Computational Optimization and Applications 15, 5–32 (2000)

An Exterior Newton Method for Strictly Convex Quadratic Programming*

THOMAS F. COLEMAN
Department of Computer Science and Center for Applied Mathematics, Cornell University, Ithaca, NY 14853, USA

JIANGUO LIU
Department of Mathematics, University of North Texas, Denton, TX 76203, USA

Received September 16, 1997; Revised September 14, 1998; Accepted September 22, 1998

Abstract. We propose an exterior Newton method for strictly convex quadratic programming (QP) problems. This method is based on a dual formulation: a sequence of points is generated which monotonically decreases the dual objective function. We show that the generated sequence converges globally and quadratically to the solution (if the QP is feasible and certain nondegeneracy assumptions are satisfied). Measures for detecting infeasibility are provided. The major computation in each iteration is to solve a KKT-like system. Therefore, given an effective symmetric sparse linear solver, the proposed method is suitable for large sparse problems. Preliminary numerical results are reported.

Keywords: convex quadratic programming, exterior methods, Newton methods, dual problems

1. Introduction

We consider convex quadratic programming (QP) problem of the form:

$$\begin{aligned} \min_{x\in\Re^n} \quad & \left\{ q(x) := \frac{1}{2}x^T Hx + c^T x \right\} \\ \text{subject to} \quad & Ax = b \\ & -e \le x \le e, \end{aligned} \tag{1.1}$$

where $H \in \Re^{n\times n}$ is symmetric and positive definite, $c \in \Re^n$, $A \in \Re^{m\times n}$ $(m < n)$ with full row rank, $b \in \Re^m$, and $e \in \Re^n$ is the vector of ones, $e = [1 \cdots 1]^T$. It should be noted that constraints of the form $l \le x \le u$ can easily be transformed to the form $-e \le x \le e$ by shifting and scaling, provided $l < u$ and all bounds are finite.

*Research partially supported by the Applied Mathematical Sciences Research Program (KC-04-02) of the Office of Energy Research of the U.S. Department of Energy under grant DE-FG02-86ER25013.A000, and by NSF, AFOSR, and ONR through grant DMS-8920550, and by the Cornell Theory Center which receives major funding from the National Science Foundation and IBM Corporation, with additional support from New York State and members of its Corporate Research Institute.

We propose an exterior Newton method which generates a sequence of points converging globally and quadratically to the solution of (1.1), assuming (1.1) is feasible. Measures for detecting infeasibility are provided.

Exterior methods have the advantage that no feasible starting point is required. They are especially suitable for problems where feasibility is uncertain. Many of the well-known penalty function methods for nonlinear programming are exterior methods (see, e.g., [7]) and [8]). However, in most cases determining an appropriate penalty parameter in each iteration can be difficult. Our proposed method is *penalty-parameter-free*, and is based on a dual formulation of problem (1.1): a sequence of points is generated which monotonically decrease the dual objective function. A feature of the dual problem is that it has no constraints (though it involves a nondifferentiable term). Hence techniques for unconstrained minimization can be adapted. Moreover, the major computation in each iteration is to solve a KKT-like system. Therefore, applicability to large sparse problems depends on the ability to effectively solve large sparse symmetric linear systems.

In [4], a new exterior method is proposed to solve convex QP with simple bounds. It is shown that such a QP problem can be transformed into an unconstrained minimization problem involving a piecewise quadratic function. Global and superlinear convergence results are established and the potential of such methods is demonstrated by the results of numerical experiments.

Our method is an extension of the exterior method proposed in [4]. However, this extension is by no means trivial. For example, feasibility is not an issue for simple bounds but is important for problem (1.1). In addition, we improve the convergence results. Specifically, we prove global convergence without assuming strict complementarity, and quadratic (instead of superlinear) convergence is established in this paper.

There are numerous studies on solving convex QP. For example, [2, 3, 9, 10, 13–17, 24], and [25]. More references may be found in [4] and [25].

Given a function $f : \Re^n \rightarrow \Re$, we use ∇f to denote the gradient of f and $\nabla^2 f$ to denote the Hessian. Given x^* and x^k, we write $f^* = f(x^*)$, $f^k = f(x^k)$, and use similar notations for ∇f and $\nabla^2 f$. We use superscripts to denote the iteration counts and use subscripts to indicate the indices of vector components. Occasionally the superscripts will be dropped when there is no confusion. The norm $\|\cdot\|$ used in this paper is the l_2 norm unless otherwise specified. Sets will be denoted by calligraphic capital letters. Given a vector $x \in \Re^n$ and a vector $l \in \Re^n$, the notation $x \geq l$ means $x_i \geq l_i$ for every $1 \leq i \leq n$. We call x *a feasible point* if $Ax = b$ and $-e \leq x \leq e$. If u and v are two vectors, we denote $(u,\ v) := [u^T\ v^T]^T$. When $M \in \Re^{n \times n}$ is a square matrix, the notation $M > 0$ indicates that M is positive definite and the notation $M \geq 0$ indicates that M is positive semidefinite. If x denotes a vector, $X = diag(x) = diag(x_1, x_2, \ldots, x_n)$ will denote the diagonal matrix whose entries are the components of x. Finally, if $x = [x_1, x_2, \ldots, x_n]^T$ and $M = (m_{ij}) \in \Re^{n \times n}$, then $|x| = [|x_1|, |x_2|, \ldots, |x_n|]^T$ and $|M| = (|m_{ij}|) \in \Re^{n \times n}$.

In the rest of this section, we give the optimality conditions for problem (1.1). In the subsequent section, we establish the equivalence of (1.1) to an unconstrained minimization problem where the objective function is a "dual" non-differentiable function. The algorithm is given in Section 3. Global and quadratic convergence are proved in Sections 4 and 5. Preliminary numerical results are reported in Section 6, followed by some concluding remarks in Section 7.

The optimality conditions for (1.1) are well known and can be expressed as follows: x^* is the solution to (1.1) if and only if there exists $w^* \in \Re^m$ such that

$$-e \leq x^* \leq e, \tag{1.2}$$

$$Ax^* = b, \tag{1.3}$$

$$(Hx^* + c - A^T w^*)_i \neq 0 \Rightarrow x_i^* = -sign((Hx^* + c - A^T w^*)_i), \tag{1.4}$$

where *sign* is the sign function defined by $sign(t) = 1$ if $t \geq 0$ and $sign(t) = -1$ if $t < 0$. The vector w^* is known as *Lagrange multiplier.*

If we let $y^* = Hx^* + c - A^T w^*$, then $x^* = H^{-1}(y^* - c + A^T w^*)$, and conditions (1.3) and (1.4) are equivalent to the condition that (y^*, w^*) satisfies the nonlinear equation

$$F(y, w) := \begin{bmatrix} Y(x + sign(y)) \\ Ax - b \end{bmatrix} = 0, \tag{1.5}$$

where $Y = diag(y) = diag(y_1, y_2, \ldots, y_n)$ and $sign(y) = (sign(y_1), sign(y_2), \ldots, sign(y_n))^T$. The nonlinear system (1.5) is important and points to the promise of a Newton-like method.

2. An equivalent dual problem

In this section, we show that (1.1) is equivalent to the following unconstrained minimization problem:

$$\min_{y,w}\left\{ f(y, w) := \frac{1}{2}\bar{y}^T \bar{H} \bar{y} + \bar{b}^T \bar{y} + \|y\|_1 + \frac{1}{2}c^T H^{-1} c\right\}, \tag{2.1}$$

where

$$\bar{y} = \begin{bmatrix} y \\ w \end{bmatrix}, \quad \bar{H} = \begin{bmatrix} H^{-1} & H^{-1}A^T \\ AH^{-1} & AH^{-1}A^T \end{bmatrix}, \quad \text{and} \quad \bar{b} = \begin{bmatrix} -H^{-1}c \\ -b - AH^{-1}c \end{bmatrix}. \tag{2.2}$$

Theorem 2.1. *If x^* is the solution to* (1.1) *and w^* is a Lagrange multiplier, then (y^*, w^*) is a solution to* (2.1) *with $y^* = Hx^* + c - A^T w^*$. On the other hand, if (y^*, w^*) is a solution to* (2.1), *then $x^* = H^{-1}(y^* - c + A^T w^*)$ is the solution to* (1.1) *and w^* is a Lagrange multiplier.*

Proof: We show that (2.1) is a reformulation of the standard dual of (1.1). Then the theorem follows from the standard duality results.

The standard dual of (1.1) can be written as

$$\begin{aligned}
\max \quad & -\left\{\frac{1}{2}x^T Hx + b^T(u_1 - u_2) + e^T(u_3 + u_4)\right\} \\
\text{subject to} \quad & Hx + A^T(u_1 - u_2) + (u_3 - u_4) = -c \\
& u_1, u_2, u_3, u_4 \geq 0.
\end{aligned} \tag{2.3}$$

Let $w = u_2 - u_1$, $y = u_4 - u_3$, and change *max* to *min*. Then problem (2.3) is equivalent to

$$\begin{aligned}
\min \quad & \left\{\frac{1}{2}x^T Hx - b^T w + \|y\|_1\right\} \\
\text{subject to} \quad & Hx - A^T w - y = -c,
\end{aligned}$$

which can be seen the same as (2.1) by using $x = H^{-1}(y + A^T w - c)$ (since $H > 0$). □

By Theorem 2.1, to solve (1.1), we may solve (2.1) instead. The next theorem gives some primal-dual properties of (1.1) and (2.1). Again it follows from the standard duality results so the proof is omitted.

