p_k^{i+1})^TH_k^ip_k^{i+1}\\&\qquad+o(\|(\beta_k^{i+1}-\beta_k^i)p_k^{i+1}\|^2)\end{aligned}$$

$$= (\beta_k^{i+1} - \beta_k^i) g(b_k^i)^T p_k^{i+1} + \tfrac{1}{2}(\beta_k^{i+1} - \beta_k^i)^2 (p_k^{i+1})^T H_k^i p_k^{i+1} + o(\|d_k\|^2)$$
$$= (\beta_k^{i+1} - \beta_k^i) g_k^T p_k^{i+1} + \tfrac{1}{2}(\beta_k^{i+1} - \beta_k^i)^2 (p_k^{i+1})^T H_k^i p_k^{i+1} + o(\|d_k\|^2)$$
$$= (\beta_k^{i+1} - \beta_k^i) g_k^T p_k^{i+1} + \tfrac{1}{2}(\beta_k^{i+1} - \beta_k^i)^2 (d_k)^T D_k^{\sigma_i+1} H_k^i D_k^{\sigma_i+1} d_k + o(\|d_k\|^2)$$
$$= (\beta_k^{i+1} - \beta_k^i) g_k^T p_k^{i+1} + o(\|d_k\|^2) .$$

Moreover, using Taylor's theorem and Lemma 12,

$$f(b_k^1) - f(x_k) = \beta_k^1 g_k^T d_k + \tfrac{1}{2}(\beta_k^1)^2 d_k^T H_k d_k + o(\|d_k\|^2)$$
$$= g_k^T d_k + \tfrac{1}{2} d_k^T H_k d_k + o(|g_k^T d_k|) + o(\|d_k\|^2) .$$

The most difficult term to deal with is $g_k^T p_k^{i+1}$; however, we can show that $|g_k^T p_k^{i+1}| = O(-g_k^T d_k)$ and this leads the way to the final result. To show this we use the fact that, due to second-order sufficiency, there exists $\mu > 0$ such that for all *sufficiently large*,

$$d_k^T B_k d_k \geqslant \mu \|d_k\|^2 , \tag{6.12}$$

and

$$\hat{d}_k^T \hat{B}_k \hat{d}_k \geqslant \mu \|\hat{d}_k\|^2 .$$

But since d_k is the Newton direction,

$$g_k = -B_k d_k = -D_k^{-1} \hat{B}_k D_k^{-1} d_k = -D_k^{-1} \hat{B}_k \hat{d}_k ;$$

therefore,

$$-g_k^T d_k = \hat{d}_k^T \hat{B}_k \hat{d}_k \geqslant \mu \|\hat{d}_k\|^2 . \tag{6.13}$$

But $p_k^{i+1} = D_k^{\sigma_i+1} d_k$ where $D_k^{\sigma_i+1}$ is a diagonal matrix with each diagonal element equal to ± 1. Hence, using the boundedness of $\{\hat{B}_k\}$,

$$|-g_k^T p_k^{i+1}| = |d_k^T D_k^{\sigma_i+1} B_k d_k| = |\hat{d}_k^T D_k^{\sigma_i+1} \hat{B}_k \hat{d}_k| = O(\|\hat{d}_k\|^2) . \tag{6.14}$$

Therefore, combining (6.13) and (6.14),

$$|-g_k^T p_k^{i+1}| = O(-g_k^T d_k) . \tag{6.15}$$

Collecting together the terms above, and applying Lemma 12, (6.11) becomes

$$f(x_k + p_k(\alpha_k)) - f(x_k) = g_k^T d_k + \tfrac{1}{2} d_k^T H_k d_k + o(|g_k^T d_k|) + o(\|d_k\|^2) .$$

But $-g_k^T d_k = d_k^T B_k d_k \geqslant \mu \|d_k\|^2$, from (6.12). Therefore,

$$\begin{aligned} f(x_k + p_k(\alpha_k)) - f(x_k) &= g_k^T d_k + \tfrac{1}{2} d_k^T H_k d_k + o(|g_k^T d_k|) \\ &= \tfrac{1}{2} g_k^T d_k - \tfrac{1}{2} d_k^T C_k d_k + o(|g_k^T d_k|) . \end{aligned} \tag{6.16}$$

