Risk Minimization Hedging Under Non-optimal Exercising

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Abstract

Many complex options (for example options embedded in insurance contracts) have an early exercising feature. It is important to evaluate the impact of the non-optimal exercising in the pricing and risk management for these options. We consider the problem of discrete hedging under irrational exercising. We propose a model to explicitly model irrational exercising and compute local risk minimization hedging strategies under this model. We evaluate quadratic and piecewise linear risk minimization approaches in this framework and compare hedging performance between different hedging strategies. In addition, we compare hedging effectiveness under irrational exercising with that of European contracts and American contracts with optimal exercising. We find that irrational exercising may have a significant impact on discrete hedging strategies and hedging costs.

1 Introduction

Pricing and risk management of options typically assume that an option holder exercises optimally to maximize the gain from the contract. This is often referred to as rational or optimal exercising. While certain groups of option holders, for example, traders, may come close to the assumption of optimal exercising, other holders typically exercise the option non-optimally. It turns out that, even for standard American put and call options traded directly on the S&P 100 index (Fernando Diz (1993)) at Chicago Board Option Exchange, there is empirical evidence of irrationality, see, e.g., (Poteshman and Serbin, 2001).

Non-optimal exercising can be particularly prevalent for insurance contracts such as variable annuities with surrender options, given the nature of the contracts. Thus the assumption of rational exercising for pricing and hedging of these insurance contracts can be more problematic. Indeed, not all contract holders are expected to use the surrender opportunity optimally, if at all. It is thus important to investigate whether irrationality of exercising plays a significant role in valuation and hedging of such variable annuities.

The variable annuity (VA) market in the US or segregated funds in Canada experienced a tremendous growth in the 1990's due to the demand for retirement savings from the baby boomers and simultaneous strong performance of the equity market. These VA contracts may include Guaranteed Minimum Death Benefits (GMDB), received when the owner of the contract dies, and these benefits may be linked to the performance of a mutual fund or a market index such as S&P 500.

Until the beginning of the 1990's, the death benefits have been simple principal guarantees (the original investment is guaranteed in case of death) or rising floor guarantees (the original investment accrued at a minimally guaranteed interest rate, possibly capped at a predetermined level). In the bullish market of the 1990's, insurance companies have started to offer the GMDB with more attractive features, such as the ratchet, which guarantees a death benefit based upon the highest anniversary account value. This exposes the annuity writers to potentially large claims during prolonged periods of weak equity market performance. Regulators and insurance companies are increasingly concerned with potentially large losses. This has increased the demand for good risk management strategies for variable annuities.

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The traditional actuarial risk management approach adopts a passive strategy of holding a sufficient reserve in the risk-free asset in order to meet liabilities with a high probability. Option theory has been applied to price and hedge embedded options in variable annuities, see e.g., Brennan and Schwartz (1976), Boyle and Schwartz (1977), Aase and Persson (1992), Persson (1993), Bacinello and Ortu (1993). With the assumption that the market is complete under both financial and mortality risk, the option value is equal to the expected value of the payoff with respect to a risk-neutral probability measure. Moreover, the option can be exactly replicated using delta hedging.

Variable annuity contracts that grant a policy holder a surrender option[1] allow the policy holder to cash out the contract from the issuing company anytime he chooses. These contracts are essentially complex embedded options with American-type exercising features. Rational policy holders surrender the option when the policy value minus the surrender charge is greater than the value of the embedded option. The surrender option adds additional complexity for pricing and hedging of variable annuity contracts.

Under different assumptions and methodologies, the problem of valuing a surrender option embedded in life insurance products has been investigated in Albizzati and Geman (1994), Bacinello (2003), Grosen and Jørgensen (1997, 2000), and Jensen et al. (2001). Mudavanhu and Zhuo (2002) also consider the fair value of GMDB with the rachet feature and surrender option. In Coleman et al. (2006) and Coleman et al. (2007b), discrete hedging of embedded European options in variable annuity contracts have been investigated under different models, including models with jump risk. In addition, it is shown that, under a jump model, hedging using liquid standard options is more effective compared with dynamic hedging using the underlying. It is important to note that, in the aforementioned studies, valuation and hedging are considered under the assumption that the policy holders *rationally* lapse out the contract.

In order to extend such frameworks to allow policy holders to surrender contracts *non-optimally* it is necessary to explicitly model the exercising behavior of the pool of policy holders for a given type of contracts. An example of such modeling for a different type of problem – refinancing behavior of a mortgage pool – is presented in Kalotay et al. (2004). Faced with a choice of refinancing or not-refinancing, a mortgage holder may refinance too early (a *Leaper*), refinance optimally (a *Financial Engineer*), or refinance too late (a *Laggard*). All people in a mortgage pool are divided into these categories according to refinancing behavior. The formal definition of Leaper and Laggard are based on a parameterization which is a natural extension of the definition of the financial engineer. An important feature of such a model is the representation of the change of the pool's structure over time. As time goes by and refinancing events occur, holders leave the pool: first the leapers (if any), then individuals who refinance optimally, and finally the laggards. The most aggressive mortgagors are the first to refinance, leaving behind the slower reacting laggards. This is referred to as the prepayment *burnout*.

In Kalotay et al. (2004), it is assumed that the difference between mortgage rates fully determines whether refinancing occurs or not. The authors show that the Leaper behavior is not typical in practice and may result in significant losses for the contract holder from a theoretical point of view. Thus, the authors concentrate on considering financial engineers and laggards only. Kalotay et al. (2004) also demonstrates that modeling irrational refinancing changes the value of the contract. For example, the authors observe up to 1% change in the contract's value. While not a particularly large deviation, it is already significant and deserves attention, particularly when the proportion of laggards increases.

Instead of modeling the pool of mortgages directly, Kalotay et al. (2004) represent the proportions of people for each refinancing behavior category in the pool by a probability distribution which is called the *laggard distribution*. If the goal is to obtain the value of a contract, the value corresponding to each refinancing behavior category can be calculated. The expected value is then computed based on the laggard distribution. Note that this approach allows for the burnout effect to take place because the laggard distribution is adjusted as refinancing events occur.

We adapt the approach presented in Kalotay et al. (2004) to the problem of discrete hedging of a pool of contracts held by individuals with different exercising behaviors. We model exercise behaviors in a pool of option holders by classifying them into different categories depending on whether they exercise optimally, too early, or too late. Also, following the same argument as in Kalotay et al. (2004), we exclude Leapers from our analysis of hedging strategies. As time goes on, the structure of the pool changes and it is necessary to model the surrender burnout. A family of laggard distributions is introduced that allows convenient modeling of

[1] This is also known as the lapsation option.

the burnout.

Once the exercising behavior of contract holders is described, an appropriate hedging strategy should be constructed. For frameworks featuring optimal exercising a variety of approaches have been explored. Under the assumption of continuous trading and market completeness, an American type option can be delta hedged in a manner similar to the European option. In the discrete hedging context, optimal discrete hedging strategies for European options have been studied based on risk minimization, see, e.g., Föllmer and Schweizer (1989)), Schäl (1994), Schweizer (1995, 2001), Mercurio and Vorst (1996), Heath et al. (2001a), Heath et al. (2001b), Bertsimas et al. (2001). Risk minimization discrete hedging of American-type options has also been considered in Coleman et al. (2007a); however the analysis is conducted under the assumption that the holder exhibits *optimal early exercise behavior*. Once again, the option holder is assumed to be capable of evaluating market conditions perfectly, constantly monitoring market information, and making an optimal exercise decision based on the current value of the contract.

Using the model proposed in this paper for exercising behavior of policy holders, we extend risk minimization discrete hedging for American-type options to incorporate irrational exercising and investigate its impact on the corresponding solutions and performance measures. The modeling approach and computation of hedging strategies can be extended to more complex embedded options, including variable annuity contracts.

Our presentation is organized as follows. In Section 2 we describe the problem for a pool of holders for a put option and show how it can be represented as a single contract with a *non-deterministic exercising* model. We introduce a *laggard distribution* model for exercising behavior. Then in Section 3 we describe quadratic and piecewise linear local risk minimization methods for calculating hedging strategies under the laggard distribution model. Finally, we analyze hedging performance based on simulations and present and discuss computational results in Section 4.

2 A Model for Irrational Exercising

2.1 Laggard Distributions

Consider a Bermudan put option on an underlying asset with the expiration date T. Without loss of generality, we assume that the underlying asset pays no dividend and this option can be exercised at discrete times $0 = t_0 < t_1 < \ldots < t_M = T$. For simplicity, we also assume that permissible exercising times are the hedging times as well. Assume that $X_k,\ k = 0, 1, \ldots, M$, denote the discounted underlying price at $t_k,\ k = 0, 1, \ldots, M$. We assume that the optimal early exercise critical prices are $\bar{X}_k,\ k = 0, \ldots, M$ (which can be precomputed in computational implementation). For an American option with exercising permissible in continuous time, the collection of the optimal exercise critical prices form the early exercise curve (see, for example, Myneni (1992)).