Theorem 2.2. *Let the feasible point set of* (1.1) *be*

$$\Omega := \{x \in \Re^n : Ax = b, -e \leq x \leq e\}.$$

Then,

(i) *If* $\Omega \neq \emptyset$ *(the empty set), then*

$$-f(y, w) \leq q(x) \text{ for every } x \in \Omega \text{ and for every } (y, w) \in \Re^{m+n}. \tag{2.4}$$

(ii) *Let* (x^*, y^*, w^*) *be a triple. Then* x^* *is the solution to* (1.1) *and* (y^*, w^*) *is a solution to* (2.1) *if and only if* $x^* \in \Omega$ *and*

$$-f(y^*, w^*) = q(x^*).$$

(iii) *Problem* (1.1) *is infeasible if and only if the function* f *is unbounded below.*

3. The algorithm

We present an algorithm to minimize $f(y, w)$. When a minimizer of f is identified, the solution to (1.1) is obtained. Our algorithm generates a sequence $\{(y^k, w^k)\}$ which monotonically decreases f. Simultaneously, a sequence $\{x^k\}$ is generated, which, under certain conditions, will converge to the solution of (1.1).

3.1. *Compute search direction*

At any point (y, w) such that $y_i \neq 0$ for all $1 \leq i \leq n$, both functions f and F (defined in (1.5)) are twice continuously differentiable (they are actually in C^∞). At such a point, we have

$$\nabla f(y,w) = \begin{bmatrix} d \\ Ax - b \end{bmatrix}, \quad \nabla^2 f(y,w) = \bar{H}, \quad (\bar{H} \text{ is defined in (2.2)}) \tag{3.1}$$

and

$$\nabla F = \begin{bmatrix} D + YH^{-1} & YH^{-1}A^T \\ AH^{-1} & AH^{-1}A^T \end{bmatrix}, \quad \text{(the Jacobian of } F\text{)}$$

where

$$d = x + sign(y), \quad D = diag(d), \quad \text{and} \quad x = H^{-1}(y - c + A^T w). \tag{3.2}$$

A Newton direction for (1.5) at a differentiable point is defined to be the solution to

$$\nabla F s = -\begin{bmatrix} Yd \\ Ax - b \end{bmatrix}, \tag{3.3}$$

which is equivalent to

$$\left(\bar{H} + \begin{bmatrix} Y^{-1}D & 0 \\ 0 & 0 \end{bmatrix}\right) s = -\nabla f.$$

To obtain fast local convergence, unit Newton steps are locally attractive. However, when the current point is far from the solution, a Newton direction may not be a descent direction for f. Therefore, we consider a "modified" Newton equation:

$$\left(\bar{H} + \begin{bmatrix} |Y|^{-1}|D| & 0 \\ 0 & 0 \end{bmatrix}\right) s = -\nabla f, \tag{3.4}$$

where $|Y| = diag(|y|) = diag(|y_1|, |y_2|, \ldots, |y_n|)$ and $|D| = diag(|d|) = diag(|d_1|, |d_2|, \ldots, |d_n|)$. It can be shown that Eq. (3.4) yields a descent direction for f and the "modified" Newton equation is identical to the Newton equation (3.3) in a small neighborhood of a solution (y^*, w^*) (see Section 5).

A Newton direction or a modified Newton direction is well-defined only at a differentiable point. Therefore the proposed algorithm generates a sequence of points strictly in the differentiable region (converging, asymptotically, to the optimal point which is typically a non-differentiable point). Differentiability is maintained with the aid of subproblem (3.5) which we describe next. Let $\Delta > 0$ be sufficiently small. Write $s = (s_y, s_w) := [s_y^T \; s_w^T]^T$ and $|Y|^{\frac{1}{2}} = diag(|y|^{\frac{1}{2}}) := diag(|y_1|^{\frac{1}{2}}, |y_2|^{\frac{1}{2}}, \ldots, |y_n|^{\frac{1}{2}})$. The solution to

$$\begin{cases} \min & \frac{1}{2} s^T \nabla^2 f s + \nabla f^T s \\ \text{subject to} & \left\| |Y|^{-\frac{1}{2}} s_y \right\| \le \Delta \quad (\Delta \text{ varies from iteration to iteration}) \end{cases} \tag{3.5}$$

is a descent direction for f, and, $(y + s_y)_i \neq 0$ if $y_i \neq 0$ $(1 \le i \le n)$.

Problem (3.5) is similar to but slightly different from a Dikin affine scaling subproblem (see [6]) in that $|Y|^{-\frac{1}{2}}$ is used in (3.5) while $|Y|^{-1}$ is usually used in a Dikin subproblem. Our idea is to compute search directions which reflect a smooth transition from the solution of (3.5) to the "modified" Newton direction, i.e., the solution of (3.4). Therefore, quadratic convergence can be expected. Since a solution to (3.5) satisfies the following linear system (see, e.g., [21]),

$$\left(\bar{H} + \begin{bmatrix} \lambda |Y|^{-1} & 0 \\ 0 & 0 \end{bmatrix}\right) s = -\nabla f \text{ for some } \lambda \geq 0,$$

we compute a search direction by solving the following equation:

$$\bar{H}_\theta s = -\nabla f, \tag{3.6}$$

where

$$\bar{H}_\theta = \bar{H} + \begin{bmatrix} |Y|^{-1} D_\theta & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad D_\theta = \theta I + (1-\theta)|D|. \tag{3.7}$$

The parameter $\theta = \theta(y, w)$ is calculated by

$$\theta = \frac{\frac{\|F(y,w)\|}{\|F(y^0,w^0)\|} + e^T \max\{|x| - e, 0\}}{\rho + \frac{\|F(y,w)\|}{\|F(y^0,w^0)\|} + e^T \max\{|x| - e, 0\}}, \tag{3.8}$$

where (y^0, w^0) is the initial guess, $x = H^{-1}(y - c + A^T w)$, $\rho > 0$ is a constant, and $|x| - e$ is componentwise subtraction. It is easy to see that $0 \leq \theta < 1$ and that $\theta = 0$ if and only if (y, w) is a solution of (2.1). When $\theta \to 0$, the solution to (3.6) approaches the modified Newton direction, the solution of (3.4).

Equation (3.6) can be conveniently written as

$$(H^{-1} + |Y|^{-1} D_\theta) s_y + H^{-1} A^T s_w = -d \tag{3.9}$$

$$A H^{-1} s_y + A H^{-1} A^T s_w = b - Ax, \tag{3.10}$$

where we have written $s = (s_y, s_w) := [s_y^T \; s_w^T]^T$. These two equations will be useful in our convergence analysis, but they will not be used to calculate the search direction s.

The next result shows that $\bar{H}_\theta > 0$ whenever $\theta \neq 0$. Hence the solution to (3.6) is a descent direction for f.

Lemma 3.1. *Suppose $\theta > 0$ and $y_i \neq 0$ for all $1 \leq i \leq n$. Then $\bar{H}_\theta$ is positive definite.*

Proof: Let $D_Y := |Y|^{-\frac{1}{2}} D_\theta^{\frac{1}{2}}$. Clearly, $D_Y > 0$ whenever $\theta > 0$ and $y_i \neq 0$ for all $1 \leq i \leq n$.

By the Sherman-Morrison-Woodbury formula (see, e.g., [11]), we have $(H^{-1}+D_Y^2)^{-1} = H - HD_Y(I+D_YHD_Y)^{-1}D_YH$. Since A has full row rank, we see that the Schur complement of $H^{-1}+D_Y^2$ in $\bar{H}_\theta$ (see [12]),

$$\begin{aligned} &AH^{-1}A^T - AH^{-1}\left(H^{-1}+D_Y^2\right)^{-1}H^{-1}A^T \\ &\quad = AD_Y(I+D_YHD_Y)^{-1}D_YA^T > 0. \end{aligned}$$

Therefore, by Corollary 1 in [12], $\bar{H}_\theta > 0$. □

Corollary 3.2. *Suppose $\theta > 0$. Then at any point (y, w) satisfying $y_i \neq 0$ for every $1 \le i \le n$, the solution s of* (3.6) *is a descent direction for f.*

Explicitly forming and solving (3.6) is impractical since $\bar{H}_\theta$ involves the usually dense matrix H^{-1} and several matrix products. However, it can be shown, using linear algebra manipulations, that (3.6) is equivalent to the following *KKT-like* system:

$$\begin{bmatrix} D_\theta^{\frac{1}{2}}HD_\theta^{\frac{1}{2}} + |Y| & D_\theta^{\frac{1}{2}}A^T \\ AD_\theta^{\frac{1}{2}} & 0 \end{bmatrix} \begin{bmatrix} \tilde{s}_y \\ s_w \end{bmatrix} = -\begin{bmatrix} D_\theta^{\frac{1}{2}}Hd \\ A\, sign(y) + b \end{bmatrix}, \tag{3.11}$$

where the notation $D_\theta^{\frac{1}{2}}$ means $D_\theta^{\frac{1}{2}}D_\theta^{\frac{1}{2}} = D_\theta$ and

$$\tilde{s}_y = |Y|^{-1}D_\theta^{\frac{1}{2}}s_y. \tag{3.12}$$

The coefficient matrix in (3.11) reflects the sparsity of H and A since D_θ and Y are diagonal.

The following vector will be used in the algorithm:

$$s_x := H^{-1}(s_y + A^T s_w), \tag{3.13}$$

which, by arranging the first (block) equation in (3.11), can be calculated by

$$s_x = -d - D_\theta^{\frac{1}{2}}\tilde{s}_y. \tag{3.14}$$

Given any initial guess y^0 such that $y_i^0 \neq 0$ for every $1 \le i \le n$, we choose w^0 and x^0 that satisfy

$$\begin{bmatrix} H & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x^0 \\ -w^0 \end{bmatrix} = \begin{bmatrix} y^0 - c \\ b \end{bmatrix}. \tag{3.15}$$

This choice implies $Ax^0 = b$ and $x^0 = H^{-1}(y^0 - c + A^T w^0)$.

We can now state the algorithm.