But, for k sufficiently large,

$$o(|g_k^T d_k|) \leqslant -\tfrac{1}{2}(1 - 2\sigma_l) g_k^T d_k \tag{6.17}$$

and $-d_k^T C_k d_k \leqslant \min(d_k^T H_k d_k, 0)$ and so, using (6.16),

$$f(x_k+p_k(\alpha_k))-f(x_k)<\sigma_1 d_k^T g_k+\tfrac{1}{2}\min(d_k^T H_k d_k, 0)$$

which implies for $\sigma_1<1$,

$$f(x_k+p_k(\alpha_k))-f(x_k)<\sigma_1(d_k^T g_k+\tfrac{1}{2}\min(d_k^T H_k d_k, 0)) . \qquad \square$$

Theorem 15. *Assume $\{x_k\}$ is generated by the interior-reflective Newton algorithm in Fig. 10. Let $\{s_k\}$ satisfy constraint-compatibility and consistency. Suppose Y_k is a matrix with orthonormal columns and let $\mathcal{S}_k=\langle Y_k\rangle$ be chosen such that, when $\lambda_{\min}(\hat{M}_k)\leqslant 0$,*

$$\lambda_{\min}(Y_k^T\hat{M}_k Y_k)\leqslant\max(-\epsilon_{nc}, \tau\lambda_{\min}(\hat{M}_k)) , \tag{6.18}$$

for some $\epsilon_{nc}>0$, $\tau>0$. Then,

- *Every limit point of $\{x_k\}$ is a first-order point.*
- *Every nondegenerate limit point satisfies the second-order necessary conditions, provided τ_ϵ is sufficiently small (see (2.12) for the definition of τ_ϵ).*
- *Assume that τ_ϵ is sufficiently small. If a nondegenerate limit point x_* satisfies second-order sufficiency conditions then $\{x_k\}$ is convergent to x_*. The convergence rate is quadratic, i.e.,*

$$\|x_{k+1}-x_*\|=O(\|x_k-x_*\|^2) .$$

Proof. By Theorem 8 every limit point satisfies the first order necessary conditions. Assume that τ_ϵ is sufficiently small, $M_k=B_k$ for sufficiently large k. Hence, from Theorem 10, the second-order necessary conditions are satisfied. Let x_* be a limit point satisfying sufficient nondegeneracy and second-order sufficiency conditions. By Theorem 14 a unit step size [6], for some constant $\chi_\alpha>0$ will satisfy (3.4) for $\|x_k-x_*\|$ sufficiently small. Therefore, for $\|x_k-x_*\|$ sufficiently small, the interior-reflective Newton method in Fig. 10 reduces to the local Newton Algorithm in Fig. 11: quadratic convergence follows from Theorem 13. $\square$

Clearly if we determine s_k by solving (6.1) at each iteration with $\mathcal{S}_k=\mathcal{R}^n$, for example, then the assumptions of Theorem 15 will be satisfied and so second-order convergence will be attained. We state this formally.

Corollary 16. *Assume $x_1\in\text{int}(\mathcal{F})$ and let $\{x_k\}$ be generated by the interior-reflective Newton method in Fig. 10 with $\{s_k\}$ determined by solving (6.1) at each iteration with $\mathcal{S}_k=\mathcal{R}^n$. Then,*

- *Every limit point of $\{x_k\}$ is a first-order point.*
- *Every nondegenerate limit point satisfies the second-order necessary conditions, provided τ_ϵ is sufficiently small.*
- *Assume that τ_ϵ is sufficiently small. If a nondegenerate limit point x_* satisfies second-*

[6] If $\alpha_k=1$ corresponds to a breakpoint then $\alpha_k=\tilde{\alpha}_k=1-\epsilon_k$ where $\tilde{\alpha}_k$ is not a breakpoint, $\tilde{\alpha}_k$ satisfies (3.4), and $\epsilon_k<\chi_\alpha\|D_k g_k\|$.

order sufficiency conditions then $\{x_k\}$ *is convergent to* x_* *and the convergence rate is quadratic, i.e.,*

$$\|x_{k+1}-x_*\| = O(\|x_k - x_*\|^2)\ .$$

Proof. By Theorems 5 and 6 the sequence $\{s_k\}$ satisfies constraint-compatibility and consistency. Since (6.1) is used to define s_k with $\mathcal{S}_k = \mathbb{R}^n$, it follows that condition (6.18) is satisfied. Therefore, the assumptions of Theorem 15 are satisfied and the result follows. □

