Appl. Math. Lett. Vol. 1, No. 2, pp. 141–145, 1988
Printed in Great Britain

0893-9659/88 $3.00 + 0.00
Pergamon Press plc

LOCAL CONVERGENCE OF THE MULTI-SECANT METHOD FOR THE PARALLEL SOLUTION OF SYSTEMS OF NONLINEAR EQUATIONS

Thomas F. Coleman [1] and Guangye Li [2]

1. Introduction. Coleman and Li [2] recently proposed several parallel algorithms for the solution of systems of nonlinear equations

$$F(x) = 0 \tag{1}$$

where $F : R^n \rightarrow R^n$ and F is differentiable with Jacobian matrix $J(x)$. The algorithms proposed in [2] are applicable to message-passing multiprocessor computers where each processor has local memory and there is no shared (global) memory; Coleman and Li [2] discussed implementation details and provided results of numerical experiments obtained on an Intel hypercube computer (iPSC). The algorithms discussed in [2] are global algorithms based on the trust region/dogleg idea first proposed by Powell [6] and then refined and implemented in Minpack [5].

In this note we analyze the *local* behaviour of one of the methods proposed in [2]: the multi-secant method. This method can be implemented on any multiprocessor topology but is most natural on a ring of processors. We assume that there are p processors, or nodes, labelled $P_0, P_1, ..., P_{p-1}$ such that P_i is connected to $P_{i+1\ (mod\ p)}$, for $i = 0 : p-1$. Further we assume that $n \geq p$ and that each processor has enough local memory to store roughly n/p columns of the Jacobian approximation B. (Of course in practise we must be able to store a factorization of the matrix B but we ignore such details here - see [2] .)

We assume the columns of B have been partitioned amongst the p nodes: define $I(j)$ to be the index set of columns of B stored on node j. Another major assumption behind the multi-secant method is that the evaluation of $F(x)$ at any point x is *not* a distributed computation. Specifically, we assume that every node has a copy of the subroutine that evaluates F: this subroutine is sequential (Coleman and Li [2] also considered algorithms for the case when $F(x)$ can be evaluated in a distributed parallel manner) and $F(x)$ can be evaluated by any node (given x) without requiring further communication with other nodes.

A high level description of the multi-secant algorithm is illustrated in Figure 1.

The implementation (including globalization) is discussed in [2]; here we are concerned only with the asymptotic analysis concerning the multi-secant update.

As mentioned in Figure 1, once s is determined F is evaluated at p points concurrently. Specifically, each node evaluates F at a different point. Node 0 evaluates

[1] Computer Science Department, Cornell University, Ithaca NY 14853. Research partially supported by Applied Mathematical Sciences Research Program (KC-04-02) of the Office of Energy Research of the U.S. Department of Energy under grant DE-FG02-86ER25013.A000.

[2] Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ontario, Canada. N2L 3G1. Permanent Address: Computer Center, Jilin University, People's Republic of China.

Guess an initial x;
Evaluate $F(x)$ and determine an initial B;

Repeat
Solve $Bs = -F(x)$;
Evaluate F at p points (including $x + s$);
Update B using the multi-secant update (rank p) ;
$x \leftarrow x + s$;

FIG. 1. *Local multi-secant algorithm*

$F(x+s)$; node j, $1 \leq j \leq p-1$, evaluates $F(x+s^j)$ where s^j is a sparse projection of s. That is, component i of s^j will be either s_i or 0. In particular, $s^0 = s$ and for $j = 1 : p-1$,

$$\text{(2)} \qquad if\ \{i-1 = j \ mod\ p\}\ \ or\ \ \{s_i^k = 0,\ 0 \leq k < j\}\ then \quad s_i^j = 0$$

$$\text{(3)} \qquad otherwise \quad s_i^j = s_i$$

After evaluation, each node sends a copy of its newly computed function value to its higher numbered neighbor on the ring. Hence, after this shift, node j will have the vectors $F(x)$, $F(x+s^j)$, and $F(x + s^{(j-1) \ mod \ p})$.

We now demand that each node satisfy its own local secant equation. Notation: For a matrix M let $M_{I(j)}$ denote the matrix of the same dimensions which matches M in columns $I(j)$ and whose other columns are zero columns. Define $d^0 = s^{p-1}$ and $d^j = s^{j-1} - s^j$. For $j = 0 : p-1$ the secant equation for node j is

$$\text{(4)} \qquad B^+_{I(j)}[d^j] = y^j$$

where $y^j \stackrel{def}{=} F(x+s^{j-1}) - F(x+s^j)$. Equation (4) is reasonable because

$$\text{(5)} \qquad \{\int_0^1 J_{I(j)}(x + s^j + \tau d^j)\partial\tau\}(d^j) = y^j.$$

In light of (4), the local secant update for node j, $j = 0 : p-1$, is

$$\text{(6)} \qquad B^+_{I(j)} \leftarrow B_{I(j)} + (d^{j^T} d^j)^+(y^j - B_{I(j)}d^j)d^{j^T}$$

where for any scalar α we define the pseudo-reciprocal:

$$\alpha^+ = \begin{cases} \alpha^{-1} & if \alpha \neq 0 \\ 0 & if \alpha = 0. \end{cases}$$

Therefore, the multi-secant method can be written as

$$\text{(7)} \qquad B^+ \leftarrow \sum_{j=0}^{p-1}(B_{I(j)} + (d^{j^T} d^j)^+(y^j - B_{I(j)}d^j)d^{j^T})$$

2. Local and Superlinear Convergence. In this section we establish the local and superlinear convergence of the multi-secant method.

