Mathematical Programming 15 (1978) 239–242.
North-Holland Publishing Company

SHORT COMMUNICATION

A NOTE ON 'NEW ALGORITHMS FOR CONSTRAINED MINIMAX OPTIMIZATION'*

T.F. COLEMAN

University of Waterloo, Waterloo, Ontario, Canada

Received 20 March 1978
Revised manuscript received 2 June 78

An unconstrained minimax algorithm of Charalambous and Conn is easily modified to solve the constrained case. Here we present some numerical results and find that this algorithm compares favourably to those of Dutta and Vidyasagar.

Key words: Minimax Optimization, Nonlinear Programming.

1. Introduction

In "New algorithms for constrained minimax optimization" [4] two algorithms for solving the constrained minimax problem are presented, along with numerical results. The method of Charalambous and Conn [1], [2] is mentioned in the context of unconstrained minimax problems, and, as is pointed out by them, it is easy to adapt their method to the constrained case. To see this, consider the problem

$$\text{minimize} \quad F(x) = \max_{i \in I} f_i(x), \tag{1}$$

where $I = \{1, \dots, m\}$ is a finite set of integers. The approach suggested in [1], [2] is based on the nonlinear programming formulation of (1):

$$\begin{aligned} &\text{minimize} \quad z, \\ &\text{subject to} \quad z - f_i(x) \geq 0, \quad i \in I. \end{aligned} \tag{2}$$

Feasible descent directions for z are generated by projecting $(-1, 0, \dots, 0)$ onto the space orthogonal to the gradients of active constraints of (2). In view of (2), the constrained minimax problem can be written as

$$\begin{aligned} &\text{minimize} \quad z, \\ &z - f_i(x) \geq 0, \quad i \in I, \\ &g_j(x) \geq 0, \quad j \in J, \\ &h_l(x) = 0, \quad l \in L, \end{aligned} \tag{3}$$

* This research was supported in part by Natural Sciences and Engineering Research Council grant number A8639.

where J and L are finite sets of integers. (3) is a nonlinear programming problem and can be solved in a manner similar to that suggested by Conn and Pietryzkowski [3]. To determine a descent direction for (3), we use the penalty function

$$p(x, \mu) = \mu \cdot z - \sum_J \min(0, g_j(x)) - \sum_L |h_l(x)|, \tag{4}$$

thus generating first-order descent directions in the space orthogonal to the gradients of active constraints of (3). We then use this direction as in [1] and [2].

This approach was tried by the author, and subsequent numerical results are listed below. The runs for the first example were performed on the Honeywell 6060, in double precision, whilst the other runs were done on the IBM 370, in double precision. Wherever possible, comparisons are made with the algorithms of Dutta and Vidyasagar. (These algorithms will be denoted by DV1 and DV2). For the penalty function approach of Charalambous and Conn, two parameters, μ and ϵ, must be initialized. These settings will be listed.

2. Numerical results and comparisons

Example 1. The problem is to minimize the maximum of

$$f_1 = x_1^2 + x_2^2, \qquad f_2 = (2 - x_1)^2 + (2 - x_2)^2,$$
$$f_3 = 2\exp(-x_1 + x_2),$$

subject to the constraints,

$$x_1 + x_2 = 2, \qquad -x_1^2 - x_2^2 + 2.25 \le 0.$$

This example is tried with 9 different starting points. The parameters μ and ϵ, are initialized to 0.1 and 0.01 respectively, in each case (see Table 1). The final values as determined by the algorithm of C & C are

$$x_1 = 1.35355, \qquad x_2 = 0.646447, \qquad F = 2.25000.$$

Table 1

Starting point	Number of function evaluations C & C	DV1	DV2
(0.5, 0.5)	14	89	37
(2, 2)	15	87	35
(2.1, 1.9)	9		
(1.9, 2.1)[a]	23		
(4, 2)	8		
(2, 4)[a]	9		
(−4, −5)	13		
(−5, −4)[a]	19		
(10, −8)	9		

[a] In these cases, convergence was to the local optimum (0.646447, 1.35355), where F = 4.05623.