7. Computing descent directions by subspace trust region subproblems

The interior-reflective Newton method described in Fig. 10 allows for some freedom in the determination of the direction s_k. As we have already remarked, if we determine s_k be solving (6.1) at each iteration with $\mathcal{S}_k = R^n$, then second-order convergence ensues (Corollary 16). However, this choice can lead to expensive subproblems (6.1), especially when n is large. Therefore, it is worthwhile exploring alternative choices for $\mathcal{S}_k$, particularly if we can maintain the strong convergence properties for small dimensions of $\mathcal{S}_k$. Below we propose a specific way to choose $\mathcal{S}_k$, restricting $|\mathcal{S}_k| \leqslant 2$, whilst retaining strong second-order convergence properties.

Constraint-compatibility plays a key role in the convergence of an interior-reflective algorithm. If a reduced trust region problem (6.1)

$$\min_s\{s^T g_k + \tfrac{1}{2}s^T M_k s\colon \|D_k^{-1}s\|_2 \leqslant \Delta_k,\ s\in\mathcal{S}_k\}$$

is used to solve for a direction s_k – which, in turn, defines the reflective path p_k – the subspace $\mathcal{S}_k$ must be chosen so that the solutions of the corresponding trust region subproblem yield constraint-compatibility. It is easy to see that if s_k solves (6.1) for some subspace $\mathcal{S}_k$ then $\{D_k^{-1}s_k\}$ is bounded. This observation leads to the following two technical results.

Lemma 17. *Let* $\{Y_k\}$ *be a sequence of matrices where each matrix* Y_k *has orthonormal columns and suppose* $\mathcal{S}_k = \langle D_k Y_k\rangle$. *Assume every column of* $D_k Y_k$ *generates a constraint-compatible sequence. Let* $u_k \in \mathcal{S}_k$; *assume the sequence* $\{D_k^{-1}u_k\}$ *is bounded. Then, the sequence* $\{u_k\}$ *is constraint-compatible.*

Proof. If $u_k\in\mathcal{S}_k$ then $u_k = D_k Y_k w_k$ for some vector w_k. But $\{D_k^{-1}u_k\}$ is bounded by assumption; therefore, $\{Y_k w_k\}$ is bounded and, by orthonormality of the columns of Y_k, the sequence $\{w_k\}$ is bounded. It is now easy to see that $\{u_k\}$ is constraint-compatible, i.e., $\{D_k^{-2}u_k\}$ is bounded. To see this notice that the sequence generated by any column of $D_k^{-2}(D_k Y_k)$ is bounded, by assumption, and we have already argued that $\{w_k\}$ is bounded. Therefore, since $u_k = D_k Y_k w_k$, the result follows. □

In the next lemma we indicate that the application of Lemma 17 is straightforward in the 2-dimensional case – subsequently we will use it in this setting.

Let $\mathscr{A}$ be a subspace and w a vector. We denote $r(\mathscr{A}, w)$ to be the *residual vector* of the orthogonal projection of w onto $\mathscr{A}$. If the columns of matrix Y form an orthonormal basis for $\mathscr{A}$, then $r(\mathscr{A},w) \stackrel{\text{def}}{=} w - YY^Tw$.

Lemma 18. *Let a_k be a unit vector and suppose the sequences $\{D_ka_k\}$ and $\{D_kb_k\}$ are constraint-compatible; assume there exists a constant $\tau > 0$ such that $r(a_k, b_k) > \tau$ for all k. Then if $u_k \in \mathscr{S}_k = \langle D_ka_k, D_kb_k\rangle$ and $\{D_k^{-1}u_k\}$ is bounded, then $\{u_k\}$ is constraint-compatible.*

Proof. Let $y_k^1 = a_k$ and so

$$r(a_k, b_k) = b_k - [(y_k^1)^Tb_k]y_k^1 .$$

Since $\{D_ky_k^1\}$ and $\{D_kb_k\}$ are both constraint-compatible, and $\{b_k\}$ is bounded due to constraint-compatibility of $\{D_kb_k\}$, $\{D_kr_k\}$ is constraint compatible. Let $y_k^2 = r_k / \|r_k\|$. From $\|r_k\| > \tau > 0$, $\{D_ky_k^2\}$ is constraint-compatible. Let $Y_k = [y_k^1, y_k^2]$. Since the columns of D_kY_k are constraint-compatible and $\{\mathscr{S}_k\} = \{\langle D_ka_k, D_kb_k\rangle\} = \{\langle D_kY_k\rangle\}$, it follows from Lemma 17 that $\{s_k\}$ is constraint-compatible. □