Algorithm Exterior-Newton

Let $0 < \rho < 1$. Let (y^0, w^0, x^0) be the initial guess.
For $k = 0, 1, 2, \ldots$ until convergence

1. Compute d^k, $\|F(y^k, w^k)\|$, and θ^k. (by (3.2), (1.5), and (3.8))
2. Compute $s^k = (s_y^k, s_w^k)$ and s_x^k. (by (3.11), (3.12), and (3.13))
3. $(y^{k+1}, w^{k+1}, x^{k+1}) = (y^k, w^k, x^k) + \alpha^k(s_y^k, s_w^k, s_x^k)$ where $\alpha^k > 0$ is the step length.

The procedure for determining α^k will be described next.

3.2. *Determine step length*

Now we show how to calculate the step length α^k for each iteration. Define for each k a univariate function

$$\psi_k(\alpha) := f\left(y^k + \alpha s_y^k, w^k + \alpha s_w^k\right), \quad \alpha \geq 0.$$

Ideally, α^k should be a global minimizer of ψ_k. However, to keep every iterate a differentiable point of f, a small perturbation might be needed occasionally. We first describe how to calculate a global minimizer of ψ_k and then describe how to make the perturbation. For notational simplicity, we will omit the superscript k for the rest of the section. So $y = y^k$, $w = w^k$, $s_y = s_y^k$, $\alpha = \alpha^k$, and $\psi = \psi_k$, etc.

Since f is convex, ψ is convex. Hence any local minimum of ψ is a global minimum. Since f is continuous and piecewise quadratic, ψ is continuous and piecewise quadratic. The nondifferentiable points of ψ are those $\alpha > 0$ such that $y_i + \alpha\,(s_y)_i = 0$ for some $1 \leq i \leq n$. We will call these nondifferentiable points of ψ the *break points* and put them in a vector $r \in \Re^n$ which is defined by

$$r_i = \begin{cases} -\dfrac{y_i}{(s_y)_i}, & \text{if } (s_y)_i \neq 0 \text{ and } -\dfrac{y_i}{(s_y)_i} > 0 \\ \infty, & \text{otherwise} \end{cases} \qquad (1 \leq i \leq n). \tag{3.16}$$

Clearly $y_i + r_i(s_y)_i = 0$ for those i such that $r_i \neq \infty$.

Let

$$[\beta, p] = sort(r), \tag{3.17}$$

where the operation *sort* returns a vector $\beta \in \Re^n$ and a permutation vector p such that

$$\beta_i = r_{p_i}\ (1 \leq i \leq n) \quad \text{and} \quad \beta_1 \leq \beta_2 \leq \cdots \leq \beta_n. \tag{3.18}$$

Let $\beta_0 = 0$ and $\beta_{n+1} = \infty$. Let

$$imax = \max\{i : 1 \leq i \leq n, \beta_i < \infty\},$$

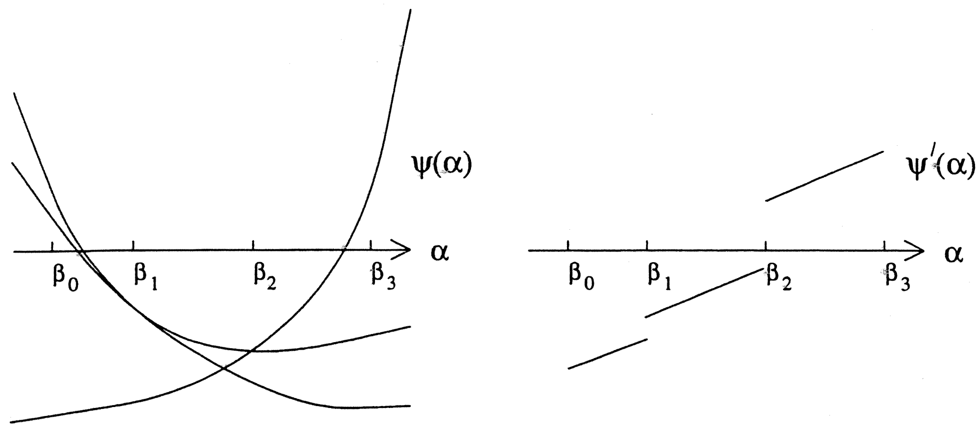

Figure 1. The 1-dimensional view.

i.e., β_{imax} is the last break point of ψ. At each point β_i $(1 \le i \le imax)$, the function ψ is nondifferentiable. But on each interval (β_i, β_{i+1}) $(0 \le i \le imax)$, ψ is a convex quadratic function. In addition, $\psi'(\alpha)$ is piecewise linear and $\psi''(\alpha) = s_x^T H s_x$ (independent of α) which follows from the definition of f in (2.1) and the definition of s_x in (3.13). Typically, ψ can be illustrated by Figure 1.

Let the right-hand derivative and left-hand derivative of ψ at β_i be

$$\psi'(\beta_i^+) := \lim_{t \to 0^+} \psi'(\beta_i + t) \quad (1 \le i \le imax) \tag{3.19}$$

and

$$\psi'(\beta_i^-) := \lim_{t \to 0^-} \psi'(\beta_i + t) \quad (1 \le i \le imax), \tag{3.20}$$

respectively. Define

$$imin := \min\left\{i : 1 \le i \le n,\ \beta_i < \infty,\ \psi'\left(\beta_i^+\right) \ge 0\right\},$$

i.e., β_{imin} is the first break point of ψ at which ψ has a nonnegative right-hand derivative. Then the "left-most" global minimizer of ψ, denoted by α_{opt}, can be calculated as follows.

$$\alpha_{opt} = \begin{cases} \infty, & \text{if } \psi'' = 0 \text{ and } imin \text{ does not exist} \quad \text{(Case 1)} \\ \beta_{imin}, & \text{if } \psi'' = 0 \text{ and } imin \text{ exists} \quad \text{(Case 2)} \\ \beta_{imax} - \dfrac{\psi(\beta_{imax}^+)}{\psi''}, & \text{if } \psi'' \ne 0 \text{ and } imin \text{ does not exist} \quad \text{(Case 3)} \\ \beta_{imin}, & \text{if } \psi'' \ne 0,\ imin \text{ exists, and } \psi'(\beta_{imin}^-) \le 0 \quad \text{(Case 4)} \\ \beta_{imin-1} - \dfrac{\psi(\beta_{imin-1}^+)}{\psi''} & \text{if } \psi'' \ne 0,\ imin \text{ exists, and } \psi'(\beta_{imin}^-) > 0. \quad \text{(Case 5)} \end{cases} \tag{3.21}$$

The five cases are shown in Figure 2 on Page 7.

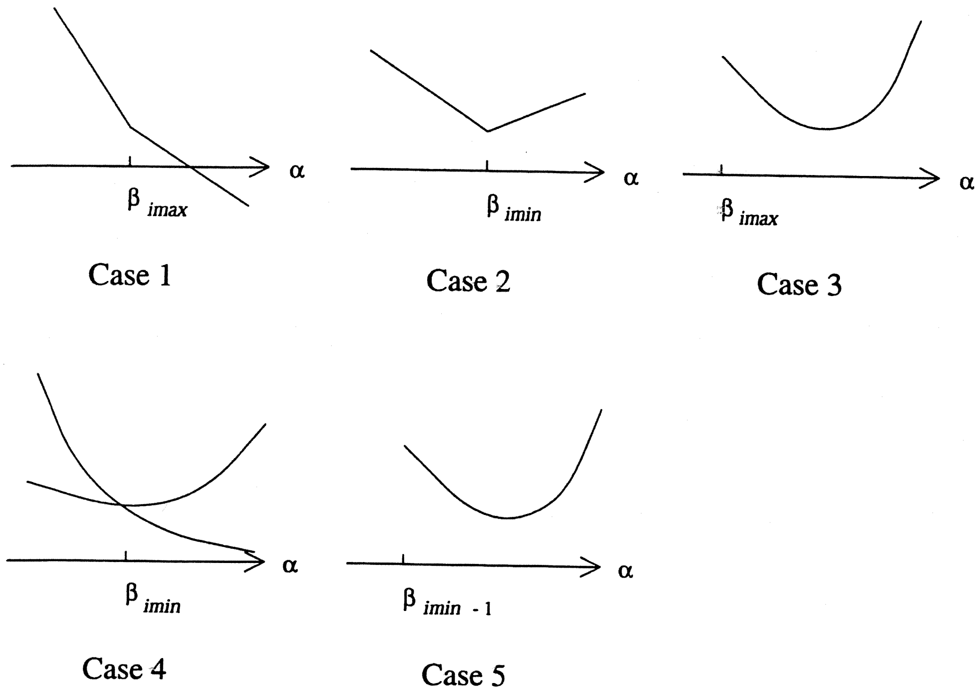

Figure 2. The 5 break point cases.

The next several lemmas tell us that we can efficiently calculate α_{opt}. The first lemma can easily be proved by induction and using (3.15) and its proof is omitted.

Lemma 3.3. *Let* $\{(y^k, w^k, x^k)\}$ *be the sequence generated by Algorithm Exterior-Newton. Then for every* k,

$$Ax^k = b, \tag{3.22}$$

$$x^k = H^{-1}(y^k - c + A^T w^k), \tag{3.23}$$

and

$$As_x^k = 0. \tag{3.24}$$

Lemma 3.4. *For any* $\alpha > 0$, $\alpha \neq \beta_i$ $(1 \leq i \leq imax)$, *we have*

$$\psi''(\alpha) = s_x^T H s_x = s_x^T s_y \tag{3.25}$$

and

$$\psi'(\alpha) = s_y^T d + \alpha s_x^T s_y + 2 \sum_{\{j : r_j < \alpha\}} |(s_y)_j|, \tag{3.26}$$

where $d = x + sign(y)$ *is defined in* (3.2).