The multi-secant method we have proposed here appears to be a member of the broad class of multiple secant methods considered by Schnabel [7]. However, the analysis given in [7] is not applicable to the method described here for an important reason: in [7] it is assumed that the matrix of "differencing vectors" - $(d^0, \ldots, d^{p-1})$ in our case - is always of full rank p. This assumption is crucial to the analysis provided in [7]; however, it is not a permissible assumption here. For example, at some point x the vector d^j may be equal to the zero vector (for some j).

Assumptions: Let $x^* \in D$, D an open convex set in R^n, such that $F(x^*) = 0$ and $J(x^*)$ is nonsingular. Assume that the Jacobian matrix $J(x)$ satisfies the following Lipschitz condition for all $x \in D$: For every $0 \leq j \leq p-1$ there exists a $\gamma_j > 0$ such that

$$\| J(x)_{I(j)} - J(y)_{I(j)} \|_F \leq \gamma_j \| x - y \|_2, \quad \forall x, y \in D. \tag{8}$$

Hence, if we define $\gamma^2 = \sum_{j=0}^{p-1} \gamma_j^2$ then we have the following Lipschitz condition on J:

$$\| J(x) - J(y) \|_F \leq \gamma \| x - y \|_2, \quad \forall x, y \in D. \tag{9}$$

The following lemma is the crucial "bounded deterioration" result needed to establish the convergence properties. Let P_j be the orthogonal projector: $P_j = (d^{jT} d^j)^+ (d^j d^{jT})$, for $j = 0 : p-1$. Notation: A vector norm is assumed to be the 2-norm unless otherwise indicated.

LEMMA 1. *Let F satisfy the assumptions listed above and let B^+ be generated by the multi-secant method (7). If $x + s^j \subset D$, $j = 0 : p-1$, then*

$$\| B^+_{I(j)} - J(x^*)_{I(j)} \|_F^2$$
$$\leq$$
$$\| [B_{I(j)} - J(x^*)_{I(j)}][I - P_j] \|_F^2 + (3\gamma_j \sigma(x^+, x))^2$$

where $\sigma(x^+, x) = max\{\| x^+ - x^* \|, \| x - x^* \|\}$.

Proof. Let $E^+ = B^+ - J(x^*)$, $E = B - J(x^*)$ and define

$$\bar{J}_{I(j)} = \int_0^1 J_{I(j)}(x + s^j + \tau d^j) \partial\tau \tag{10}$$

and therefore

$$\bar{J}_{I(j)} d^j = y^j \tag{11}$$

Hence, from (6) and (11), it follows that

$$\begin{aligned} E^+_{I(j)} &= B_{I(j)} + (d^{jT} d^j)^+ (y^j - B_{I(j)} d^j) d^{jT} - J(x^*)_{I(j)} \\ &= E_{I(j)}[I - P_j] + (d^{jT} d^j)^+ [y^j - J(x^*)_{I(j)} d^j] d^{jT} \\ &= E_{I(j)}[I - P_j] + [\bar{J}_{I(j)} - J(x^*)_{I(j)}] P_j. \end{aligned} \tag{12}$$

But the matrix P_j is an orthogonal projector and therefore, from (12),

$$\| E^+_{I(j)} \|_F^2 = \| E_{I(j)}[I - P_j] \|_F^2 + \| [\bar{J}_{I(j)} - J(x^*)_{I(j)}]P_j \|_F^2 . \tag{13}$$

Using Lipschitz condition (8), we have

$$\begin{aligned}
\| [\bar{J}_{I(j)} - J(x^*)_{I(j)}]P_j \|_F^2 &\leq \| \int_0^1 [J_{I(j)}(x + s^j + \tau d^j) - J(x^*)_{I(j)}]\partial\tau \|_F^2 \\
&\leq (\int_0^1 \gamma_j(\| x + s^j + \tau(s^{j-1} - s^j) - x^* \|)\partial\tau)^2 \\
&\leq (\int_0^1 \gamma_j(\| x - x^* \| + \tau \| s^{j-1} \| + (1-\tau)\|s^j\|)\partial\tau)^2 \\
&\leq \gamma_j^2(\|x - x^*\| + \|s\|)^2 \\
&\leq (3\gamma_j\sigma(x^+, x))^2. \qquad (14)
\end{aligned}$$

Substituting (14) into (13) yields the desired inequality. ∎

THEOREM 2. *Let F satisfy the assumptions stated above. Let $\{x^{(k)}\}$ be generated by the multi-secant method(Fig.1, (7)). If there exist $\epsilon, \delta > 0$ such that if $x^{(0)} \in D$ and $B^{(0)}$, a nonsingular $n \times n$ matrix , satisfy*

$$\|x^{(0)} - x^*\| < \epsilon, \quad \|B^{(0)} - J(x^*)\|_F < \delta \tag{15}$$

then $\{x^{(k)}\}$ is well-defined and converges q-superlinearly to x^*.