The subspace descent direction procedure in Fig. 12 describes a particular way to choose s_k (and $\mathscr{S}_k$, when appropriate) with the large-scale setting in mind. Each subspace $\mathscr{S}_k$ satisfies $|\mathscr{S}_k| \leqslant 2$ and so problem (6.1) is inexpensive to solve.

Two technical results pertaining to the subspace descent direction procedure in Fig. 12 are needed before establishing the main theorem. Let χ_M be the maximum spectral radius of $\hat{M}(x)$ on $\mathscr{L} = \{x: x \in \mathscr{F} \text{ and } f(x) \leqslant f(x_1)\}$. Since the spectral radius $\rho(\hat{M}(x))$ of $\hat{M}(x)$ is continuous on $\mathscr{L}$, a compact set, the maximum χ_M exists.

Lemma 19. *Assume $\{x_k\}$ is generated by the interior-reflective Newton method described in Fig. 10 with $\{s_k\}$ calculated by the subspace descent direction procedure in Fig. 12. Then,*

(1) *the subsequence $\{\|D_k\text{sgn}(g_k)\| : \lambda_{\min}(\hat{M}_k) < 0\}$ is bounded away from zero,*

(2) *the subsequence $\{z_k = D_k\hat{z}_k : \lambda_{\min}(\hat{M}_k) < 0\}$ is constraint-compatible, where $\hat{z}_k = D_k\text{sgn}(g_k) / \|D_k\text{sgn}(g_k)\|$.*

Moreover, if we assume that $\tau_2 < 1/(5\chi_M)$, and that corresponding to any subsequence $\{\mathscr{S}_k\} = \{\langle D_k^2\text{sgn}(g_k)\rangle\}$, $\{D_kg_k : \mathscr{S}_k = \langle D_k^2\text{sgn}(g_k)\rangle$ and $\lim_{k\to\infty}\lambda_{\min}(\hat{M}_k) < 0\}$ converges to zero, then

$$\hat{z}_k^T\hat{M}_k\hat{z}_k < \tfrac{1}{2}\hat{w}_k^T\hat{M}_k\hat{w}_k$$

for sufficiently large k.

A subspace descent direction procedure [Let $\tau < 1$, τ_1, and τ_2 be small positive constants.]

Case 0: $\hat{M}_k$ is positive definite and $\|\hat{s}_k^N\| \leqslant \Delta_k$.

Set $s_k = s_k^N = -D_k \hat{M}_k^{-1} \hat{g}_k = D_k \hat{s}_k^N$.

Case 1: $\hat{M}_k$ is positive definite and $\|\hat{s}_k^N\| > \Delta_k$.

if $\|r(\hat{s}_k^N, \hat{g}_k)\| > \tau_1$

$\mathscr{S}_k = \langle D_k^2 g_k, s_k^N \rangle$, solve (6.1) to get s_k.

else

set $s_k = -D_k^2 g_k$

end

Case 2: $\hat{M}_k$ is not positive definite. Compute $w_k = D_k \hat{w}_k$, where $\hat{w}_k$ is a unit vector such that $\{w_k\}$ is constraint-compatible and

$$\hat{w}_k^T \hat{M}_k \hat{w}_k \leqslant \max\{ -\epsilon_{nc}, \tau \lambda_{\min}(\hat{M}_k) \} .$$

Let $\hat{z}_k = D_k \mathrm{sgn}(g_k) / \|D_k \mathrm{sgn}(g_k)\|$.

if $\|r(\hat{w}_k, \hat{z}_k)\| < \max(\|D_k g_k\|, -\tau_2 \hat{w}_k^T \hat{M}_k \hat{w}_k)$

$\mathscr{S}_k = \langle D_k^2 \mathrm{sgn}(g_k) \rangle$, solve (6.1) to get s_k.

else

$\mathscr{S}_k = \langle D_k^2 \mathrm{sgn}(g_k), D_k \hat{w}_k \rangle$, solve (6.1) to get s_k.

end

Fig. 12. Determination of the descent direction s_k.