Proof: First, by (3.13) and (3.24), $\psi''(\alpha) = s_x^T H s_x = s_x^T s_y + s_x^T A^T s_w = s_x^T s_y$. Then by (3.1) and (3.22),

$$\begin{aligned}\psi'(\alpha) &= s^T \nabla f(y + \alpha s_y, w + \alpha s_w) \\ &= s_y^T\, (x + \alpha s_x + sign(y + \alpha s_y)) \\ &= s_y^T x + \alpha s_y^T s_x + s_y^T sign(y + \alpha s_y). \end{aligned} \tag{3.27}$$

Since

$$sign((y + \alpha s_y)_j) = \begin{cases} sign(y_j), & \text{if } r_j > \alpha \\ -sign(y_j), & \text{if } r_j < \alpha \end{cases} \quad (1 \le j \le n), \tag{3.28}$$

we have

$$sign(y + \alpha s_y) = sign(y) - 2 \sum_{\{j : r_j < \alpha\}} sign(y_j) e_j,$$

where e_j is the jth column of the identity matrix. Therefore (3.26) follows from (3.16).

□

Lemma 3.5. *For each iteration, $O(n \log n)$ comparisons and $O(n)$ arithmetic operations are required to calculate α_{opt}.*

Proof: First, we can sort the vector r to get β and p in $O(n \log n)$ comparisons (see, e.g., Page 271 of [1]). Then by (3.26), (3.19), (3.20), and the fact that $\beta_j = r_{p_j}$ $(1 \le j \le imax)$ (see (3.18)), for each $i = 1, 2, \ldots, imax$,

$$\psi'(\beta_i^+) = s_y^T d + \beta_i s_y^T s_x + 2 \sum_{j=1}^{i} |(s_y)_{p_j}| \tag{3.29}$$

and

$$\psi'(\beta_i^-) = s_y^T d + \beta_i s_y^T s_x + 2 \sum_{j=1}^{i-1} |(s_y)_{p_j}|. \tag{3.30}$$

Therefore we can calculate all the required quantities in (3.21) in $O(n)$ arithmetic operations.

□

We now show how to make the perturbation when α_{opt} falls on a break point. Our purpose is to avoid nondifferentiability while obtaining quadratic local convergence. With the intention of taking unit steps, we choose α as follows. For a given $\tau_1 > 0$, let

$$\tilde{\alpha} = \min(\alpha_{opt}, 1 + \theta \tau_1), \tag{3.31}$$

where θ is defined in (3.8). Then for a given $\tau_2 \in (0, 1)$,

$$\alpha = \begin{cases} \beta_{i-1} + \max(\tau_2, 1-\theta)(\beta_i - \beta_{i-1}), & \text{if } \tilde{\alpha} = \beta_i \text{ for some } 1 \le i \le imax, \\ \tilde{\alpha}, & \text{otherwise.} \end{cases} \tag{3.32}$$

In other words, if $\tilde{\alpha}$ is not a break point, then it will be taken as the step length; otherwise, a small step back from $\tilde{\alpha}$ is used to get the step length.

It is clear that $\alpha \le \tilde{\alpha} \le \alpha_{opt}$. Since α_{opt} is the "left-most" global minimizer of ψ, we must have

$$\psi'(\alpha) \le 0. \tag{3.33}$$

Moreover, by (3.26) and (3.13),

$$-\psi'(0) = -s_y^T d = \psi'' + s_y^T |Y|^{-1} D_\theta s_y \ge \psi'', \tag{3.34}$$

which implies that $\alpha_{opt} \ge \min(1, \beta_1)$, $\tilde{\alpha} \ge \min(1, \beta_1)$, and

$$\alpha \ge \min(1, \max(\tau_2, 1-\theta)\beta_1). \tag{3.35}$$

Conditions (3.33) and (3.35) will be useful in the next section.

Remark. It is clear from (3.31) that

$$\tilde{\alpha} = \max\{\alpha \le 1 + \theta\tau_1 : \psi'(\alpha) \le 0\}.$$

Therefore, we may calculate $\tilde{\alpha}$ by bisection on the interval $[0, 1 + \theta\tau_1]$ using the sign of $\psi'(\alpha)$. This procedure is usually more efficient in practice since the break points often appear in clusters.

4. Global convergence and detecting infeasibility

To prove global convergence, a *primal nondegeneracy* assumption will be needed. Given an index set $\mathcal{I}$, let $A_{\mathcal{I}}$ denote the submatrix of A consisting of those columns of A whose indices are in $\mathcal{I}$. Let

$$\mathcal{C} = \{x \in \Re^n : F(y, w) = 0 \text{ for some } w \in \Re^m, \text{ where } y = Hx + c - A^T w\}.$$

Definition. We say that problem (1.1) is *primal nondegenerate* if $rank(A_{\mathcal{I}}) = m$ for every $x \in \mathcal{C}$, where $\mathcal{I} = \mathcal{I}(x) := \{i : 1 \le i \le n, |x_i| \ne 1\}$.

A direct consequence of primal nondegeneracy is the following result.

Lemma 4.1. *Suppose problem* (1.1) *is primal nondegenerate. Then the set* $\mathcal{C}$ *is finite.*

Proof: Let $x \in \mathcal{C}$. Let $\mathcal{I} = \mathcal{I}(x)$ and let $\mathcal{I}^c$ denote the complement of $\mathcal{I}$. By the definition of $\mathcal{C}$ and the definition of F in (1.5), $y_i d_i = 0$ for every $i = 1, 2, \ldots, n$. For each $i \in \mathcal{I}$, $|x_i| \neq 1$, which implies that $d_i \neq 0$. Therefore $y_{\mathcal{I}} = 0$, i.e.,

$$H_{\mathcal{I}\times\mathcal{I}} x_{\mathcal{I}} + H_{\mathcal{I}\times\mathcal{I}^c} x_{\mathcal{I}^c} + c_{\mathcal{I}} - A_{\mathcal{I}}^T w = 0, \tag{4.1}$$

where $H_{\mathcal{I}\times\mathcal{I}^c}$ denotes the submatrix of H consisting of those entries of H whose row indices are in $\mathcal{I}$ and column indices are in $\mathcal{I}^c$. Similar notation is used for other terms. Since $H_{\mathcal{I}\times\mathcal{I}} > 0$,

$$x_{\mathcal{I}} = H_{\mathcal{I}\times\mathcal{I}}^{-1}\left(A_{\mathcal{I}}^T w - H_{\mathcal{I}\times\mathcal{I}^c} x_{\mathcal{I}^c} - c_{\mathcal{I}}\right). \tag{4.2}$$

Then using (4.2) and the facts that $Ax = b$ (since $x \in \mathcal{C}$) and that $rank(A_{\mathcal{I}}) = m$ (by the primal nondegeneracy), we have

$$w = \left(A_{\mathcal{I}} H_{\mathcal{I}\times\mathcal{I}}^{-1} A_{\mathcal{I}}^T\right)^{-1}\left(b - A_{\mathcal{I}^c} x_{\mathcal{I}^c} + A_{\mathcal{I}} H_{\mathcal{I}\times\mathcal{I}}^{-1}\left(x_{\mathcal{I}^c} + c_{\mathcal{I}}\right)\right).$$

Therefore $x_{\mathcal{I}}$ is uniquely determined when $x_{\mathcal{I}^c}$ has been given. Since there are only finitely many choices for $x_{\mathcal{I}^c}$ ($x_i = 1$ or -1 if $i \in \mathcal{I}^c$), the number of points in $\mathcal{C}$ is finite. □

Let $\{(x^k, y^k, w^k)\}$ be the sequence generated by Algorithm Exterior-Newton and let α^k be calculated by (3.32) for each $k = 0, 1, 2, \ldots$. The convergence results are presented in the following order. We first show that if problem (1.1) is primal nondegenerate and if the sequence $\{\|y^k\|\}$ is bounded above, the sequence $\{x^k\}$ will converge to the solution of (1.1). We then prove that if problem (1.1) is feasible and primal nondegenerate, the sequence $\{\|y^k\|\}$ will be bounded above. Therefore, given that problem (1.1) is feasible and primal nondegenerate, the sequence $\{x^k\}$ will converge to the solution of (1.1). Measures for detecting infeasibility are provided at the end of the section.

Lemma 4.2. *Suppose $\{\|y^k\|\}$ is bounded. Then $\{\|w^k\| + \|x^k\|\}$ is bounded above and $\{f(y^k, w^k)\}$ is bounded below. Consequently, $\{f(y^k, w^k)\}$ converges.*

Proof: By (3.22) and (3.23), we have

$$AH^{-1}A^T w^k = b - AH^{-1}(y^k - c).$$

Since A is assumed to have full row rank, it follows that $\{\|w^k\|\}$ is bounded. Then by (3.23) again, $\{\|x^k\|\}$ is bounded. Therefore, $\{f(y^k, w^k)\}$ is bounded below since f can be written as $f(y, w) = -b^T w + \frac{1}{2}x^T Hx + \|y\|_1$. Moreover, by (3.33) and (3.35), $\{f(y^k, w^k)\}$ is monotonically decreasing. Hence $\{f(y^k, w^k)\}$ converges. □

Lemma 4.3. *Suppose $\{\|y^k\|\}$ is bounded. Then* $\min(1, \tau_2\beta_1^k)(\nabla f^k)^T s^k \to 0$.