Suppose that the pool initially consists of P option holders. We classify the holders in the pool according to a parameterization based on the optimal exercising critical price $\bar{X}_k$. Each holder $p,\ p = 1, \ldots, P$, in the pool is assigned an irrationality parameter ℓ (which depends on p); the critical exercise price for a holder with this parameter is $(1 - \ell)\bar{X}_k$. This irrationality parameter determines whether the person exercises or not at a given relative difference between the current underlying price and the optimal early exercise critical price. More precisely, suppose that at hedging time t_k the discounted underlying price is X_k and the optimal exercise critical value is $\bar{X}_k$. Unless the option has been exercised earlier, the holder with the parameter ℓ makes a decision to exercise the option at time t_k if

$$Z_k \stackrel{def}{=} 1 - \frac{X_k}{\bar{X}_k} \geq \ell.$$

Under this parameterization, option holders with $\ell = 0$ exercise optimally, $\ell < 0$ corresponds to leapers and $\ell > 0$ to laggards. The value of Z_k shows the relative difference between the current underlying price and the optimal early exercise critical price. If $Z_k < 0$ then X_k is above $\bar{X}_k$ and only leapers may exercise in this case. If Z_k becomes zero, the underlying price hits the optimal exercise boundary and rational people ($\ell = 0$) exercise the option. Subsequently we refer to ℓ as the laggard spread. Figure 2.1 illustrates the situation.

We assume that an initial distribution of the exercising types of option holders is given. For example, consider a pool consisting of 100 rational people (type I), 70 irrational people with $\ell = 0.1$ (type II), and 30 irrational people (type III) with $\ell = 1$. The initial pool distribution is given by the frequency (proportion) of each type in the pool. For example, since there are 100 Type I individuals in the pool of total 200, the probability for $\ell = 0$ is $1/2$. Clearly, this distribution of types can be equivalently expressed as a distribution of the laggard spread ℓ.

Analysis of hedging of a pool of the put contracts can be done by considering hedging of a single contract with the holder randomly selected with an equal probability from the pool. Equivalently we can consider one single representative contract with non-deterministic exercising with probability of exercising defined from the laggard distribution.

To see this equivalence, we note that the probability of any exercise event for these two approaches are equal. Consider the example of the pool distribution given before. When the underlying price first hits the optimal exercise boundary, a random selected holder in the first approach will exercise with probability $100/200 = \frac{1}{2}$. On the other hand, the laggard distribution following the second approach prescribes that the contract will be exercised with probability $\frac{1}{2}$ as well. Distributions for all quantities determined by exercise events (e.g., payoffs, incurred hedging costs, etc.) for these two approaches will be the same. Thus hedging of a pool of put options can be considered equivalently as hedging of a single representative contract in which the holder has a non-deterministic exercising strategy.

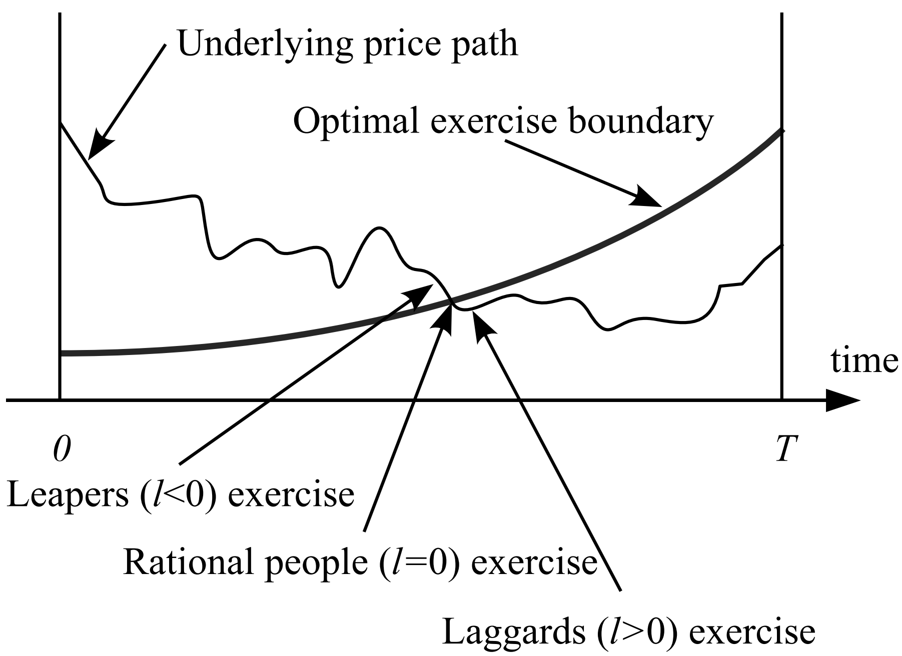

Figure 2.1: Illustration of exercising behavior of people in the pool. Here ℓ denotes the individual's irrational behavior parameter.

However, as time progresses, the structure of the pool of holders changes. This means that the laggard distribution in terms of the laggard spread changes as well. Typically $Z_0 < 0$ initially. As Z_k increases to zero, the financial engineers with a laggard spread $l = 0$ will exercise and thus leave the pool. If Z_k further increases, laggards with larger irrationality parameter $\ell \leq Z_k$ exercise and leave the pool as well. We assume that there are no leapers[2]; however, extending the model to allow leapers in the pool is

[2] As already noted, leaper behavior is not typical in practice and we limit our analysis to the case $\ell \geq 0$, thus, allowing only rational people and laggards into the pool.

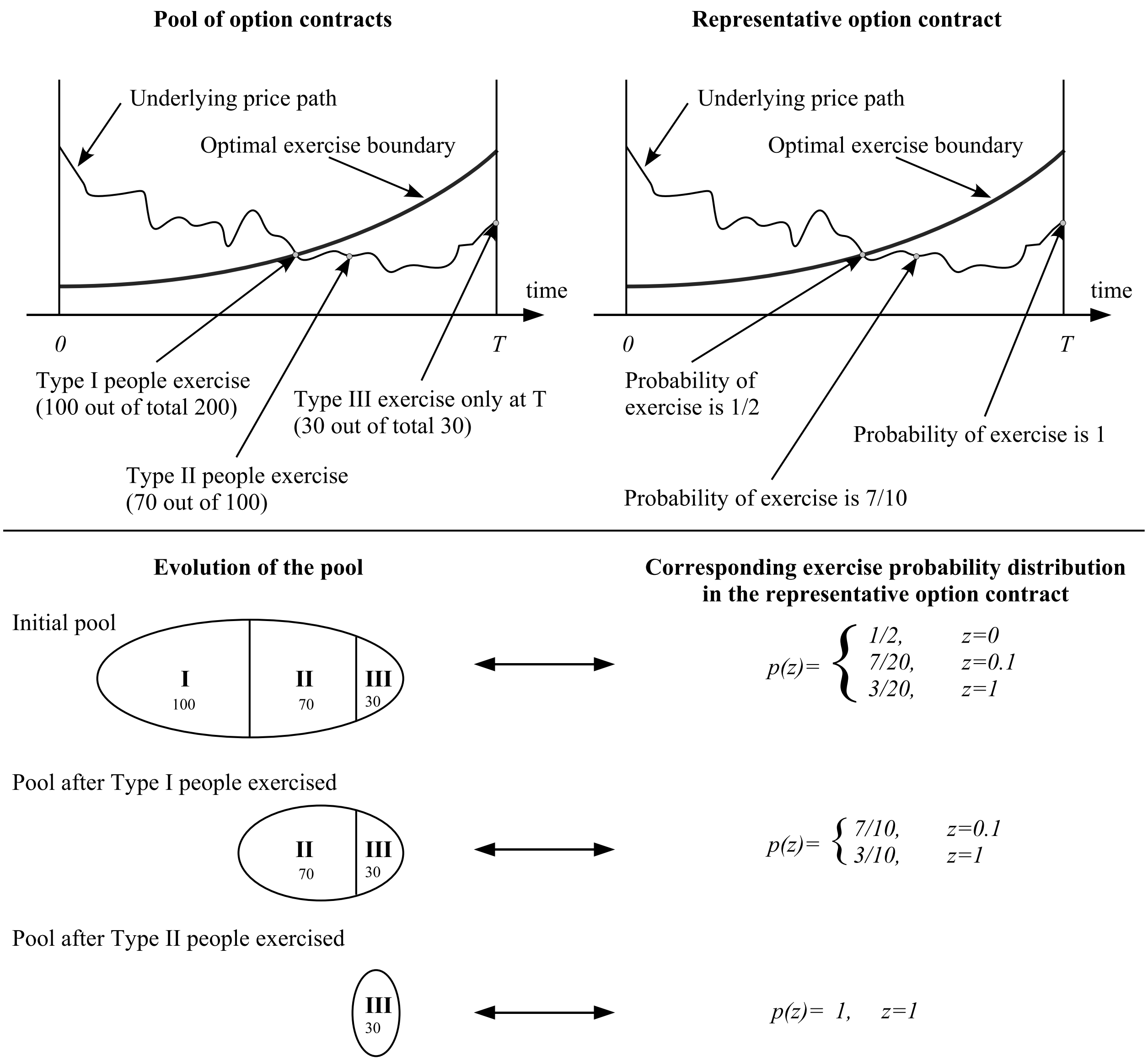

Figure 2.2: Correspondence between the pool of contracts and the representative contract. Type I individuals are rational. Type II are irrational with parameter $\ell = 0.1$. Type III are irrational with parameter $\ell = 1$.

straightforward. In the event of exercising at hedging moment t_k the put option holder receives discounted payoff $H_k = \max(0, Ke^{-rt_k^H} - X_k)$.