Example 2. This is the 3-section transmission line problem, as described in [4], subject to the constraint

$$z_1 + z_2 + z_3 - 10 = 0.$$

This example was run with the starting points (see Table 2).
(1) $z_1 = 1.5$, $z_2 = 3.0$, $z_3 = 6.0$, $l_1 = 0.8$, $l_2 = 1.2$, $l_3 = 0.8$, and
(2) $z_1 = 1.0$, $z_2 = 3.16228$, $z_3 = 10.0$, $l_1 = 1.0$, $l_2 = 1.0$, $l_3 = 1.0$.
The parameter μ was initialized to unity in all cases.

Table 2

	Number of function evaluations			
Starting point	C & C ($\epsilon = 0.01$)	C & C ($\epsilon = 0.001$)	*DV*1	*DV*2
1	82	55	145	78
2	48	65	150	73

Final values achieved by the algorithm of Charalambous and Conn are

$$z_1 = 1.5113, \qquad z_2 = 2.8859, \qquad z_3 = 5.6028,$$
$$l_1 = l_2 = l_3 = 1.0000, \qquad F = 0.20475.$$

Example 3. This is the 3-section transmission line problem, subject to the constraints

$$0 \le z_i \le 5, \qquad i = 1, 2, 3.$$

The two starting points of Example 2 are used again here, and μ is initialized to unity (see Table 3).

Table 3

	Number of function evaluations			
Starting point	C & C ($\epsilon = 0.01$)	C & C ($\epsilon = 0.001$)	DV1	DV2
1	21	33	146	96
2	99	54	149	95

Final values achieved by the algorithms of Charalambous and Conn are

$$z_1 = 1.3825, \qquad z_2 = 2.6295, \qquad z_3 = 5.0000,$$
$$l_1 = l_2 = l_3 = 1.0000, \qquad F = 0.23056.$$

3. Conclusions

These results indicate that the algorithm of Charalambous and Conn, extended to handle constraints, is competitive with the algorithms of Dutta and Vidyasagar. It also appears that the solution generated by the former algorithm is more accurate than the solution produced by the latter. For example, if we determine the minimax value using the 4-decimal solution given by Charalambous and Conn, in Example 3, we have that $F = 0.23057$. Using the 4-decimal solution given by Dutta and Vidyasagar in [4], results in $F = 0.23064$. This better accuracy probably stems from the fact that ϕ_k (in the notation of Dutta and Vidyasagar), is a parameter that does not reflect the true minimax value, except in the limit. In [1], however, z is always the true minimax value.

Housekeeping expense comparisons are difficult to make with any accuracy. A few general remarks are in order here, however. The minimization of (4) has a work-per-iteration cost comparable to that involved in a step of the minimization of the smooth objective function of Dutta and Vidyasagar. In each case a descent direction for the respective objective functions must be found, and then a line 'minimization' must be performed. To determine a descent direction for (4), we must formulate a projection matrix, using the gradients of active constraints/functions. Finding a descent direction for the objective function of Dutta and Vidyasagar involves evaluating the gradients of some of the constraints/functions and formulating and updating a matrix approximating the Hessian. A specialized line search is used to minimize (4) along a line, to take advantage of the derivative discontinuities, as was done by Charalambous and Conn in [2]. The order of work is equivalent to that of the more usual type of line search used for smooth functions. Note, however, that the objective function (4) is no more costly to evaluate than the original functions/constraints. Most smooth approaches have an objective function of increased complexity. In [4], for example, the terms are squared.

The actual implementation will, of course, affect the overhead costs. If one chooses a numerically stable implementation, as was done here, a little is lost in terms of work-per-iteration in order to gain numerical accuracy and trustworthiness.

References

[1] C. Charalambous and A.R. Conn, "Optimization of microwave networks", *IEEE Transactions on Microwave Theory & Techniques* MTT-21 (1973) 815–818.

[2] C. Charalambous and A.R. Conn, "An efficient method to solve the minimax problem directly", *SIAM Journal on Numerical Analysis* 15 (1) (1978) 162–187.

[3] A.R. Conn and T. Pietrzykowski, "A penalty function method converging directly to a constrained optimum", *SIAM Journal on Numerical Analysis* 14, No. 2, April (1977), 348–375.

[4] S.R.K. Dutta and M. Vidyasagar, "New algorithms for constrained minimax optimization", *Mathematical Programming* 13 (1977) 140–155.