Proof: Let $\alpha_m^k = \min(1, \tau_2 \beta_1^k)$, $y_+^k = y^k + \alpha_m^k s_y^k$, and $w_+^k = w^k + \alpha_m^k s_w^k$. We have, using the definition of f in (2.1),

$$\begin{aligned} & f(y^k, w^k) - f(y_+^k, w_+^k) \\ &= -\alpha_m^k \left[(\nabla f^k)^T s^k - sign(y^k)^T s_y^k \right] - \frac{(\alpha_m^k)^2}{2} (s^k)^T \bar{H} s^k - \|y_+^k\|_1 + \|y^k\|_1 \\ &= -\alpha_m^k (\nabla f^k)^T s^k - \frac{(\alpha_m^k)^2}{2} (s^k)^T \bar{H} s^k, \end{aligned} \tag{4.3}$$

as well as

$$\begin{aligned} & f(y^k, w^k) - f(y_+^k, w_+^k) \\ &= -\alpha_m^k \left[\nabla f(y_+^k, w_+^k)^T s^k - sign(y_+^k)^T s_y^k \right] + \frac{(\alpha_m^k)^2}{2} (s^k)^T \bar{H} s^k + \|y^k\|_1 - \|y_+^k\|_1 \\ &= -\alpha_m^k \nabla f(y_+^k, w_+^k)^T s^k + \frac{(\alpha_m^k)^2}{2} (s^k)^T \bar{H} s^k. \end{aligned} \tag{4.4}$$

By Lemma 3.4, (3.33) and (3.35), $\nabla f(y_+^k, w_+^k)^T s^k \le 0$. Hence adding (4.3) and (4.4) yields

$$2[f(y^k, w^k) - f(y_+^k, w_+^k)] \ge -\alpha_m^k (\nabla f^k)^T s^k \ge 0.$$

Since $\alpha_m^k \le \alpha^k \le \alpha_{opt}^k$ by (3.35), $f(y^{k+1}, w^{k+1}) \le f(y_+^k, w_+^k)$. Therefore

$$2[f(y^k, w^k) - f(y^{k+1}, w^{k+1})] \ge -\alpha_m^k (\nabla f^k)^T s^k \ge 0$$

and the convergence of $\{f(y^k, w^k)\}$ forces that $\alpha_m^k (\nabla f^k)^T s^k \to 0$. □

Theorem 4.4. *Suppose $\{\|y^k\|\}$ is bounded. Then*

$$\|F(y^k, w^k)\| \to 0.$$

The proof of Theorem 4.4 is a proof by contradiction and is postponed until we have proved the following several lemmas.

Lemma 4.5. *Suppose $\{\|y^k\|\}$ is bounded. If any subsequence $\{\theta^{k_l}\}$ of $\{\theta^k\}$ is bounded away from zero, then $\{\|\bar{H}_{\theta^{k_l}}^{-1}\| + \|s_y^{k_l}\| + \|s_w^{k_l}\| + \|s_x^{k_l}\|\}$ is bounded.*

Proof: We may omit the superscript k_l in the proof when there is no confusion.

First, by Lemma 4.2, $\{\|x^k\|\}$ is bounded. Then by (3.1), $\{\|\nabla f^k\|\}$ is bounded.

Using the definition of $\bar{H}_\theta$ in (3.7), and the notations $s = (s_y, s_w)$ and $s_x = H^{-1}(s_y + A^T s_w)$, we have

$$s^T \bar{H}_\theta s = s_x^T H s_x + s_y^T |Y|^{-1} D_\theta s_y.$$

Since $\{\|y^k\|\}$ is bounded, we may assume that $\|y^k\|_\infty \le M$ for some constant $M \ge 1$ and for all k. Then by (3.6), the definition of D_θ in (3.7), the fact that $H > 0$, and the fact that $\{\theta^{k_l}\}$ is bounded away from zero, there exist $\epsilon_1,\ \epsilon_2 > 0$, such that

$$s^T \bar{H}_\theta s \ge \epsilon_1 \|s_x\|^2 + \frac{\epsilon_2}{M}\|s_y\|^2. \tag{4.5}$$

Since A has full row rank and $s_x = H^{-1}(s_y + A^T s_w)$, there exists $\epsilon_3 > 0$ such that

$$s^T \bar{H}_\theta s \ge \epsilon_3 \|s\|. \tag{4.6}$$

Thus

$$\left\|\bar{H}_{\theta^{k_l}}^{-1}\right\| \le \frac{1}{\epsilon_3} \quad \text{for every } k_l. \tag{4.7}$$

The boundedness of $\{\|s_y^{k_l}\|\}$, $\{\|s_w^{k_l}\|\}$, and $\{\|s_x^{k_l}\|\}$ then follows from the boundedness of $\{\|\nabla f^k\|\}$, (3.6) and (3.13). □

Lemma 4.6. *Suppose $\{\|y^k\|\}$ is bounded. If any subsequence $\{\theta^{k_l}\}$ of $\{\theta^k\}$ is bounded away from zero, then $(\nabla f^{k_l})^T s^{k_l} \to 0$.*

Proof: If the lemma is not true, then by Lemma 4.3, there exists a subsequence, still denoted by $\{\beta_1^{k_l}\}$, which converges to zero. By (3.17) and (3.16), we may assume without loss of generality that

$$\beta_1^{k_l} = -\frac{y_1^{k_l}}{\left(s_y^{k_l}\right)_1}.$$

From (3.9), we have (using (3.13))

$$\frac{\theta^{k_l} + (1-\theta^{k_l})\left|d_1^{k_l}\right|}{sign\left(y_1^{k_l}\right)\left[x_1^{k_l} + \left(s_x^{k_l}\right)_1 + sign\left(y_1^{k_l}\right)\right]} = -\frac{y_1^{k_l}}{\left(s_y^{k_l}\right)_1} \to 0. \tag{4.8}$$

Therefore, we must have $x_1^{k_l} + (s_x^{k_l})_1 \to \infty$ since $\{\theta^{k_l}\}$ is bounded away from zero. This is a contradiction to the fact that the sequences $\{\|x^k\|\}$ and $\{\|s_x^k\|\}$ are bounded. □

Lemma 4.7. *Suppose $\{\|y^k\|\}$ is bounded. If any subsequence $\{\theta^{k_l}\}$ of $\{\theta^k\}$ is bounded away from zero, then $\|s_y^{k_l}\| + \|s_w^{k_l}\| + \|s_x^{k_l}\| \to 0$.*

Proof: By (3.6), $-(\nabla f^k)^T s^k = (s^k)^T \bar{H}_{\theta^k} s^k$. Then similar to (4.6),

$$-\left(\nabla f^{k_l}\right)^T s^{k_l} \geq \epsilon_3 \left\|s^{k_l}\right\|^2. \tag{4.9}$$

Therefore the lemma follows from the fact that $s^{k_l} = (s_y^{k_l}, s_w^{k_l})$, Lemma 4.6 and (3.13). □

Lemma 4.8. *Suppose* $\{\|y^k\|\}$ *is bounded. If any subsequence* $\{\theta^{k_l}\}$ *of* $\{\theta^k\}$ *is bounded away from zero, then* $\|F(y^{k_l}, w^{k_l})\| \to 0$.

Proof: The Eq. (3.9) yields

$$(|Y|H^{-1} + D_\theta)s_y + |Y|H^{-1}A^T s_w = -|Y|d. \tag{4.10}$$

The boundedness of $\{\|x^k\|\}$ implies that $\{\|D_{\theta^k}\|\}$ is bounded. Therefore $\|\,|Y^{k_l}|d^{k_l}\| \to 0$ since $\|s_y^{k_l}\| + \|s_w^{k_l}\| \to 0$ (Lemma 4.7) and $\{\|Y^k\|\}$ is bounded. Then the lemma follows from the definition of F in (1.5) and (3.22). □

Proof of Theorem 4.4: If Theorem 4.4 is false, then there is a subsequence $\{\|F(y^{k_l}, w^{k_l})\|\}$ that is bounded away from zero. Therefore, definition (3.8) shows that the subsequence $\{\theta^{k_l}\}$ is bounded away from zero. Consequently $\|F(y^{k_l}, w^{k_l})\| \to 0$ by Lemma 4.8. That is a contradiction. Therefore, Theorem 4.4 must hold. □

Corollary 4.9. *Suppose problem* (1.1) *is primal nondegenerate and* $\{\|y^k\|\}$ *is bounded. Then the number of limit points of* $\{x^k\}$ *is finite.*

Proof: Let $\bar{x}$ be any limit point of $\{x^k\}$. Since all the sequences $\{\|x^k\|\}$, $\{\|y^k\|\}$, and $\{\|w^k\|\}$ are bounded, we may assume that $x^{k_l} \to \bar{x}$, $y^{k_l} \to \bar{y}$, and $w^{k_l} \to \bar{w}$. By Theorem 4.4, $F(\bar{y}, \bar{w}) = 0$ and $\bar{x} \in C$. Therefore every limit point of $\{x^k\}$ is in the set C and the result follows from Lemma 4.9. □

Lemma 4.10. *Suppose problem* (1.1) *is primal nondegenerate and* $\{\|y^k\|\}$ *is bounded. Then* $\{x^k\}$ *converges.*

Proof: By Lemma 4.2, $\{x^k\}$ has at least one limit point. Let $\bar{x}$ and $\hat{x}$ be any two limit points of $\{x^k\}$. We show that $\bar{x}$ and $\hat{x}$ must be the same.

If $|\bar{x}_i| \leq 1$ $(1 \leq i \leq n)$ and $|\hat{x}_i| \leq 1$ $(1 \leq i \leq n)$. Then by Theorem 4.4, both $\bar{x}$ and $\hat{x}$ are solutions of (1.1). Since (1.1) has a unique solution, we must have $\bar{x} = \hat{x}$.

If $|\bar{x}_i| > 1$ for some $1 \leq i \leq n$ and if $\{x^k\}$ does not converge to $\bar{x}$, then by Lemma 4.10 in [18], there exists a subsequence $\{x^{k_l}\}$ which converges to $\bar{x}$ and $\|x^{k_l+1} - x^{k_l}\| \geq \epsilon$ for some fixed $\epsilon > 0$ (for all k_l). Moreover, by the definition of θ in (3.8), the subsequence θ^{k_l} must be bounded away from zero since $x_l^k \to \bar{x}$ and $|\bar{x}_i| > 1$. Then Lemma 4.7 shows $\|s_x^{k_l}\| \to 0$. Therefore, $\|x^{k_l+1} - x^{k_l}\| = \|\alpha^{k_l} s_x^{k_l}\| \to 0$ which results in a contradiction.