The change of the pool structure for the simple example described previously is depicted in Figure 2.2 First Type I holders exercise. This is followed by departure of the Type II holders. Note that, for each new distribution of the exercising type of the remaining pool, there is a corresponding distribution of the laggard spread. Moreover, new distribution at t_k corresponds to a conditional distribution from the initial distribution, conditional on excluding holders who have already exercised before t_k.

Next we discuss a family of probability distributions that we use to represent laggard distributions.

2.2 A Family of Laggard Distributions

In Kalotay et al. (2004), a discretized exponential distribution is used for specifying the initial state of the pool of mortgage holders. While discrete distributions might be more convenient for illustrative and computational purposes, we consider continuous or mixed distributions which can be useful when considering a large pool of holders. Such distributions may correspond to pools in which individuals have a large variety of exercise behaviors rather than a few distinct ones. Also, we would like the distribution to be described by only a few parameters yet be flexible enough. For example, the distribution used in Kalotay et al. (2004) depends only on one parameter, and, as a result, the proportion of rational people is linked to the behavior of the tail of the distribution. For our model we would like to be able to specify the proportion of rational people and proportions for irrational ones separately.

To achieve these goals we consider a family of laggard distributions which are described as follows:

- There is an atom at point 0 with a weight $\rho \in [0, 1]$, specifying the proportion of rational individuals;
- The right tail of the distribution declines exponentially with rate $\lambda > 0$, describing the laggards.

In other words, a random variable Y with such a distribution has the CDF below:

$$\mathcal{LD}_0(\ell) = P(Y \leq \ell) = \begin{cases} \rho + (1-\rho)(1-e^{-\lambda \ell}), & \ell \geq 0 \\ 0, & \ell < 0 \end{cases}.$$

The value of $\mathcal{LD}_0(\ell)$ represents the proportion of people in the corresponding pool at time 0 with the laggard spread parameter less than or equal to ℓ. Figure 2.3 shows a typical graph of such a CDF.

The mean and variance of the random variable Y are given by the following formulae:

$$E[Y] = \frac{1-\rho}{\lambda}, \qquad Var[Y] = \frac{1-\rho^2}{\lambda^2}.$$

It is interesting to note that this family of distributions include the American option with optimal exercising as well as European option as special cases: $\rho = 1$ corresponds to the optimal exercise (which is referred to as 'Bermudan option' later), because it means that all individuals are rational in the initial pool. The limiting case of $\rho = 0, \lambda \to 0$ corresponds to the European option since the pool is comprised of people who never exercise until the expiry.

In our computational analysis, we focus on values of $\mathcal{LD}_0$ at 5% and 10% points, i.e., $\mathcal{LD}_0(0.05)$ and $\mathcal{LD}_0(0.1)$. They could be described as proportions of people with irrationality levels less than 5% and 10% respectively. Table 2.1 presents the values $\mathcal{LD}_0(0.05)$ and $\mathcal{LD}_0(0.1)$ for some distribution examples. We will reference this table when discussing the influence of parameters ρ and λ on risk minimization solutions.

2.3 Evolution of the Laggard Distribution

As time evolves, the composition of the option holder pool changes. As financial engineers and less passive holders leave the pool, slower reacting exercising holders are left in the remaining pool. In the mortgage backed security (MBS) market, the similar change in the mortgage pool is referred to as the prepayment burnout, or simply burnout. Similar to Kalotay et al. (2004), this burnout is easily captured in our model by proper updating of the laggard distribution.

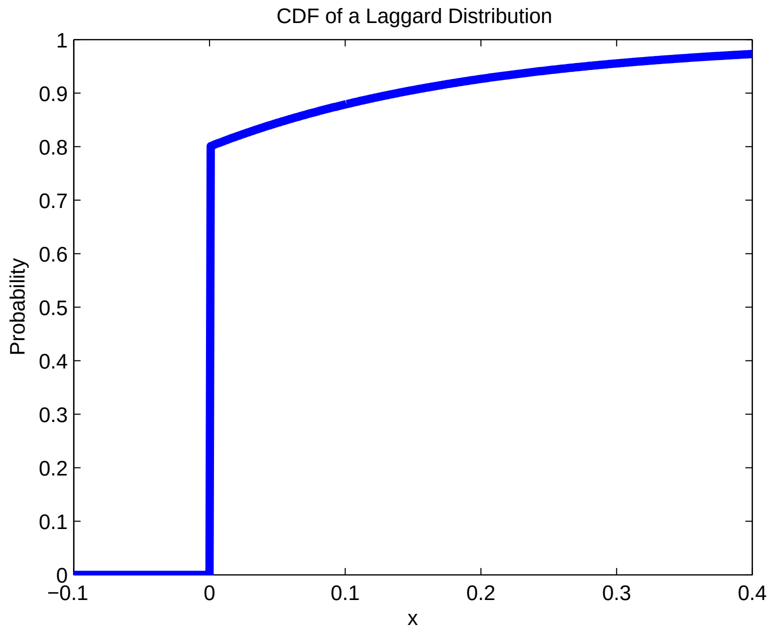

Figure 2.3: CDF of a typical laggard distribution in the family.

Table 2.1: Properties of laggard distributions: values of $\mathcal{LD}_0$ at points 5 and 10% ($\mathcal{LD}_0(0.05)$ and $\mathcal{LD}_0(0.1)$). ρ and λ are parameters of the laggard distribution.

$\mathcal{LD}_0(0.05)$		λ		
		3	5	10
ρ	0.8	0.83	0.84	0.88
	0.5	0.57	0.61	0.7
	0.2	0.31	0.38	0.51
	0	0.14	0.22	0.39

$\mathcal{LD}_0(0.1)$		λ		
		3	5	10
ρ	0.8	0.85	0.88	0.93
	0.5	0.63	0.7	0.82
	0.2	0.41	0.51	0.71
	0	0.26	0.39	0.63

Suppose we are interested in the laggard distribution $\mathcal{LD}_k^{\mathbf{B}}$ with burnout at the hedging moment $0 < k < M$ (The superscript **B** here emphasizes that the distribution characterizes the burn-out effect). We know that, at each hedging moment $k' < k$, people with the laggard spread parameter ℓ less than or equal to $Z_{k'} = 1 - X_{k'}/\bar{X}_{k'}$ have left the pool. Thus, by hedging time k, all people with the exercise behavior parameter $\ell \leq \hat{Z}_k \stackrel{def}{=} \max_{k' \in \{0,...,k-1\}} Z_{k'}$ have departed.

The laggard distribution with burnout can be easily described by specifying a conditional distribution. If $\hat{Z}_k$ has value $\hat{z}_k$ then the laggard distribution at moment k will be given by

$$\mathcal{LD}_k^{\mathbf{B}}(\ell) = P(Y \leq \ell | Y > \hat{z}_k) = \frac{P(\hat{z}_k < Y \leq \ell)}{P(Y > \hat{z}_k)} = \begin{cases} 1 - e^{-\lambda(\ell - \hat{z}_k)}, & \ell > \hat{z}_k \geq 0 \\ 0, & \ell \leq \hat{z}_k \\ \mathcal{LD}_0(\ell), & \hat{z}_k < 0, \end{cases} \tag{2.1}$$

where random variable Y has the CDF $\mathcal{LD}_0$.

Note that, for $\hat{z}_k \geq 0$, $\mathcal{LD}_k^{\mathbf{B}}$ is the CDF of an exponential distribution with parameter λ shifted right by the value $\hat{z}_k$. In addition, note that $\mathcal{LD}_k^{\mathbf{B}}$ depends on the evolution of the underlying price only through the path-dependent parameter $\hat{z}_k$. Thus $\hat{Z}_k$ is a *path-dependent burn-out variable*.

If $\hat{Z}_k = 0$, it means that all rational people have already exercised and the distribution becomes the exponential distribution with a parameter λ. As the underlying price decreases below the optimal exercise curve the value of $\hat{Z}_k$ increases and the distribution shifts right, i.e., we observe the burn-out effect.

To evaluate the impact of ignoring burn-out in the model, we consider the model in which the burn-out effect is not present. While this model may not be valid from the practical point of view, resulting risk minimization hedging optimization problems are much easier to solve. Thus the no burn-out model might serve as an attractive approximation to the more complex but realistic burn-out model. For the no burn-out model the evolution of laggard distribution is simple:

$$\mathcal{LD}_k^{\mathbf{NB}}(\ell) = \mathcal{LD}_0(\ell) \quad \forall \ell.$$

For a given underlying price path we can determine the critical exercise point for each type of individuals in the pool, see Figure 2.2.