Proof. First assume there exists a subsequence with $\lim_{k\to\infty}\{D_k \mathrm{sgn}(g_k)\} = 0$ and $\lambda_{\min}(\hat{M}_k) < 0$. This implies $\lim_{k\to\infty}\{v_k\} = 0$ which implies that for k sufficiently large, $\hat{M}_k$ is positive definite (by virtue of the definition of $\hat{M}_k$), a contradiction. Hence the subsequence $\{\|D_k \mathrm{sgn}(g_k)\| \colon \lambda_{\min}(\hat{M}_k) < 0\}$ is bounded away from zero and it follows, using Theorem 6, that the corresponding subsequence $\{z_k\}$ is constraint-compatible.

To prove that $\hat{z}_k^T \hat{M}_k \hat{z}_k < \frac{1}{2} \hat{w}_k^T \hat{M}_k \hat{w}_k$ for sufficiently large k, first notice that by the definition of s_k in Fig. 12, $\mathscr{S}_k = \langle D_k^2 \mathrm{sgn}(g_k) \rangle$ only when

$$\|r(\hat{w}_k, \hat{z}_k)\| < \max(\|D_k g_k\|, -\tau_2 \hat{w}_k^T \hat{M}_k \hat{w}_k) .$$

Since $\{D_k g_k\}$ converges to zero and $\lim_{k\to\infty} \lambda_{\min}(\hat{M}_k) < 0$, $\|D_k g_k\| < -\tau_2 \hat{w}_k^T \hat{M}_k \hat{w}_k$ for sufficiently large k. Hence

$$\|r_k\| = \|r(\hat{w}_k, \hat{z}_k)\| < -\tau_2 \hat{w}_k^T \hat{M}_k \hat{w}_k \leqslant \tau_2 \rho_M .$$

From $r_k = \hat{w}_k - (\hat{z}_k^T \hat{w}_k) \hat{z}_k$, we have

$$(\hat{z}_k^T \hat{w}_k)^2 \hat{z}_k^T \hat{M}_k \hat{z}_k = \hat{w}_k^T \hat{M}_k \hat{w}_k - 2 r_k^T \hat{M}_k \hat{w}_k + r_k^T \hat{M}_k r_k .$$

But

$$|r_k^T \hat{M}_k \hat{w}_k| \leqslant \rho_M \|r_k\| < \rho_M \tau_2 |\hat{w}_k^T \hat{M}_k \hat{w}_k| ,$$

and

$$|r_k^T \hat{M}_k r_k| \leqslant \rho_M \|\hat{r}_k\|^2 < \rho_M^2 \tau_2^2 |\hat{w}_k^T \hat{M}_k \hat{w}_k| ,$$

and so

$$(\hat{z}_k^T \hat{w}_k)^2 \hat{z}_k^T \hat{M}_k \hat{z}_k < \hat{w}_k^T \hat{M}_k \hat{w}_k + (2\rho_M \tau_2 + \rho_M^2 \tau_2^2) |\hat{w}_k^T \hat{M}_k \hat{w}_k| .$$

But $\tau_2 < 1/(5\rho_M)$; Therefore,

$$(\hat{z}_k^T \hat{w}_k)^2 \hat{z}_k^T \hat{M}_k \hat{z}_k < \hat{w}_k^T \hat{M}_k \hat{w}_k + \tfrac{1}{2} |\hat{w}_k^T \hat{M}_k \hat{w}_k| = \tfrac{1}{2} \hat{w}_k^T \hat{M}_k \hat{w}_k .$$

Finally, since $\hat{z}_k$ and $\hat{w}_k$ are unit vectors, $|\hat{z}_k^T \hat{w}_k| \leqslant 1$; moreover, $\hat{w}_k^T \hat{M}_k \hat{w}_k < 0$ which implies $\hat{z}_k^T \hat{M}_k \hat{z}_k < 0$. Therefore,

$$\hat{z}_k^T \hat{M}_k \hat{z}_k \leqslant \tfrac{1}{2} \hat{w}_k^T \hat{M}_k \hat{w}_k . \qquad \square$$

Theorem 20. *Assume $\{x_k\}$ is generated by the interior-reflective Newton method in Fig.* 10 *with $\{s_k\}$ generated by the subspace descent direction procedure in Fig.* 12 *and $\tau_2 < 1/(5\rho_M)$. Then every subsequence $\{s_k\}$ satisfies the consistency condition. Moreover, for any subsequence, if either $\{\|D_k g_k\|\}$ or $\{\max(0, \lambda_{\min}(\hat{M}_k))\}$ is bounded away from zero, then the corresponding subsequence $\{s_k\}$ is constraint-compatible.*

Proof. Applying Theorem 6 to each case in Fig. 12, it is easy to see that $\{s_k\}$ satisfies consistency.