Similarly, it cannot happen that $|\hat{x}_i| > 1$ for some $1 \leq i \leq n$ and that $\{x^k\}$ does not converge to $\hat{x}$. This completes the proof. □

Theorem 4.11. *Suppose problem* (1.1) *is primal nondegenerate* $\{\|y^k\|\}$ *is bounded. Then* $\{x^k\}$ *converges to the solution of* (1.1).

Proof: By Lemma 4.10, we may assume that $x^k \to x^*$. Then x^* satisfies the optimality conditions (1.3) and (1.4) (Theorem 4.4). Therefore, it suffices to show that $|x_i^*| \leq 1$ for every $i = 1, 2, \ldots, n$. Without loss of generality, we will only show that if $x_1^* > 1$, a contradiction will result. We can show in a similar way that $x_i^* \leq 1$ and $x_i^* \geq -1$ for every $i = 1, 2, \ldots, n$.

First, by (4.10) and (3.13), for each $i = 1, 2, \ldots, n$,

$$\left(s_y^k\right)_i = -\frac{|y_i^k|\left(d_i^k + \left(s_x^k\right)_i\right)}{\theta^k + (1-\theta^k)\left|d_i^k\right|}, \quad \text{where } d_i^k = x_i^k + sign\left(y_i^k\right). \tag{4.11}$$

Now assume $x_1^* > 1$. Then the sequence $\{\theta^k\}$ is bounded away from zero. Hence $\|s_x^k\| \to 0$ (Lemma 4.7) and there exists $\epsilon_1 > 0$ such that for all k sufficiently large,

$$d_1^k \geq \epsilon_1, \quad d_1^k + \left(s_x^k\right)_1 \geq \epsilon_1\, \theta^k x_1^k + \left(s_x^k\right)_1 > \epsilon_1 \quad \text{and} \quad \left(s_y^k\right)_1 < 0. \tag{4.12}$$

By Theorem 4.4, $y_1^k d_1^k \to 0$ which implies $y_1^k \to 0$. Noticing that $y_1^{k+1} = y_1^k + \alpha^k (s_y^k)_1$, we must have, for all k sufficiently large,

$$y_1^k > 0 \quad \text{and} \quad d_1^k = 1 + x_1^k > 2. \tag{4.13}$$

Therefore, by (3.16) and (4.11), for all k sufficiently large,

$$r_1^k = \frac{\theta^k + (1-\theta^k)\left|d_1^k\right|}{sign\left(y_1^k\right)\left(d_1^k + \left(s_x^k\right)_1\right)} = 1 - \frac{\theta^k x_1^k + \left(s_x^k\right)_1}{d_1^k + \left(s_x^k\right)_1} < 1, \tag{4.14}$$

and

$$0 < \alpha^k = \frac{y_1^{k+1} - y_1^k}{\left(s_y^k\right)_1} < -\frac{y_1^k}{\left(s_y^k\right)_1} = r_1^k. \tag{4.15}$$

One the other hand, let

$$\mathcal{J}^k := \left\{j : 1 \leq j \leq n, r_j^k \leq r_1^k\right\}$$

which is not empty since $1 \in \mathcal{J}^k$. Then similar to (4.14),

$$0 < \frac{\theta^k + (1-\theta^k)\left|d_j^k\right|}{sign\left(y_j^k\right)\left(d_j^k + \left(s_x^k\right)_j\right)} = r_j^k \leq r_1^k = \frac{\theta^k + (1-\theta^k)\left|d_1^k\right|}{sign\left(y_1^k\right)\left(d_1^k + \left(s_x^k\right)_1\right)} \quad \text{for every } j \in \mathcal{J}^k. \tag{4.16}$$

Hence when k is sufficiently large, $sign(y_j^k)(d_j^k + (s_x^k)_j) = |d_j^k + (s_x^k)_j|$ for every $j \in \mathcal{J}^k$. Thus (4.16) can be rewritten as

$$0 < \frac{\theta^k + (1-\theta^k)|d_j^k|}{|d_j^k + (s_x^k)_j|} \le \frac{\theta^k + (1-\theta^k)d_1^k}{d_1^k + (s_x^k)_1} \quad \text{for every } j \in \mathcal{J}^k.$$

Therefore, for every k sufficiently large,

$$|d_j^k + (s_x^k)_j| \ge d_1^k + \left(1 - \frac{1-\theta^k}{\theta^k} d_1^k\right)(s_x^k)_1 + \frac{1-\theta^k}{\theta^k}|d_j^k|(s_x^k)_j \quad \text{for every } j \in \mathcal{J}^k.$$

Consequently, since $\|s_x^k\| \to 0$, $d_1^k > 2$ (by (4.13)), and $\|d^k\|$ is bounded above, when k is sufficiently large,

$$|d_j^k + (s_x^k)_j| > 2 \quad \text{for every } j \in \mathcal{J}^k.$$

Now using (3.29), (3.14), (3.12), and (3.25),

$$\begin{aligned}
\psi'((r_1^k)^+) &= (s_y^k)^T d^k + r_1^k (s_x^k)^T s_y^k + \sum_{j\in\mathcal{J}^k} |(s_y^k)_j| \\
&= -\psi'' - (s_y^k)^T |Y^k|^{-1} D_{\theta^k} s_y^k + r_1^k \psi'' + \sum_{j\in\mathcal{J}^k} |(s_y^k)_j| \\
&= (r_1^k - 1)\psi'' - (s_y^k)^T |Y^k|^{-1} D_{\theta^k} s_y^k + 2\sum_{j\in\mathcal{J}^k} |(s_y^k)_j| \\
&\le -\sum_{j=1}^{n} \frac{(D_{\theta^k})_{jj}(s_y^k)_j^2}{|y_j^k|} + 2\sum_{j\in\mathcal{J}^k} |(s_y^k)_j| \quad \text{(by (4.14))} \\
&\le \sum_{j\in\mathcal{J}^k} \left(2 - \frac{(D_{\theta^k}^k)_{jj}|(s_y^k)_j|}{|y_j^k|}\right)|(s_y^k)_j| \quad \text{(by dropping some negative terms)} \\
&= \sum_{j\in\mathcal{J}^k} (2 - |d_j^k + (s_x^k)_j|)|(s_y^k)_j| \quad \text{(by (4.11))} \\
&< 0
\end{aligned} \tag{4.17}$$

where the last inequality is strict because $1 \in \mathcal{J}^k$ and $(s_y^k)_1 \ne 0$. Therefore, $\alpha_{opt}^k > r_1^k$, $\tilde{\alpha}^k > r_1^k$ (by the definition of $\tilde{\alpha}$ in (3.31) and by the fact that $r_1^k < 1$), and $\alpha^k > r_1^k$ (by the definition of α in (3.32)). This is a contradiction to (4.15). □

Next, we show that if (1.1) is feasible and primal nondegenerate, then the sequence $\{\|y^k\|\}$ must be bounded. To this end, we first give a general result (Lemma 4.12) which is a restatement of Corollary 8.7.1 in [20].

Let $\phi(z) : \Re^n \to \Re$ be any convex function. Let the set of minimizers of ϕ be

$$\mathcal{S}_\phi := \{z^* : z^* \text{ is a global minimizer of } \phi\}.$$

Since ϕ is convex, any local minimizer of ϕ is also a global minimizer.

Lemma 4.12. *Suppose ϕ is a convex function and the set $\mathcal{S}_\phi$ is nonempty and bounded. Then for any given z^0, the level set*

$$\mathcal{L}(z^0) := \{z : \phi(z) \le \phi(z^0)\}$$

is bounded.

The following result is well-known (see, e.g., Lemma 6.7 in [2]) and the proof is omitted.

Lemma 4.13. *Suppose* (1.1) *is feasible and primal nondegenerate. Then problem* (2.1) *has a unique solution.*

Corollary 4.14. *Suppose* (1.1) *is feasible and primal nondegenerate. Then the sequence* $\{\|y^k\|\}$ *is bounded.*

Proof: Since $f(y, w)$ is convex, by Lemma 4.12 and Lemma 4.13, the level set

$$\mathcal{L}_0 := \{(y, w) : f(y, w) \le f(y^0, w^0)\}$$

is bounded. Then the corollary follows from the fact that $\{f(y^k, w^w)\}$ is monotonically decreasing. □

To sum up, we have the following theorem:

Theorem 4.15. *Suppose* (1.1) *is primal nondegenerate. If* (1.1) *is feasible, then* $(y^k, w^k, x^k) \to (y^*, w^*, x^*)$*, where x^* is the solution of* (1.1)*, w^* is the Lagrange multiplier, and (y^*, w^*) solves* (2.1)*. On the other hand, if* (1.1) *is infeasible, then* $\{\|y^k\|\}$ *must be unbounded.*

The following condition can be used for detecting infeasibility.

Lemma 4.16. *Let $H = (h_{ij})$ and $f_{ub} = \frac{1}{2}\sum_{i,j} |h_{ij}| + \|c\|_1$. If $-f(y^k, w^k) > f_{ub}$ for any k, then problem* (1.1) *is infeasible.*

Proof: By (2.4), if $\Omega \ne \emptyset$, then for any $(y, w) \in \Re^{m+n}$,

$$\begin{aligned} -f(y, w) &\le \max\{q(x) | x \in \Omega\} \\ &\le \max\{q(x) | \|x\|_\infty \le 1\} \\ &\le f_{ub}. \end{aligned} \tag{4.18}$$

Hence the lemma follows. □

In our numerical experiments, Lemma 4.16 was used and was seen effective for detecting infeasibility.

5. Quadratic convergence

In this section, we assume that problem (1.1) is feasible and primal nondegenerate. Then $(x^k, y^k w^k) \to (x^*, y^* w^*)$ where x^* is the solution of (1.1) and (y^*, w^*) is the solution of (2.1). In addition, $y^* = Hx^* + c - A^T w^*$.