As mentioned before, we can determine a hedging strategy and analyze hedging performance by considering a single contract with nondeterministic hedging. Once the laggard distribution is specified, the holder of the single contract with nondeterministic exercising strategy exercises the option in the following way.

Consider now a single contract with one option holder who, at time $k \in \{0, ..., M\}$, *exercises nondeterministically* according to a specified probability distribution (the *laggard distribution*) with CDF denoted as $\mathcal{LD}_k(.)$. Note that the laggard distribution $\mathcal{LD}_k(.)$ is updated as in (2.1) for a model with burn-out.

At the expiry, we assume that holder has the exercising probability of 1. This means that the laggard distribution $\mathcal{LD}_M$ has a singleton of $\ell = 0$ with probability 1.

At each exercising time k, the value of $Z_k = 1 - X_k/\bar{X}_k$ is calculated. Given $\mathcal{LD}_k(.)$, the CDF of the laggard distribution at time k, $\mathcal{LD}_k(Z_k)$ gives the probability of holder has the critical exercise price of $(1 - Z_k)\bar{X}_k = X_k$. Thus $\mathcal{LD}_k(Z_k)$ gives the probability of holder exercising at time t_k, where X_k is the discounted price at time t_k. We can imagine that the option holder, when the discounted underlying price is X_k at time t_k, flips an asymmetric coin to decide whether to exercise or not. These flips and and the underlying price evolution are assumed to be independent.

Formally speaking, let $\{D_k \sim U[0, 1],\ k \in \{0, ..., M\}\}$ be a collection of uniform random variables which are independent of the underlying price stochastic process $\{X_k, k \in \{0, ..., M\}\}$. Then the exercising moment is given by

$$M^* = \min\{k \in \{0, ..., M\} : D_k < \mathcal{LD}_k(z_k))\} \tag{2.2}$$

3 Local Risk Minimization Hedging Under Irrational Exercising

Because our model assumes discrete hedging the market is incomplete and the optimal hedging strategy depends on the choice of the risk measure of the option writer. *Local risk minimization* is computationally simple and it has been successfully used for constructing hedging strategies in incomplete markets (see, e.g., Föllmer and Schweizer (1989), Schäl (1994), Schweizer (1995, 2001), Mercurio and Vorst (1996), Heath et al.

(2001a), Heath et al. (2001b), Bertsimas et al. (2001)). The local risk minimization hedging strategy is determined through backward iterations. Starting from the hedging portfolio matching the liability at the terminal time, the optimal hedging portfolio and hedging strategy at the preceding time are determined to minimize the expected quadratic incremental cost. For European options, the terminal time is the expiry of the option. For an American option, the terminal time is the random optimal stopping time determined by the rational (optimal) exercising strategy.

We now generalize the local risk minimization to hedging the option irrational exercising, which becomes more complex since exercising time now depends on the laggard distribution.

Suppose that the discounted underlying asset price is a square integrable process on a probability space $(\Omega, \mathcal{F}, P)$, with a filtration $(\mathcal{F}_k)_{k=0,1,\ldots,M}$, where $\mathcal{F}_k$ corresponds to the hedging time t_k and, w.l.o.g., $\mathcal{F}_0 = \{\emptyset, \Omega\}$ is trivial.

Assume that the holder exercises irrationally with the stochastic exercising strategy determined from the laggard distribution as described in the previous section. Specifically the probability of exercising at time t_k when the underlying price is X_k is given by $\mathcal{LD}_k(1 - \frac{X_k}{\bar{X}_k})$.

Assume that the hedging portfolio can be rebalanced at time t_k, $k = 0, 1, \cdots, M$. For simplicity of discussion, we assume here that the hedging times are the same as permitted exercising times.

For a European option, the stopping moment is always the expiry T, i.e., $M^* \equiv M$. For an American option with rational exercising M^* is the first time underlying price reaches the critical exercise price. For irrational exercising, the stopping time M^* depends on laggard distributions. In contrast to these two types of options, the main difference for hedging a put with a irrational (stochastic) exercising determined by a laggard distribution is that the stopping time M^* is now defined by (2.2). The stopping time M^* depends on the path-dependent process $\{\hat{Z}_k\}$.

Consider a *trading strategy* represented by two stochastic processes $(\xi_k)_{k=0,\ldots,M^*}$ and $(\eta_k)_{k=0,\ldots,M^*}$, which are adapted to the filtration $\{\mathcal{F}_k\}$, $\mathcal{F}_k = \sigma(X_j | j \in \{0, \ldots, k\})$. Here ξ_k is the number of shares held at time t_k^H, and η_k is the amount invested in the bond at time t_k^H. Let $\xi_{M^*} = 0$ and $\eta_{M^*} = H_{M^*}$, which means that we liquidate our portfolio at the stopping moment to cover the payoff of the option.

The (discounted) value of the portfolio at hedging time t_k, $k = 0, 1, \ldots, M^*$, is given by

$$V_k = \xi_k X_k + \eta_k$$

Let $G_k = \sum_{j=0}^{k-1} \xi_j (X_{j+1} - X_j)$, $k = 1, \ldots, M^*$. Thus G_k is the *accumulated gain* of the dynamic trading strategy due to changes of asset prices up to time t_k^H. At time moment 0, G_0 is set to zero.

The *cumulative cost* C_k is then given by

$$C_k = V_k - G_k, \quad k = 0, \ldots, M^*.$$

The local risk minimization optimization problems can be solved backwards in time, starting at the expiry time t_M with $\xi_M = 0$ and $\eta_M = H_M$.

At each hedging time t_k, $k = 0, \ldots, M-1$, provided that the contract has not been exercised yet ($k < M^*$), the local risk minimization hedging strategy $\{\xi_k, \eta_k\}$ can be computed using one of the following risk minimization formulation, see e.g., Coleman et al. (2007a).

- Quadratic: $\min_{\xi_k, \eta_k} \mathbb{E}_{X,l}\left[(C_{k+1} - C_k)^2 | \mathcal{F}_k\right]$;
- Piecewise Linear: $\min_{\xi_k, \eta_k} \mathbb{E}_{X,l}\left[|C_{k+1} - C_k| | \mathcal{F}_k\right]$;
- Constrained Piecewise Linear:

$$\begin{aligned} & \min_{\xi_k, \eta_k} \mathbb{E}_{X,l}\left[|C_{k+1} - C_k| | \mathcal{F}_k\right] \\ \text{s.t.} \quad & \mathbb{E}_{X,l}\left[(C_{k+1} - C_k) | \mathcal{F}_k\right] = 0. \end{aligned}$$

Note that $\mathbb{E}_{X,l}(\cdot)$ is expectation with respect to both the underlying price distribution and the laggard distribution. In addition, different risk minimization formulations lead to different optimal strategies.

The objective function of the risk minimization problem at time t_k depends on the incremental risk $C_{k+1} - C_k$. Assume that the discounted underlying price at time t_k is X_k and $z_k = 1 - X_k / \bar{X}_k$, where $\bar{X}_k$

is the critical price at t_k. Suppose that the value of path-dependent burn-out parameter $\hat{z}_k$ is given. Then $\hat{z}_{k+1}$ is calculated as $\hat{z}_{k+1} = \max(z_k, \hat{z}_k)$. Given this information, the change in cumulative cost $C_{k+1} - C_k$ can take the following values:

- $(X_{k+1}\xi_{k+1} + \eta_{k+1}) - (X_{k+1}\xi_k + \eta_k)$, in the case exercise does not happen. In this case, the incremental risk equals the difference between the new portfolio value and the value of the current portfolio in the next period;
- $H_{k+1} - (X_{k+1}\xi_k + \eta_k)$, when exercise happens. The incremental risk equals the difference between the payoff and the value of the current portfolio at the end of the period.

The time of exercising under the irrational exercising model is determined from laggard distributions. Since laggard distributions are defined from the path dependent quantity $\hat{Z}_k \stackrel{def}{=} \max_{k' \in \{0,\ldots,k-1\}} Z_{k'}$, computation becomes more expensive since the hedging strategies now depends on the state value of $\hat{Z}_k$.

Next we describe more precisely the local risk minimization hedging strategy computation under a binomial lattice. We assume that the continuous underlying price process $S_t,\ t \in [0, T]$, satisfies a stochastic differential equation,

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \tag{3.1}$$

where W_t is a standard Brownian motion.

3.1 Implementation Under a Binomial Lattice

We consider a discretization of the continuous asset price process (3.1) using a binomial lattice with N periods, see, e.g., Cox et al. (1979). Denote $\tau = T/N$. Node (i, j) of the binomial lattice corresponds to time $i\tau$ with the underlying price $u^{2j-i}S_0$ and the discounted underlying price $e^{-ri\tau}u^{2j-i}X_0$, for $i = 0, \ldots, N,\ j = 0, \ldots, i$, where $u = 1/d = e^{\sigma\sqrt{\tau}}$ and $X_0 = S_0$. For each node the (discounted) underlying price goes up with probability p and down with probability $1 - p$ where $p = \frac{e^{\mu\tau} - d}{u - d}$.