Proof. From Lemma 1,

$$\|B^+_{I(j)} - J(x^*)_{I(j)}\|_F^2 \leq \|B_{I(j)} - J(x^*)_{I(j)}\|_F^2 + (3\gamma_j\sigma(x^+, x))^2 \tag{16}$$

and therefore, summing both sides of (16) as $j = 0 : p-1$, and then taking square roots, the following bound is obtained:

$$\|B^+ - J(x^*)\|_F \leq \|B - J(x^*)\|_F + 3\gamma(\sigma(x^+, x)). \tag{17}$$

Therefore, using Theorem 5.1 of [3], $\{x^{(k)}\}$ converges at least q-linearly to x^*.

To prove q-superlinear convergence, Theorem 3.1 of [3] states that we need only show

$$\lim_{k\to\infty} \frac{\|(B^{(k)} - J(x^*))s^{(k)}\|}{\|s^{(k)}\|} = 0. \tag{18}$$

For a given $j \in \{0, 1, ..., p-1\}$, if there is a k_0 such that $((d^j)^{(k)^T}(d^j)^{(k)})^+ = 0$ for all $k > k_0$, then $\|[B^{(k)}_{I(j)} - J(x^*)_{I(j)}](d^j)^{(k)}\| = 0$ for all $k > k_0$. Otherwise let $\{(d^j)^{(k_i)}\}$ be the subsequence of all points satisfying $((d^j)^{(k_i)^T}(d^j)^{(k_i)})^+ > 0$. Using Lemma 1 and essentially the same argument used in ([3], p. 58) or ([4], p.183) we have

$$\lim_{i\to\infty} \frac{\|[B^{(k_i)}_{I(j)} - J(x^*)_{I(j)}](d^j)^{(k_i)}\|}{\|(d^j)^{(k_i)}\|} = 0. \tag{19}$$

Hence, in either case,

$$\lim_{k\to\infty} \|(d^j)^{(k)}\|^+ \|[B_{I(j)}^{(k)} - J(x^*)_{I(j)}](d^j)^{(k)}\| = 0. \tag{20}$$

But,

$$\begin{aligned} \frac{\|[B^{(k)} - J(x^*)]s^{(k)}\|}{\|s^{(k)}\|} &= \frac{\|[B^{(k)} - J(x^*)]\sum_{j=0}^{p-1}(d^j)^{(k)}\|}{\|s^{(k)}\|} \\ &= \frac{\|\sum_{j=0}^{p-1}[B_{I(j)}^{(k)} - J(x^*)_{I(j)}](d^j)^{(k)}\|}{\|s^{(k)}\|} \\ &\leq \sum_{j=0}^{p-1}\{\|[B_{I(j)}^{(k)} - J(x^*)_{I(j)}](d^j)^{(k)}\| \cdot \|(d^j)^{(k)}\|^+\} \end{aligned} \tag{21}$$

Therefore, (18) follows from (20) and (21). ∎

Note: Coleman and Li [2] briefly discussed the *generalized* multi-secant method in which each processor performs a multiple rank update, say rank q. Our analysis above can be directly applied to this situation as well: it is merely necessary to define a conceptual multiprocessor with $\bar{p} = qp$ nodes.

REFERENCES

[1] C. Broyden, *A class of methods for solving nonlinear simultaneous equations*, Mathematics of Computation, 19 (1965), pp. 577–593.

[2] T. F. Coleman and G. Li, *Solving systems of nonlinear equations on a message-passing multiprocessor*, Tech. Rep. CS–87–887, Computer Science Department, Cornell University, 1987.

[3] J. E. Dennis and J. J. Moré, *Quasi-newton methods, motivation and theory*, SIAM Review, 19 (1977), pp. 46–89.

[4] J. E. Dennis and R. B. Schnabel, *Numerical Methods for Unconstrained Optimization*, Prentice-Hall, 1983.

[5] J. J. Moré, B. S. Garbow, and K. E. Hillstrom, *User guide for Minpack-1*, Tech. Rep. ANL–80–74, Argonne National Laboratory, 1980.

[6] M. J. D. Powell, *A hybrid method for nonlinear equations*, in Numerical Methods for Nonlinear Algebraic Equations, P. Rabinowitz, ed., Academic Press, 1970, pp. 87–114.

[7] R. B. Schnabel, *Quasi-Newton methods using multiple secant equations*, Tech. Rep. CU-CS-247-83, Department of Computer Science, University of Colorado, 1983.