The following strict complementarity condition will be assumed for establishing quadratic convergence:

$$|x_i^*| = 1 \Rightarrow y_i^* \neq 0 \quad \text{for every } i = 1, 2, \ldots, n. \tag{5.1}$$

Lemma 5.1. *Suppose condition* (5.1) *holds. Let*

$$G^k := \begin{bmatrix} |Y^k|H^{-1} + |D_{\theta^k}| & |Y^k|H^{-1}A^T \\ AH^{-1} & AH^{-1}A^T \end{bmatrix}.$$

Then

$$G^* := \begin{bmatrix} |Y^*|H^{-1} + |D^*| & |Y^*|H^{-1}A^T \\ AH^{-1} & AH^{-1}A^T \end{bmatrix} = \lim G^k$$

is nonsingular, where $D^* = diag(x^* + sign(y^*))$. Consequently,

$$s^k = \left(s_y^k, s_w^k\right) \to 0. \tag{5.2}$$

Proof: Let $G^*z = 0$. We show that z must be zero. Then it follows that G^* is nonsingular. Write $z = (z^1, z^2)$ where $z^1 \in \Re^n$ and $z^2 \in \Re^m$. By (5.1), $y_{\mathcal{I}_*}^* = 0$ and $d_{\mathcal{I}_*}^* \neq 0$. Hence $G^*z = 0$ implies $z_{\mathcal{I}_*}^1 = 0$. Therefore, the equation $G^*z = 0$ can be reduced to

$$\begin{bmatrix} H^{-1}_{\mathcal{I}_*^c \times \mathcal{I}_*^c} & (H^{-1}A^T)_{\mathcal{I}_*^c} \\ (H^{-1}A^T)^T_{\mathcal{I}_*^c} & AH^{-1}A^T \end{bmatrix} \begin{bmatrix} z^1_{\mathcal{I}_*^c} \\ z^2 \end{bmatrix} = 0. \tag{5.3}$$

Using Corollary 1 in [12] and the fact that $rank(A_{\mathcal{I}_*}^T) = m$, we see that the coefficient matrix in (5.3) is positive definite. Consequently $(z^1_{\mathcal{I}_*^c}, z^2) = 0$.

To show (5.2), we note that (3.6) is equivalent to

$$G^k s^k = -\begin{bmatrix} |Y^k|d^k \\ 0 \end{bmatrix}. \tag{5.4}$$

Since $G^k \to G^*$ and $|Y^k|d^k \to 0$, (5.2) follows from the nonsingularity of G^*. □

To show quadratic convergence, we cite a standard result (see, e.g., [5]), used subsequently.

Theorem 5.2. *Let $\mathcal{D} \subseteq \Re^l$ be an open convex set. Let $v^* \in \mathcal{D}$, $F : \Re^l \to \Re^l$, $F(v^*) = 0$, $\nabla F(v^*)$ be nonsingular, and ∇F be Lipschitz continuous at v^* in $\mathcal{D}$. Let $\{T_k\}$ be a sequence of nonsingular matrices in $\Re^{l\times l}$. Suppose for some $v^0 \in \mathcal{D}$ that the sequence of points generated by $v^{k+1} = v^k - T_k^{-1}F(v^k)$ remains in $\mathcal{D}$, $v^k \neq v^*$ for every k, and $v^k \to v^*$.*

If $\|T_k - \nabla F(v^*)\| \to 0$, then $\{v^k\}$ converges superlinearly to v^*.

If $\|T_k - \nabla F(v^*)\| = O(\|v^k - v^*\|)$, then $\{v^k\}$ converges quadratically to v^*.

Theorem 5.2 can not be directly applied to the function F defined in (1.5) since F is not differentiable at some points. Instead, we apply Theorem 5.2 to the following auxiliary function:

$$\hat{F}(y, w) := \begin{bmatrix} Y\hat{d} \\ Ax - b \end{bmatrix} : \Re^{m+n} \to \Re^{m+n},$$

where

$$\hat{d} := x + sign(y^*), \quad \text{and} \quad x = H^{-1}(y - c + A^T w). \tag{5.5}$$

In other words, the nondifferentiable term $sign(y)$ in F is frozen to a constant vector $sign(y^*)$ to form an everywhere-differentiable function $\hat{F}$. Clearly, $\hat{d}^* = d^* = x^* + sign(y^*)$ and

$$\hat{F}(y^*, w^*) = F(y^*, w^*) = 0.$$

Moreover, the Jacobian of $\hat{F}$,

$$\nabla\hat{F} = \begin{bmatrix} YH^{-1} + \hat{D} & YH^{-1}A^T \\ AH^{-1} & AH^{-1}A^T \end{bmatrix},$$

exists and is continuous everywhere in $\Re^{m+n}$. Similar to Lemma 5.1, we have the following lemma without proof:

Lemma 5.3. *Suppose condition* (5.1) *holds. Then the Jacobian $\nabla\hat{F}(y^*, w^*)$ is nonsingular. Moreover, there exist $\delta > 0$ and a neighborhood $B_\delta(y^*, w^*)$ of (y^*, w^*), such that $\nabla\hat{F}$ is Lipschitz continuous at (y^*, w^*) in $B_\delta(y^*, w^*)$, and $\nabla\hat{F}$ is nonsingular at every $(y, w) \in B_\delta(y^*, w^*)$.*

Lemma 5.4. *Suppose condition* (5.1) *holds. Then for all k sufficiently large,*

$$sign(y_i^k) = sign(y_i^*) \text{ and } \hat{d}_i^k = d_i^k \quad \text{for all } i \in \mathcal{I}_*^c, \tag{5.6}$$

$$(s_y^k)_i \neq 0 \text{ and } sign((s_y^k)_i) = -sign(y_i^k) = -sign(d_i^k) \quad \text{for all } i \in \mathcal{I}_*. \tag{5.7}$$

Proof: Let $i \in \mathcal{I}_*^c$. Then $d_i^* = 0$, $y_i^* \neq 0$ and (5.6) follows immediately.

Now let $i \in \mathcal{I}_*$. Then $d_i^* \neq 0$ and $y_i^* = 0$. By (5.1), it must be true that $|x_i^*| < 1$. Hence for all k sufficiently large, $sign(d_i^*) = sign(d_i^k) = sign(x_i^k + sign(y_i^k)) = sign(y_i^k)$. Using (4.11) and the fact that $s_x^k \to 0$ (by (5.2) and (3.13)), we have (5.7). □

Lemma 5.5. *Suppose condition* (5.1) *holds. Then*

$$\lim_{k\to\infty} r_i^k = \begin{cases} \infty, & \text{if } i \in \mathcal{I}_*^c \\ 1, & \text{otherwise.} \end{cases}$$

Proof: Let $i \in \mathcal{I}_*^c$. Then $y_i^* \neq 0$. Since $s_y^k \to 0$, it follows from (3.16) that $r_i^k \to \infty$.

If $i \in \mathcal{I}_*$, then $d_i^* \neq 0$. Since $\theta^k \to 0$, by (3.16), (4.11) and (5.7), we see $r_i^k \to 1$. □

By (5.6) and (5.7), for all k sufficiently large, either $\hat{d}_i^k = d_i^k$ if $i \in \mathcal{I}_*^c$, or $d_i^k \neq 0$ if $i \in \mathcal{I}_*$ (the vector $\hat{d}$ is defined in (5.5)). Define a vector $\bar{d}^k \in \Re^n$ for each k by

$$\bar{d}_i^k = \begin{cases} sign(y_i^k) & \text{if } i \in \mathcal{I}_*^c \\ \dfrac{\hat{d}_i^k}{d_i^k} sign(y_i^k) & \text{otherwise,} \end{cases}$$

and let $\bar{D}^k = diag(\bar{d}^k)$. We may rewrite (5.4) as

$$T_\alpha^k s_\alpha^k = -\hat{F}(y^k, w^k), \tag{5.8}$$

where

$$T_\alpha^k := \begin{bmatrix} \frac{1}{\alpha^k}\bar{D}^k(|Y^k|H^{-1} + |D_{\theta^k}|) & \frac{1}{\alpha^k}\bar{D}^k|Y^k|H^{-1}A^T \\ \frac{1}{\alpha^k}AH^{-1} & \frac{1}{\alpha^k}AH^{-1}A^T \end{bmatrix} \quad \text{and} \quad s_\alpha^k := \alpha^k s^k. \tag{5.9}$$

By Theorem 5.2, if for all k sufficiently large, T_α^k is nonsingular and

$$\left\|T_\alpha^k - \nabla\hat{F}(y^*, w^*)\right\| = O(\|(y^k, w^k) - (y^*, w^*)\|); \tag{5.10}$$

here $(y^k, w^k) \to (y^*, w^*)$ quadratically. Now we establish these results.

Lemma 5.6. *Suppose condition* (5.1) *holds. Then the following will hold*:

(i) $\theta^k = O(\|(y^k, w^k) - (y^*, w^*)\|)$;

(ii) $\|\bar{D}^k|Y^k| - Y^*\| = O(\|(y^k, w^k) - (y^*, w^*)\|)$;

(iii) $\|\bar{D}^k|D_{\theta^k}| - \hat{D}^*\| = O(\|(y^k, w^k) - (y^*, w^*)\|)$;

(iv) $|1 - \alpha^k| = O(\|(y^k, w^k) - (y^*, w^*)\|)$.

Proof: Result (i) is straightforward by (3.8), (1.5) and the fact that

$$x^* = H^{-1}(y^* - c + A^T w^*).$$

For (ii), consider first $i \in \mathcal{I}_*$. Then $y_i^* = 0$ and $d_i^* \neq 0$. Hence

$$\left|\bar{d}_i^k |y_i^k| - y_i^*\right| = \left|\bar{d}_i^k |y_i^k|\right| = \frac{|\hat{d}_i^k|}{|d_i^k|}|y_i^k - y_i^*| = O(\|(y^k, w^k) - (y^*, w^*)\|). \tag{5.11}$$

If $i \in \mathcal{I}_*^c$, then by (5.6),

$$\left|\bar{d}_i^k |y_i^k| - y_i^*\right| = \left|sign(y_i^k)|y_i^k| - y_i^*\right| = \left|y_i^k - y_i^*\right| = O(\|(y^k, w^k) - (y^*, w^*)\|).$$

Therefore, (ii) follows.