Nodes of the binomial lattice corresponding to hedging times are referred to as hedging nodes. For hedging with rebalancing every n_H periods, the hedging time (which is assumed to be the same as exercising time for simplicity) $t_k = t_k^H = kn_H\tau$. The total number of hedging periods will be an integer $M = N/n_H$. A hedging node at t_k is denoted as $[k, j]$, where j denotes the jth possible discounted values at time t_k. We use X_k^j to denote the discounted underlying price at hedging node $[k, j]$ and X_k still denotes the discounted price at hedging time t_k. For a given binomial lattice, the probability of reaching each node is uniquely determined. Starting from hedging node $[k, j]$, probability of arriving at node $[k + 1, j + l],\ l = 0, \ldots, n_H$, is $p_l = \binom{n_H}{l} p^l(1-p)^{n_H - l}$. For node $[k + 1, j + l]$, $z_{k+1} = z_{k+1}^{j+l} = 1 - X_{k+1}^{j+l}/\bar{X}_{k+1}$. The probability of exercising is determined by the laggard distribution, which is $\mathcal{LD}_{k+1}(z_{k+1})$. For distributions with burn-out, $\mathcal{LD}_{k+1}(z_{k+1})$ is defined in (2.1) in Section 2.3.

3.2 Quadratic Local Risk Minimization

The objective function in the quadratic local risk minimization can be written as

$$\begin{aligned} & \mathbb{E}_{X,l}\left[(C_{k+1} - C_k)^2 | \mathcal{F}_k\right] = \mathbb{E}_{X,l}\left[(C_{k+1} - C_k)^2 | X_k = X_k^j, \hat{Z}_k = \hat{z}_k\right] \\ = & \sum_{l=0}^{n_H} p_l \left[(1 - \pi_{k+1}^{j+l})\left(X_{k+1}^{j+l}(\xi_{k+1}^{j+l}(\hat{z}_{k+1}) - \xi_k^j) + (\eta_{k+1}^{j+l}(\hat{z}_{k+1}) - \eta_k^j)\right)^2 + \pi_{k+1}^{j+l}\left(H_{k+1}^{j+l} - X_{k+1}^{j+l}\xi_k^j - \eta_k^j\right)^2\right], \end{aligned}$$

where $\pi_{k+1}^{j+l} = \mathcal{LD}_{k+1}(z_{k+1}^{j+l})$ and $z_{k+1}^{j+l} = 1 - X_{k+1}^{j+l}/\bar{X}_{k+1}$. For a model with burnout consideration, $\mathcal{LD}_{k+1} = \mathcal{LD}_{k+1}^{\mathbf{B}}$. Otherwise, $\mathcal{LD}_{k+1} = \mathcal{LD}_{k+1}^{\mathbf{NB}}$. Note that $\mathcal{LD}_{k+1}^{\mathbf{B}}$ depends on the value of $\hat{z}_{k+1}$.

The minimization problem for this objective function can be rewritten in a matrix-vector form as

$$\min_{x\in\mathbb{R}^2} \|Ax-b\|_2 \qquad x = \begin{pmatrix} \eta_k^j \\ \xi_k^j \end{pmatrix},$$

$$A = \begin{bmatrix} \sqrt{p_0} & \sqrt{p_0(1-\pi_{k+1}^j)}X_{k+1}^j \\ \vdots & \vdots \\ \sqrt{p_{n_H}} & \sqrt{p_{n_H}(1-\pi_{k+1}^{j+n_H})}X_{k+1}^{j+n_H} \\ \sqrt{p_0} & \sqrt{p_0\pi_{k+1}^j}X_{k+1}^j \\ \vdots & \vdots \\ \sqrt{p_{n_H}} & \sqrt{p_{n_H}\pi_{k+1}^{j+n_H}}X_{k+1}^{j+n_H} \end{bmatrix} \quad b = \begin{pmatrix} \sqrt{p_0(1-\pi_{k+1}^j)}(X_{k+1}^j\xi_{k+1}^j(\hat{z}_{k+1}) + \eta_{k+1}^j(\hat{z}_{k+1})) \\ \vdots \\ \sqrt{p_{n_H}(1-\pi_{k+1}^{j+n_H})}(X_{k+1}^{j+n_H}\xi_{k+1}^{j+n_H}(\hat{z}_{k+1}) + \eta_{k+1}^{j+n_H}(\hat{z}_{k+1})) \\ \sqrt{p_0\pi_{k+1}^j}H_{k+1}^j \\ \vdots \\ \sqrt{p_{n_H}\pi_{k+1}^{j+n_H}}H_{k+1}^{j+n_H} \end{pmatrix}$$

3.3 Piecewise Linear Local Risk Minimization

The objective function in the piecewise linear local risk minimization can be written as

$$\begin{aligned} & \mathbb{E}_{X,l}\left[|C_{k+1}-C_k|\,|\mathcal{F}_k\right] = \mathbb{E}_{X,l}\left[|C_{k+1}-C_k|\,|X_k = X_k^j, \hat{Z}_k = \hat{z}_k\right] \\ = & \sum_{l=0}^{n_H} p_l\left[(1-\pi_{k+1}^{j+l})\left|X_{k+1}^{j+l}(\xi_{k+1}^{j+l}(\hat{z}_{k+1})-\xi_k^j)+(\eta_{k+1}^{j+l}(\hat{z}_{k+1})-\eta_k^j)\right| + \pi_{k+1}^{j+l}\left|H_{k+1}^{j+l}-X_{k+1}^{j+l}\xi_k^j-\eta_k^j\right|\right], \end{aligned}$$

where π_{k+1}^{j+l} is defined in the same way as for the quadratic local risk minimization above.

The minimization problem for this objective function can be rewritten in a matrix-vector form as

$$\min_{x\in\mathbb{R}^2} \|Ax-b\|_1 \qquad x = \begin{pmatrix} \eta_k^j \\ \xi_k^j \end{pmatrix},$$

$$A = \begin{bmatrix} p_0 & p_0(1-\pi_{k+1}^j)X_{k+1}^j \\ \vdots & \vdots \\ p_{n_H} & p_{n_H}(1-\pi_{k+1}^{j+n_H})X_{k+1}^{j+n_H} \\ p_0 & p_0\pi_{k+1}^jX_{k+1}^j \\ \vdots & \vdots \\ p_{n_H} & p_{n_H}\pi_{k+1}^{j+n_H}X_{k+1}^{j+n_H} \end{bmatrix} \quad b = \begin{pmatrix} p_0(1-\pi_{k+1}^j)(X_{k+1}^j\xi_{k+1}^j(\hat{z}_{k+1}) + \eta_{k+1}^j(\hat{z}_{k+1})) \\ \vdots \\ p_{n_H}(1-\pi_{k+1}^{j+n_H})(X_{k+1}^{j+n_H}\xi_{k+1}^{j+n_H}(\hat{z}_{k+1}) + \eta_{k+1}^{j+n_H}(\hat{z}_{k+1})) \\ p_0\pi_{k+1}^jH_{k+1}^j \\ \vdots \\ p_{n_H}\pi_{k+1}^{j+n_H}H_{k+1}^{j+n_H} \end{pmatrix}$$

3.4 Constrained Piecewise Linear Local Risk Minimization

For the constrained piecewise linear local risk minimization problem, the constraint can be rewritten as:

$$\begin{aligned} & \mathbb{E}_{X,l}\left[C_{k+1}-C_k\,|\mathcal{F}_k\right] = \mathbb{E}_{X,l}\left[C_{k+1}-C_k\,|X_k = X_k^j, \hat{Z}_k = \hat{z}_k\right] \\ = & \sum_{l=0}^{n_H} p_l\left[(1-\pi_{k+1}^{j+l})\left(X_{k+1}^{j+l}(\xi_{k+1}^{j+l}(\hat{z}_{k+1})-\xi_k^j)+(\eta_{k+1}^{j+l}(\hat{z}_{k+1})-\eta_k^j)\right) + \pi_{k+1}^{j+l}\left(H_{k+1}^{j+l}-X_{k+1}^{j+l}\xi_k^j-\eta_k^j\right)\right] \\ = & \sum_{l=0}^{n_H} p_l\left[(1-\pi_{k+1}^{j+l})\left(X_{k+1}^{j+l}\xi_{k+1}^{j+l}(\hat{z}_{k+1})+\eta_{k+1}^{j+l}(\hat{z}_{k+1})\right) + \pi_{k+1}^{j+l}H_{k+1}^{j+l}-X_{k+1}^{j+l}\xi_k^j\right]-\eta_k^j \\ = & \; 0 \end{aligned}$$

Substituting the expression for η_k^j into the objective function results in a one-dimensional L_1 problem for ξ_k^j.