Assume that either a subsequence $\{\|D_k g_k\|\}$ or a subsequence $\{\max(0, \lambda_{\min}(\hat{M}_k))\}$ is bounded away from zero. We prove next that the corresponding subsequence $\{s_k\}$ is constraint-compatible.

(i) Suppose there is a subsequence $\{\|D_k g_k\|\}$ bounded away from zero. If $\lambda_{\min}(\hat{M}_k) > 0$ then by the subspace descent direction procedure in Fig. 12, there are three possible ways to compute s_k. All three possibilities clearly yield constraint-compatible sequences $\{s_k\}$ using Theorem 5 and Lemma 18. Assume then that $\lambda_{\min}(\hat{M}_k) \leqslant 0$. Fig. 12 gives two possible ways to compute s_k in this case: i.e., $\mathcal{S}_k = \langle D_k^2 \text{sgn}(g_k) \rangle$ and solve (6.1) to get s_k, or $\mathcal{S}_k = \langle D_k^2 \text{sgn}(g_k), D_k \hat{w}_k \rangle$ and solve (6.1) to get s_k. In the first case constraint-compatibility of $\{\mathcal{S}_k\}$ follows from the fact that $\{\|D_k \text{sgn}(g_k)\|\}$ is bounded away from zero. In the second case, since $\|r(\hat{w}_k, \hat{z}_k)\| \geqslant \|D_k g_k\|$, it follows from Lemmas 17 and 18 that $\{s_k\}$ is constraint-compatible.

(ii) Assume $\{D_k g_k\}$ converges to zero, $\lim_{k\to\infty} \lambda_{\min}(\hat{M}_k) < 0$, and $\tau_2 < 1/(5\rho_M)$. Again there are two possible ways in which the subspace descent direction procedure will determine the search direction. Either $\mathcal{S}_k = \langle D_k^2 \text{sgn}(g_k) \rangle$ and solve (6.1) to get s_k, or $\mathcal{S}_k = \langle D_k^2 \text{sgn}(g_k), D_k \hat{w}_k \rangle$ and solve (6.1) to get s_k. In the first case constraint-compatibility of $\{s_k\}$ follows from the fact that $\{\|D_k \text{sgn}(g_k)\|\}$ is bounded away from zero. In the second case, since $\|r(\hat{w}_k, \hat{z}_k)\| \geqslant -\tau_2 \hat{w}_k^T \hat{M}_k \hat{w}_k > 0$, $\{s_k\}$ is constraint-compatible from Lemmas 17 and 18. $\square$

The main result of this section follows.

Theorem 21. *Let* $\{x_k\}$ *be generated by the interior-reflective Newton method in Fig.* 10 *with* $\{s_k\}$ *generated by the method in Fig.* 12 *with* $\tau_2 < 1/(5\rho_M)$. *Then*

- *Every limit point of* $\{x_k\}$ *is a first-order point.*
- *Every nondegenerate limit point satisfies the second-order necessary conditions, provided* τ_ϵ *is sufficiently small.*
- *Assume that* τ_ϵ *is sufficiently small. If a nondegenerate limit point* x_* *satisfies second-order sufficiency conditions, then* $\{x_k\}$ *is convergent to* x_* *and the convergence rate is quadratic, i.e.,*

$$\|x_{k+1} - x_*\| = O(\|x_k - x_*\|^2) .$$

Proof. Let $\{s_k\}$ correspond to any subsequence such that either $\{\max(0, \lambda_{\min}(\hat{M}_k))\}$ or $\{\|D_k g_k\|\}$ is bounded away from zero. Then by Theorem 20, the corresponding subsequence $\{s_k\}$ is constraint-compatible and satisfies the consistency condition. Therefore, by Theorem 15, the result holds for such a subsequence.