Result (iii) can be proved similarly by using (i) and Lemma 5.4.

To show (iv), by (3.31), (3.32) and (3.35),

$$\min\left(1, (1-\theta^k)\,\beta_1^k\right) \le \alpha^k \le 1 + \theta^k \tau_1, \text{ for all } k \text{ sufficiently large.}$$

Hence

$$\min\left(0, \beta_1^k - 1 - \theta^k\,\beta_1^k\right) \le \alpha^k - 1 \le \theta^k \tau_1.$$

By (i), it suffices to show that

$$\left|\beta_1^k - 1\right| = O(\|(y^k, w^k) - (y^*, w^*)\|).$$

In fact, since $rank(A_{\mathcal{I}_*}^T) = m$, $\mathcal{I}_*$ is not empty. Therefore, Lemma 5.5 shows that for every k sufficiently large,

$$\beta_1^k = r_{p_1^k}^k \quad \text{and} \quad p_1^k \in \mathcal{I}_*.$$

Moreover, by (5.4), it is easy to see that

$$\|s^k\| = O(\|(y^k, w^k) - (y^*, w^*)\|),$$

which implies

$$\left\|s_x^k\right\| = O(\|(y^k, w^k) - (y^*, w^*)\|).$$

Therefore, by (3.16), (4.11) and (5.7),

$$\left|\beta_1^k - 1\right| = \left|r_{p_1^k}^k - 1\right| = \left|-\frac{y_{p_1^k}^k}{(s_y^k)_{p_1^k}} - 1\right|$$

Table 4. Dense and degenerate problems.

Size		Cond = 10^3	
n	m	Max	Avg
200	50	38	31.3
200	100	67	55.7
200	150	48	44

Table 5. A comparison with infeasible interior point method (IIPM).

	Size			
Name	n	m	IIPM (IBM OSL)	Exterior-Newton
afiro	51	27	11	6
agg2	558	516	9	16
blend	114	74	9	7

$m = l^2$ and $n > m$, we partition the unit square $[0, 1] \times [0, 1]$ into an $(l + 2) \times (l + 2)$ evenly spaced grid, and obtain $m = l^2$ interior nodes (mesh points). For each node there is a first order spline basis function (the pyramid function) ψ_i $(1 \leq i \leq m)$. Let $z_j \in \Re^2$ $(1 \leq j \leq n)$ be randomly scattered on the unit square. Then

$$A = (a_{ij}) \quad \text{where } a_{ij} := \psi_i(z_j).$$

Such matrices appear in various applications. One example is to determine the spline function which is a linear combination of the m basis functions that fits best with the observed values at n points, usually in the least squares sense. It should be noted that problems in this set may be degenerate.

Table 3 reports on another set of sparse problems, where the matrix H was generated the same way as in the second set, while the matrix A was "imported" from the set of linear programming test problems. We used three matrices A from "agg2", "boeing1", and "fffff800". Again, problems in this set may be degenerate. "Cond" is the condition number of H.

Problems in the fourth set are again dense problems but they are highly degenerate. The matrix H and A were chosen to be well-conditioned ($cond(H) = cond(A) = 1000$) so that we can clearly see the effect of degeneracy.

Since an infeasible interior point method does not require a feasible starting point for convex quadratic programming problems (including LP), it would be interesting to compare one with our method. Hence, in Table 5, a comparison is made between our algorithm and an infeasible primal-dual interior point method, implemented in the IBM Optimization Subroutine Library (OSL) [23], for three problems. The three test problems were generated as follows. We first obtain a linear programming test problem from Netlib (afiro, agg2, and blend), change the bound constraints from $x \geq 0$ to $-1 \leq x \leq 1$, set the right hand side to zero (so the problem is guaranteed to be feasible), and add a quadratic term $\frac{1}{2}x^T Hx$ with H being the identity matrix.

The numerical experiment results indicate that the proposed method is very effective for nondegenerate problems and the number of iterations is reasonably small for degenerate problems. It is clear that the number of iterations is not very sensitive to problem dimension. The number of iterations in Table 5 shows that our algorithm is comparable to the infeasible primal-dual interior point method for certain QP problems. We also tested several infeasible problems and the infeasibility was always detected in less than 10 iterations. Finally, we remark that given an infeasible point near to the solution, the proposed method can be used to identify the solution very quickly. For example, we considered the test problems in Table 1, Row 1. For each problem, the starting point was taken by perturbing the solution by a random vector with l_∞ norm less than 0.01. The average number of iterations required was 8.8.

7. Concluding remarks

We have proposed an exterior Newton method for convex QP with equality constraints and lower and upper bounds. Based on a dual formulation, the proposed algorithm monotonically decreases a dual objective function. Strong convergence results are established. Preliminary numerical experiments demonstrate the potential of this algorithm. We have also applied the algorithm to some (known) infeasible problems, and infeasibility was always detected at an early stage. One open question is whether the nondegeneracy assumption can be removed from the convergence proof following the work in [22]. Another open question is whether this approach can be adapted to solve QP problems when H is positive semidefinite. This extension will be interesting and important since linear programming problems will be included. We plan to investigate this issue.

Acknowledgments

We thank the referee for several useful references and for many helpful comments that improved the presentation of this paper.

References

1. A.V. Aho, J.E. Hopcroft, and J.D. Ullman, The Design and Analysis of Computer Algorithms, Addison-Wesley Publishing Company, 1974.
2. A.B. Berkelaar, C. Roos, and T. Terlaky, "The optimal set and optimal partition approach to linear and quadratic programming," Chapter 6 in Advances in Sensitivity Analysis and Parametric Programming, H.J. Greenberg and T. Gal (Eds.), Kluwer Academic Publishers: Dordrecht, 1997.
3. T.J. Carpenter and D.F. Shanno, "An interior point method for quadratic programs based on conjugate projected gradients," Computational Optimization and Applications, vol. 2, pp. 5–28 (1993).
4. T.F. Coleman and L.A. Hulbert, "A globally and superlinearly convergent algorithm for convex quadratic programming with simple bounds," SIAM Journal on Optimization, vol. 3, pp. 298–321 (1993).
5. J.E. Dennis, Jr. and R.B. Schnabel, Numerical Methods for Unconstrained Optimization and Nonlinear Equations, Prentice-Hall: Englewood Cliffs, NJ, 1983.

6. I.I. Dikin, "Iterative solution of problems of linear and quadratic programming," Soviet Mathematics Doklady, vol. 8, pp. 674–675 (1967).
7. R. Fletcher, Practical Methods of Optimization, 2nd edition, John Wiley and Sons: New York, 1987.
8. P.E. Gill, W. Murray, and M.H. Wright, Practical Optimization, Academic Press: London, 1981.
9. D. Goldfarb, "Extensions of Newton's method and simplex methods for solving quadratic programs," in Numerical Methods for Nonlinear Optimization, F.A. Lootsman (Ed.), Academic Press: London, 1972.
10. D. Goldfarb and S. Liu, "An $O(n^3 L)$ primal interior point algorithm for convex quadratic programming," Mathematical Programming, vol. 49, pp. 325–340 (1991).
11. G.H. Golub and C.F. Van Loan, Matrix Computations, 2nd edition, The Johns Hopkins University Press, 1989.
12. E.V. Haynsworth, "Determination of the inertia of a partitioned Hermitian matrix," Linear Algebra and its Applications, vol. 1, pp. 73–81 (1968).
13. D. Hertog, C. Roos, and T. Terlaky, "A polynomial method of weighted centers for convex quadratic programming," J. Inform. Optim. Sci., vol. 12, pp. 187–205 (1991).
14. W. Li and J. Swetits, "A new algorithm for solving strictly convex quadratic programs," SIAM Journal on Optimization, to appear.
15. K. Madsen, H. Nielsen, and M. Pinar, "A new finite continuation algorithm for bound constrained quadratic programming," Tech. Report, IMM-REP-1995-22, Institute of Mathematical Modeling, Technical University of Denmark, 1995.
16. S. Mehrotra and J. Sun, "An algorithm for convex quadratic programming that requires $O(n^{3.5} L)$ arithmetic operations," Math. Oper. Res., vol. 15, pp. 342–363 (1990).
17. R. Monteiro and I. Adler, "Interior path-following primal-dual algorithms, part II: Convex quadratic programming," Mathematical Programming, vol. 44, pp. 43–66 (1989).
18. J.J. Moré and D. Sorensen, "Computing a trust region step," SIAM Journal on Scientific and Statistical Computing, vol. 4, pp. 553–572 (1983).
19. J.J. Moré and G. Toraldo, "Algorithms for bound constrained quadratic programming problems," Numerische Mathematik, vol. 55, pp. 377–400 (1989).
20. R.T. Rockafellar, Convex Analysis, Princeton University Press, 1970.
21. D. Sorensen, "Trust region methods for unconstrained optimization," SIAM Journal on Numerical Analysis, vol. 19, pp. 409–426 (1982).
22. T. Tsuchiya, "Affine scaling algorithm," Interior Point Methods of Mathematical Programming, T. Terlaky (Ed.), Kluwer Academic Publishers: Dordrecht, 1996.
23. D.G. Wilson, "A brief introduction to the IBM optimization subroutine library," SIAG/OPT Views and News, vol. 1, pp. 9–10 (1992).
24. P. Wolfe, "The simplex method for quadratic programming," Econometrica, vol. 27, pp. 382–398 (1959).
25. Y. Ye, "Interior point algorithms for quadratic programming," in Recent Developments in Mathematical Programming, S. Kumar (Ed.), Gordon & Beach Scientific Publishers: Philadelphia, 1991.