For the model with burn-out, i.e., $\mathcal{LD}_k^{\mathbf{B}}$ is used, optimization problems must be solved for each possible value of $\hat{z}_k$. Thus, for a continuous laggard distribution, discretization is necessary. The discretization for the laggard distribution used in subsequent computational results is described in Appendix A. The optimization problem is solved for all $\tilde{z}^{(i)}, \quad i = -1, \ldots, m-1$, which is the discretization for $\hat{z}_k$. For holdings at other values of $\hat{z}_k$, we use a linear interpolation as follows:

- if $\hat{z}_k < 0$ choose the holdings that corresponds to $\tilde{z}^{(-1)}$ (the underlying price path has never crossed the optimal exercise curve);
- if $\tilde{z}^{(i-1)} \leq \hat{z}_k \leq \tilde{z}^{(i)}$ for some $i \in \{1, ..., m-1\}$, choose a weighted average of holdings that correspond to $\tilde{z}^{(i-1)}$ and $\tilde{z}^{(i)}$, with weights $\frac{\tilde{z}^{(i)}-\hat{z}_k}{\tilde{z}^{(i)}-\tilde{z}^{(i-1)}}$ and $\frac{\hat{z}_k-z^{(i-1)}}{\tilde{z}^{(i)}-\tilde{z}^{(i-1)}}$, respectively;
- if $\hat{z}_k > \tilde{z}^{(m-1)}$, choose the holding that corresponds to $\tilde{z}^{(m-1)}$.

Note that higher order interpolation can also be used.

Hedging strategy computation and analysis is more complicated for a model with burn-out than the model without burn-out. For a model without burn-out, the optimization is carried out only for the initial laggard distribution $\mathcal{LD}_k^{\mathbf{NB}} = \mathcal{LD}_0$, which dramatically simplifies the optimization step. In the case of the model with burn-out, the laggard distribution has to be updated along each stock price path and $\hat{z}_k$ needs to be determined for each value of the burn-out variable $\hat{z}_k$.

Compared to the rational exercising case, hedging computation and analysis under the laggard distribution for irrational exercising is much more complex. In particular, calculations have to be carried out for *all* nodes $[(k, j)]$, because exercise does not necessarily occur below the optimal exercise boundary.

4 Computational Results

We now compare performance of different local risk minimization strategies for hedging American put options under irrational exercising. In addition, we compare hedging performance under irrational exercising with hedging performance for an American option under rational exercising as well as for a European option.

We compute expected hedging costs based on simulation, similar to the investigation in Coleman et al. (2007a) for American options with rational exercising. We generate 100,000 independent underlying price paths for (3.1). At each hedging time along each path we determine if the underlying price is below the optimal exercise value (or critical value); whether the early exercise occurs or not is determined by the current laggard distribution. If no exercise occurs, we update the value of the burn-out variable $\hat{Z}_k$, update the portfolio to values corresponding to the new $\hat{Z}_k$, and proceed to the next hedging time. In the event of exercising, the payoff's value is added to the cumulative cost.

Using simulation we obtain a distribution of the cumulative cost at the moment of exercise C_{M^*}. In particular, we compute *the average cumulative cost*, which is an approximation to the expected value $\mathbb{E}[C_{M^*}]$. First we consider the model without the burn-out effect, i.e., $\mathcal{LD}_k^{\mathbf{NB}}$ is used, and compare it to the rational exercise and European cases. Then we consider the model with the burn-out effect, i.e., $\mathcal{LD}_k^{\mathbf{B}}$ is used. We compare hedging performance for these two laggard distribution models. We focus on the average cumulative cost as the performance measure, since it is often considered a proxy for the value of the option.

4.1 Results for the Model Without the Burn-out Effect

We solve optimization problems arising from the piecewise linear (L_1), quadratic (L_2), and piecewise linear constrained (L_1c) local risk minimization formulations to determine the corresponding dynamic hedging strategies. We examine hedging performance characteristics via simulations for the underlying price and laggard distributions with various values of parameters ρ and λ. We assume the strike price for the put is fixed at $K = 100$ and $n_H = n_E = 50$. The results are summarized in Table 4.1.

Firstly, we observe that, when the weight parameter ρ of a single atom at $t = 0$ in the initial laggard distribution is $\rho = 0.8$, the results roughly coincide with the results for the Bermudan option with rational exercising regardless of tail declining rate parameter λ. Indeed, from Table 2.1, we see that at least 83% of the people in the pool exercise if the discounted underlying price falls below the optimal exercise value by 5%. Even if around 20% of the people are not rational, the difference in average cumulative costs between our irrational exercising model with a laggard distribution and the case of rational exercising still seems to be insignificant.

However, if ρ falls down to 0.5, we observe a significant deviation from the rational case - around 4% for L_1, and 2% for L_2 and L_1c. However, the influence of λ is again minimal.

Table 4.1: Results for the Non-Burn-out Laggard Model: Average Cumulative Costs for L_1, L_2, and L_1c local risk minimization methods for different values of ρ and λ. $K = 100, n_H = n_E = 50$.

AvgCumCost

European	
L_1	3.12
L_2	3.65
L_1c	3.52

			ρ			
			0	0.2	0.5	0.8
λ	3	L_1	3.50	3.73	3.91	4.00
		L_2	4.14	4.35	4.52	4.60
		L_1c	3.97	4.18	4.35	4.43
	5	L_1	3.61	3.78	3.92	4.00
		L_2	4.26	4.41	4.53	4.60
		L_1c	4.09	4.23	4.36	4.43

Rational Bermudan	
L_1	4.03
L_2	4.63
L_1c	4.45

Table 4.2: Using different irrationality indicators for explaining Average Cumulative Cost for Non-Burn-out Laggard Model: R^2 for L_1 and L_2 local risk minimization methods. $K = 110, n_H = n_E = 50$.

R^2	$\mathcal{LD}_0(0\%)$	$\mathcal{LD}_0(5\%)$	$\mathcal{LD}_0(10\%)$	expected value
L_1	0.94	0.97	0.97	0.83
L_2	0.88	0.92	0.93	0.8

If $\rho = 0.2$, the results are in between the European and the rational exercising cases. This is expected since only around 30% and 40% of the people (for $\lambda = 3$ and 5, respectively) exercise if the underlying price falls below the optimal exercise value by 5%. Note that λ now plays much bigger role since ρ is smaller - average cumulative costs differ by more than 1% for $\lambda = 3$ and 5.

For $\rho = 0$, the numbers move closer to the European case. However, since the European case is a limiting case with $\lambda \to 0$, the numbers are still significantly different from the European case for $\lambda = 3$ or $\lambda = 5$.

As discussed in Section 2.2, we can consider several properties of the laggard distribution, for example, the expectation and values for $\mathcal{LD}_0(0\%)$, $\mathcal{LD}_0(5\%)$, and $\mathcal{LD}_0(10\%)$. We investigate which ones, if any, can serve as indicators of overall irrationality in the corresponding pool for explaining the cumulative hedging cost, even if the exact shape of the laggard distribution is not known. We investigate this using linear interpolation as follows. For values of $\rho = 0, 0.2, 0.5, 0.8$ and $\lambda = 3, 5$, we compute the expectation, $\mathcal{LD}_0(0\%)$, $\mathcal{LD}_0(5\%)$, $\mathcal{LD}_0(10\%)$, and corresponding average cumulative costs for L_1 and L_2 methods. We then apply ordinary least squares to explain the average cumulative cost by each of the indicators. We are interested in the in-sample goodness-of-fit R^2 for each of the runs.

The regression results are summarized in Table 4.2. From this table, we can see that all $\mathcal{LD}_0(0\%)$, $\mathcal{LD}_0(5\%)$, and $\mathcal{LD}_0(10\%)$ explain average cumulative costs well for both methods, with L_1 being explained better. However, the expectation explains the average cumulative cost the least. This suggests that the information about the proportion of people with irrationality parameter ℓ less than 5% or 10% is more valuable than the expectation of the laggard distribution. In addition, while $\mathcal{LD}_0(0\%)$, the proportion of rational people, explains the cumulative cost quite well, $\mathcal{LD}_0(5\%)$ and $\mathcal{LD}_0(10\%)$ provide better explanation. This shows that the tail-behavior of the laggard distribution does have a significant impact on the properties of the hedging cost.

4.2 Results for the Model with the Burn-out Effect

We now examine hedging performance under the irrational exercising with the burn-out effect. If we assume that the laggard distribution changes over time as described in Section 2.3 and run simulations of the underlying price, we can observe a change in the proportions of each type of laggards over time. Figure 4.1 shows the effect for the model with the following parameters: $K = 100, n_H = n_E = 50, \rho = 0.5, \lambda = 5$. We observe that the proportion of people with irrationality parameter ℓ below 5% declines by approximately

4% by the end of time horizon $[0, T]$. While this number may look small, recall that costs associated with exercising are significant. Even such small numbers are likely to impact the solution and its properties.