Clearly then every subsequence satisfies $\{\|D_k g_k\|\} \to 0$ and $\{\max(0, \lambda_{\min}(\hat{M}_k))\} \to 0$. Hence every limit point of $\{x_k\}$ satisfies first-order necessary conditions. Moreover, the second-order necessary conditions are satisfied at every nondegenerate point, provided that τ_ϵ is sufficiently small. Let x_* be a limit point satisfying nondegeneracy and second-order sufficiency conditions. By Theorem 14 a unit step size [7] will satisfy (3.4) for $\|x_k - x_*\|$ sufficiently small. Therefore, for $\|x_k - x_*\|$ sufficiently small, the interior-reflective Newton method in Fig. 10 reduces to the local algorithm in Fig. 11: quadratic convergence follows from Theorem 13. □

The only question yet to be addressed is the determination of a sequence of directions $\{w_k\}$ which is constraint-compatible and contains sufficient negative curvature information when $\hat{M}_k$ is indefinite. In [3], the Lanczos process has been used to obtain negative curvature directions. Similarly, the Lanczos process can be used to obtain constraint-compatible negative curvature directions.

8. Concluding remarks

We have proposed a new method, an interior-reflective Newton method, for solving nonlinear minimization problems where some of the variables have upper and/or lower bounds. We have established strong convergence properties. In particular, interior-reflective Newton methods can achieve global and quadratic convergence.

[7] If $\alpha_k = 1$ corresponds to a breakpoint then $\alpha_k = \tilde{\alpha}_k = 1 - \epsilon$ where $\tilde{\alpha}_k$ is not a breakpoint, $\tilde{\alpha}_k$ satisfies (3.4), and $\epsilon < \chi_\alpha \|D_k g_k\|$, for some $\chi_\alpha > 0$.

The proposed interior-reflective Newton method involves the solution of a reduced trust region problem, (6.1). In (6.1), subspace $\mathcal{S}_k$ must be chosen with care to ensure the second-order convergence properties and to maintain practical viability in the large-scale setting. In this paper we show that a small dimensional subspace can be used, i.e., $|\mathcal{S}_k| \leqslant 2$, and yet the attractive convergence properties obtained with $\mathcal{S}_k = \mathbb{R}^n$ can be maintained.

Experimental results for the case when the objective function is quadratic are provided in [10]. These computational results are extremely encouraging and indicate that interior-reflective Newton methods have strong potential for large-scale computations. Experimentation on general nonlinear functions is a current research activity and results will be available in a future paper.

Research on two extensions of this work is underway. First, we are studying *inexact* interior-reflective Newton methods for problem (1.1). Our current implementation rests on a (partial) sparse Cholesky factorization of $\hat{M}_k$. A limitation with this approach is that a (partial) sparse Cholesky factorization is not always economical. Therefore, we are considering an interior-reflective Newton procedure that only requires the iterative use of $\hat{M}_k$.

Second, we are studying the adaptation of interior reflective Newton methods to bound-constrained problems with additional equality constraints:

$$\min_x \{f(x): Ax = b,\ l \leqslant x \leqslant u\}\ . \tag{8.1}$$

If we assume that x_k is a feasible point then, following the lines in this paper, a feasible descent direction can be obtained by solving

$$\min_s \{s^T g_k + \tfrac{1}{2} s^T M_k s:\ \|D_k^{-1} s\|_2 \leqslant \Delta_k,\ s \in \mathcal{S}_k\} \tag{8.2}$$

where $M_k = H_k + C_k$ and $\mathcal{S}_k$ is contained in the null space of matrix A. We have already sketched a technique in this paper for solving such problems; however, this approach may not be practical here (in general) since in this case $|\mathcal{S}_k|$ is not necessarily small. Therefore, a different sparsity-preserving method must be used to solve (8.2) – Coleman and Hempel [4] have developed a technique based on the use of an ''augmented'' system that may have some potential here.

A possible interior-reflective Newton approach to problem (8.1) is clear from a geometric point of view. After generating a search direction from a strictly feasible point x_k, using (8.2), a reflective path can be searched to find a new (improved) point. Nevertheless, despite this clear geometric picture, many research issues remain, not the least of which is the efficient calculation of this reflective path (while exploiting and maintaining sparsity).

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