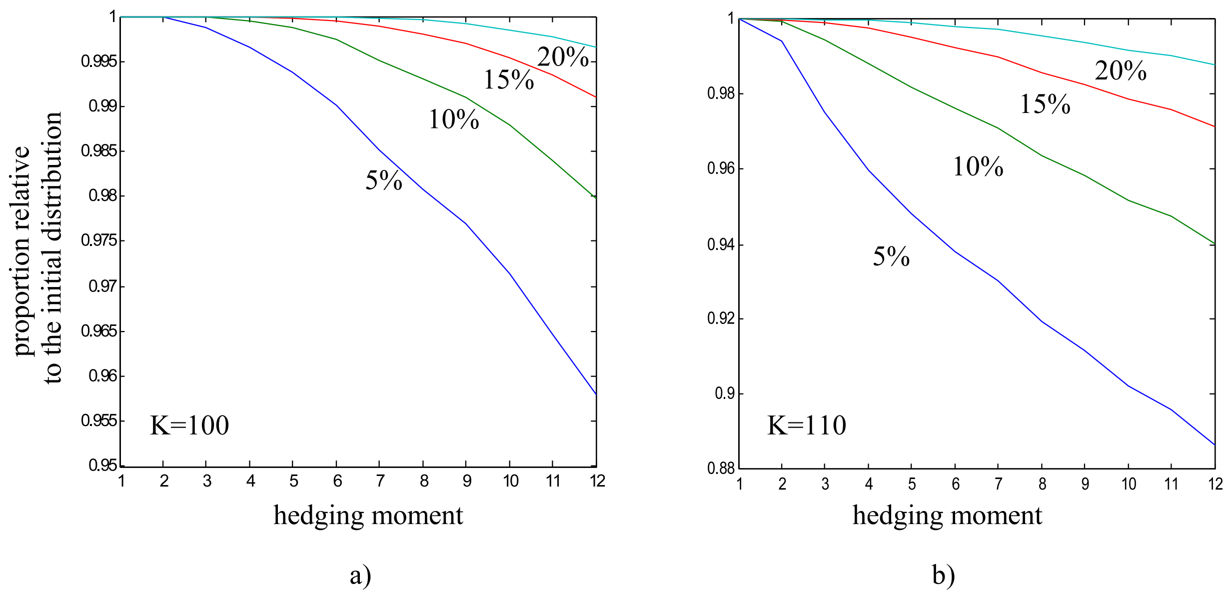

Figure 4.1: Illustration of the burn-out effect for put options with the strike $K = 100$ and $K = 110$. The graph depicts the decline in the ratio $\mathcal{LD}_k^{\mathbf{B}}(x)/\mathcal{LD}_0(x)$ for $x = 5\%, 10\%, 15\%$, and 20% as hedging time k increases. $(n_H = n_E = 50, \rho = 0.5, \lambda = 5)$

Figure 4.1 illustrates that, as the number of exercise opportunities increases, the influence of irrational exercising on the solution becomes larger. The probability of the underlying price path diving below the optimal exercise value is higher for a larger strike K; this is reflected in Figure 4.1 b). We observe a more significant change of the pool's structure for $K = 110$ (plot b)) than for $K = 100$ (plot a)) and expect more pronounced effects of irrational exercising in the first case. In addition, since we consider the model for which hedging and exercise moments coincide, i.e., $n_H = n_E$, we can expect the effect to depend on hedging frequency and be stronger for more frequent hedging (or small n_H). Thus, we run experiments for $n_H = 25, 50,$ and 100.

We first examine whether hedging under a burn-out model is significantly different from hedging under the non-burn-out model. To answer this question, we compare hedging performance under a model with burn-out with the hedging performance under a model without burn-out.

From Table 4.3, we observe that differences between the non-burn-out model and the model with burn-out (panels I and III in the table) are about 12% for L_1 method and 4-7% for L_2 and L_1c. The difference tends to grow for L_2 and L_1c as the hedging (and, thus, the number of exercise opportunities) becomes more frequent, which is consistent with our claim that more early exercise opportunities will underscore the difference between the models. However, the behavior with respect to n_H is inconclusive for L_1 method - the difference does not seem to drop down as the hedging becomes less frequent; we will discuss this issue later.

Since calculating hedging strategies for a model with burn-out is much more computationally intensive than calculating strategies under the non-burn-out model, we further investigate what reduction in performance we can expect if we use the non-burn-out strategy in the model with burn-out. To achieve this objective, we compute hedging strategies under a non-burn-out model and evaluate hedging performance under the burn-out model using simulations. We then compare its hedging performance with the performance from the strategy computed under the burn-out model, see II and III in Table 4.3.

From Table 4.3 we can see that the results for L_2 and L_1c follow the same pattern as in 'I vs III', however, the differences are smaller and range between 1% and 3%. Again, the differences for L_1 are significant (8-9%)

Table 4.3: Comparisons between Burn-out and Non-Burn-out Laggard Models: Average Cumulative Costs for L_1, L_2, and L_1c local risk minimization methods for different values of $n_H(=n_E)$. $\rho = 0.5, \lambda = 5$.

Average Cumulative Costs ($K = 110$)

0. Rational Bermudan			
	$n_H(=n_E)$		
	25	50	100
L_1	10.65	10.60	10.48
L_2	10.59	10.45	10.10
L_1c	10.48	10.35	10

I. Non-burnout model			
	$n_H(=n_E)$		
	25	50	100
L_1	10.53	10.37	10.00
L_2	10.46	10.22	9.75
L_1c	10.35	10.12	9.66

II. Non-burnout Strategy for Burn-out model			
	$n_H(=n_E)$		
	25	50	100
L_1	10.10	10.04	9.78
L_2	10.00	9.84	9.49
L_1c	9.88	9.74	9.41

III. Burn-out model			
	$n_H(=n_E)$		
	25	50	100
L_1	9.76	9.61	9.44
L_2	9.75	9.63	9.36
L_1c	9.58	9.50	9.27

Table 4.4: Comparisons between Burn-out and Non-Burn-out Laggard Models: Average Cumulative Costs for L_1, L_2, and L_1c local risk minimization methods for different values of $n_H(=n_E)$. $\rho = 0.5, \lambda = 5$.

Average Cumulative Costs ($K = 100$)

0. Rational Bermudan			
	$n_H(=n_E)$		
	25	50	100
L_1	4.53	4.03	3.70
L_2	4.72	4.63	4.46
L_1c	4.57	4.45	4.26

I. Non-burnout model			
	$n_H(=n_E)$		
	25	50	100
L_1	4.46	3.92	3.56
L_2	4.66	4.53	4.30
L_1c	4.50	4.36	4.12

II. Non-burnout Strategy for Burn-out model			
	$n_H(=n_E)$		
	25	50	100
L_1	4.35	3.83	3.50
L_2	4.53	4.43	4.23
L_1c	4.38	4.26	4.06

III. Burn-out model			
	$n_H(=n_E)$		
	25	50	100
L_1	4.20	3.72	3.45
L_2	4.41	4.34	4.18
L_1c	4.26	4.16	4.02

for all values of n_H. These results suggest that, for given parameters, *if one is using L_2 or L_1c and can tolerate the performance reduction in the average cumulative cost of 3%, it is possible to use the non-burn-out model for producing the hedging strategy even if it is known that the burn-out takes place. However, if L_1 risk measure is used, this approach may not be justified as the difference can be quite siginificant.*

Table 4.4 shows that, for $K = 100$, performance comparison follows the same pattern as for $K = 110$, but with smaller differences between the models.

It is interesting to understand why L_1 risk measure produces such different results from the quadratic risk measure L_2 . Thus we compare portfolio holdings ξ_k for L_1 and L_2 risk measures. For $K = 110, n_H = n_E = 50$, we consider hedging time $k = 6$ out of a total $M = 12$, i.e., the midlife of an option which expires in one year and is hedged monthly. For all binomial nodes corresponding to this hedging time, we graph in Figure 4.2 the holding ξ_k obtained by L_1 and L_2 risk minimization methods for:

- The European option;
- The rational Bermudan option;
- The model with burn-out assuming that the value of $\hat{z}_k < 0$ at the node of interest (underlying price path has been above the optimal exercise boundary);
- The model with burn-out assuming that the value of $\hat{z}_k = 0$ at the node of interest (all rational individuals have already exercised);
- The model with burn-out assuming that the value of $\hat{z}_k = 15\%$ at the node of interest (individuals with $\ell < 0.15$ have already exercised).

From the bottom plot in Figure 4.2, we observe that, for L_2 risk minimization, as the value of $\hat{z}_k$ grows, the holding gradually shifts to the holding for the European put. It is interesting to note that, for small underlying prices, stock holdings still differ significantly from the European ones, even falling under -1 for very small underlying price. This is because that, in the L_2 local risk minimization, the underlying holding is determined to best match both future portfolio value in case no early exercise happens and large payoff values in case of the early exercise. Since L_2-norm is sensitive to the large cost, the resulting stock holding becomes different from those for European put and Bermudan put with rational exercising.

In contrast, L_1 risk minimization under a burn-out model yields holdings close to the European case for most values of $\hat{z}_k$. It is somewhat expected because L_1 measure does not penalize large costs as much as L_2 measure does. It is thus more likely to take into account what happens in the case when early exercise does not occur, rather than to large payoffs happening with small probabilities. Note the non-smoothness of the holding, which is typical for solutions of L_1 optimization problems.

Proximity of L_1 solutions to the solutions in the European case suggests that the method will produce lower average cumulative costs for a model which is closer to the European case. In our framework this happens when either burn-out is present in the model or the number of early exercise opportunities is low. This observation explains behavior of L_1 method with respect to n_H in Table 4.3.

5 Conclusions

In this paper we consider a discrete hedging model for American options under irrational exercising. A family of probability distributions has been proposed to model irrationality, with irrationality of each option holder represented by a laggard spread. This probability distribution is referred as the *laggard distributions.* Parameters of the family of laggard distributions can be estimated from historical data by tracking the proportion of people surrendering the option after the underlying price dips below the optimal exercise boundary.

We consider two variants of irrational exercising models. In the simpler non-burn-out model, the distribution does not change over time. The more complex model accounts for the burn-out effect, i.e., the change of the pool of option holder composition over time. While the model can be regarded as a blend of the European and rational exercise cases, it has its own special properties and presents unique computational challenges.

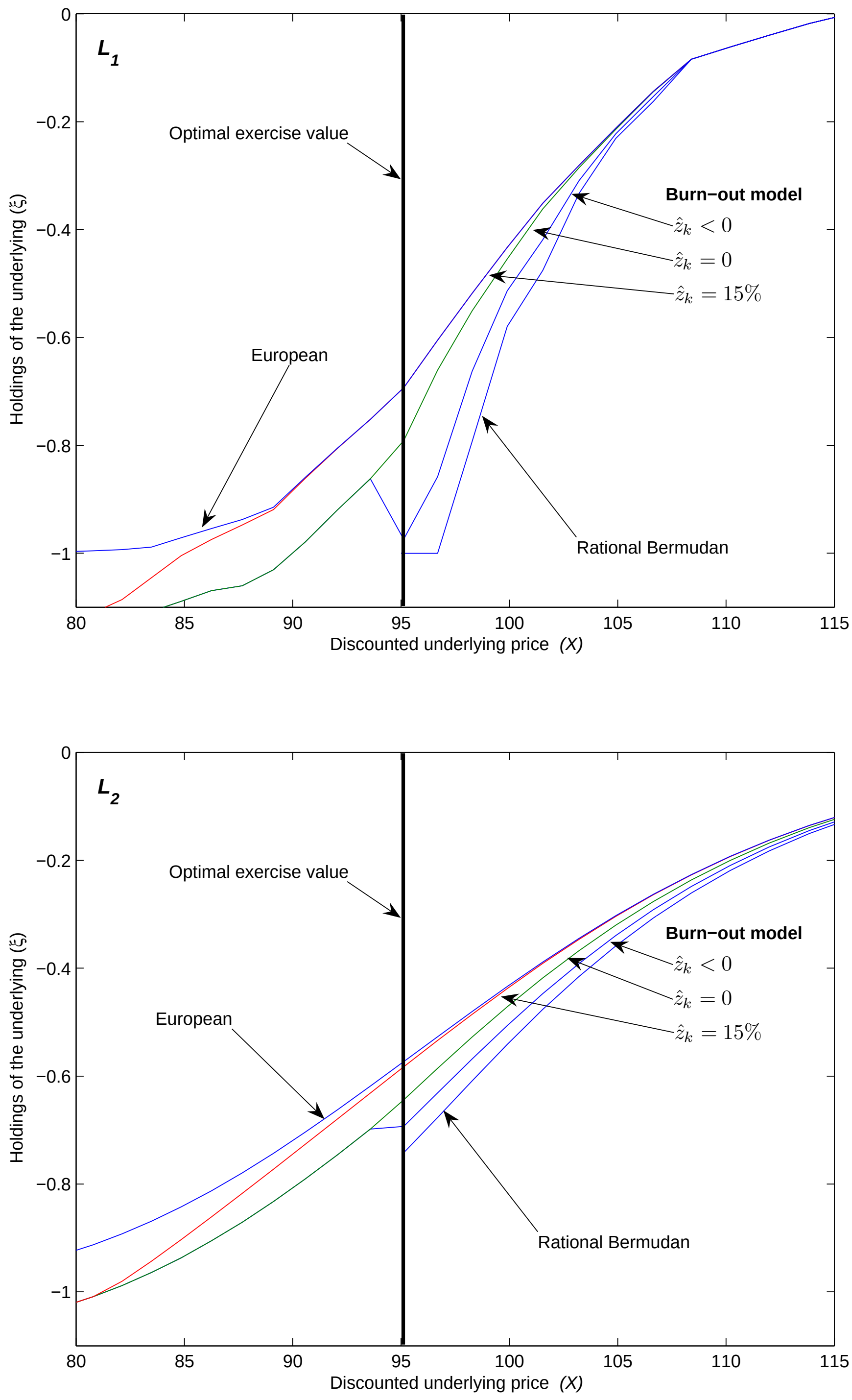

Figure 4.2: Portfolio holdings for binomial nodes correspoding to hedging time $k = 6$ for various models and methods. $K = 110, n_H = n_E = 50$. $\rho = 0.5, \lambda = 5$ for the burn-out model.

Under a model with burn-out effect, computation of hedging strategies and analysis are significantly more intensive. Our computational results suggest that the proportion of people who exercise rationally plays a significant role. If this proportion is high enough, the hedging solution under irrational exercising resembles the hedging solution for the Bermuda option under optimal exercising. However, when the proportion of the rational exercising option holders is smaller, the tail of the laggard distribution impacts the properties of the hedging strategy and its hedging performance.

In addition, we have shown that average cumulative costs for the optimal hedging strategies can potentially be explained using properties of the laggard distribution such as CDF values at 5% or 10%, which is much better than the expected value of the laggard distribution. With respect to differences between different risk minimization formulations, piecewise linear L_1 local risk minimization seems to be more sensitive to the presence of the burn-out effect, with up to 12% difference in the average cumulative costs. However, relative performance comparison between L_1, L_2, and L_1c risk minimizations is similar to the case for the European and Bermuda option under rational exercising, with L_1 risk minimization producing lower average cumulative costs than L_2 and L_1c.

Our results and analysis suggest explicit modeling of irrational exercise in risk minimization may be important. We provide different risk minimization formulations, analyze hedging performance characteristics, and compare their hedging performance. This study can be applied to risk management of standard American options as well as surrender option features embedded in various complex insurance contracts.

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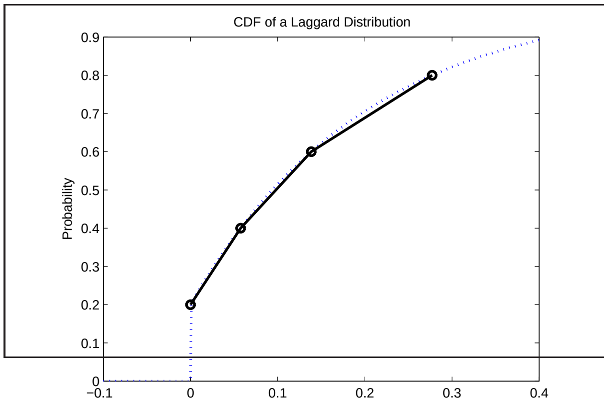
A Discretization of the Initial Laggard Distribution

The family of laggard distributions that we consider has an atom at 0 and a continuous right tail. As the underlying price decreases below the optimal exercise curve, the value of $\hat{Z}_k$ rises. If we set up a local risk minimization problem at hedging moment k, $\hat{Z}_k$ will have to be a parameter in that problem. With $\hat{Z}_k$ being continuous, its range has to be discretized for the solution to be obtained numerically. Since $\hat{Z}_k$ is used to determine the evolution of the laggard distribution, a reasonable approximation can be achieved by discretizing the initial laggard distribution. This way we can obtain discrete values for $\hat{Z}_k$ corresponding to different levels of exercise behavior parameter for people in the pool.



Cut-off values for the discretized distribution

i	-1	0	1	2	3
$\tilde{z}^{(i)}$	$-\epsilon$	0	0.058	0.139	0.277
CDF	0	0.2	0.4	0.6	0.8

Figure A.1: Discretization of the laggard distribution. $\rho = 0.2, \lambda = 5$. ϵ stands for the machine epsilon.

We divide the probability range $[\rho, 1]$ into m equal-sized intervals and applying inverse CDF to them to get points on the x-axis. The procedure is illustrated on Figure A.1 for $m = 4, \rho = 0.2, \lambda = 5$. For our distribution family cut-off points should be defined as $\tilde{z}^{(i)} = -\ln(1 - i/m)/\lambda, \quad i = 0, .., m-1$, to satisfy the definition. Since the CDF is discontinuous at point 0, it is convenient to define one more point, $\tilde{z}^{(-1)} = -\epsilon$, where ϵ is the machine epsilon.

The bigger the number of cut-off points m is, the better the approximation is. However, as we will see later, this comes at steep computational costs. We typically use $m = 10$ in our experiments.