AMATH 732 - Asymptotic Analysis and Perturbation Theory

The Method of Steepest Descent

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November 22, 2015

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1 Introduction

Towards the end of the 19th Century, the Dutch mathematician Gustav de Vries and his supervisor Diederik Korteweg formulated an equation that was to become one of the most famous equations in non-linear wave dynamics. Despite this breakthrough, it was not until the mid 20th Century that further progress was made on the so-called Korteweg-de Vries (KdV) equation. Many beautiful properties were then discovered, such as the large time decomposition of the solutions into solitons and in 1967 an analytic solution was found using inverse scattering theory.

This project focuses on the method of steepest descent and its application to the long-time asymptotics of the Korteweg-de Vries equation. We begin by briefly reviewing some of the fundamental ideas and methods for the asymptotic behaviour of integrals. Once the foundation is laid we move onto the theory behind the method of steepest descent and how it may be applied. Chapter 4 then uses direct application of this method on the linearised KdV equation to work out the long-time asymptotic behaviour of the solutions. Chapter 5 approaches the more involved problem of finding the asymptotics for the non-linear KdV equation using the non-linear variant of the method of steepest descent.

2 A Review of Asymptotic Methods for Integrals

We begin with a quick review of the methods of asymptotic evaluation of integrals. The weaknesses and applicability of each method are analysed. This leads on nicely to the method of steepest descent which exhibits powerful properties and can be applied to a more diverse range of problems.

2.1 Integration of the Taylor Series

The easiest way to form an asymptotic series of an integral is to first expand the integrand into an asymptotic series, and then integrate term by term. Consider the function $f(x)$ and its Taylor series:

$$f(x) = \sum_{n=1}^{\infty} u_n(x).$$

If $u_n(x)$ is integrable on interval $[a, b]$, and $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly to $f(x)$, then we can integrate the Taylor series term by term to obtain the asymptotic behaiviour of the following integral:

$$\int_a^b f(x)dx = \sum_{n=1}^{\infty} \left(\int_a^b u_n(x)dx \right)$$

There is in fact a weaker condition sufficient for our needs, called Lebesque's dominated convergence theorem. It says that the partial sums should be bounded, i.e

$$\left| \sum_{n=1}^{M} u_n(x) \right| \leq K$$

for all M and for all $a \leq x \leq b$.

Example:

$$f(x) = \int_x^1 \frac{\cos t}{t} dt$$

$$\frac{\cos t}{t} \sim \frac{1}{t} - \frac{t}{2} + \frac{t^3}{24} - + \ldots \to \int \frac{\cos t}{t} dt \sim \ln(t) - \frac{t^2}{2.2!} + \frac{t^4}{4.4!} - \frac{t^6}{6.6!} + - \ldots$$

For the endpoint $t = 1$ we have:

$$-\frac{1}{4} + \frac{1}{4.4!} + \frac{1}{6.6!} - + \ldots = C \approx -0.23981.$$

Thus, we will have:

$$\to \int_x^1 \frac{\cos t}{t} dt \sim -\ln(x) - C + \frac{x^2}{2.2!} - \frac{x^4}{4.4!} + \frac{x^6}{6.6!} - + \ldots$$

Note that, integrating an asymptotic series term by term gives an asymptotic series, but, this does not always work for differentiation.

2.2 What Does Major (Dominant) Contribution Mean?

To obtain the asymptotic behaviour of integrals, we are concerned with the terms in the integrand that give the major contribution to the value of the integral. To clarify further, consider the following integral:

$$I(x) = \int_1^x \frac{e^t}{t} dt$$

Here, the integral is dominated by e^x, that is, in the integrand the term e^t is dominant to other terms. In fact, it is the exponential function that grows most rapidly in general. In cases where we have terms like $a^{u(t)}e^t$ as the integrand (so that $a^{u(x)}$ grows more rapidly), the e^x term is no longer dominant. However, we may write $a^{u(t)}$ as $e^{\ln a^{u(t)}}$, so, the integrand would be $e^{t+\ln a^{u(t)}}$. Hence, for integrals involving an exponential function, the only significant part of the integral is the neighbourhood where the exponent is at its maximum.

For our example, the exponential term reaches its maximum when $t = x$. The lower bound at $t = 1$ is irrelevant, since the constants are subdominant to the series. Hence, we can write:

$$I(x) = \int_1^x \frac{e^t}{t} dt = \int_1^{x-\epsilon} \frac{e^t}{t} dt + \int_{x-\epsilon}^x \frac{e^t}{t} dt \to \int_1^x \frac{e^t}{t} dt \sim \int_{x-\epsilon}^x \frac{e^t}{t} dt$$

With the linear transformation: $t = x - s$, we will have:

$$\to \int_0^\epsilon \frac{e^{x-s}}{x-s} ds = e^x \int_0^\epsilon \frac{e^{-s}}{x-s} ds$$

Now, by using the Maclaurin series of the term $\frac{1}{x-s}$,

$$\frac{1}{x-s} = \frac{1}{x} + \frac{s}{x^2} + \frac{s^2}{x^3} + \ldots,$$

and knowing that $\int_0^\infty s^n e^{-s} ds = n!$, we will have:

$$I(x) = \int_1^x \frac{e^t}{t} dt \sim e^x \left(\frac{1}{x} + \frac{1}{x^2} + \ldots \right)$$

This is a rigorous method of finding the asymptotic behaviour of an integral. It is actually fundamental to the Laplace method, which we shall talk about in detail later.

2.3 Repeated Integration by Parts

$$I(x) = \int_a^b udv = uv|_a^b - \int_a^b vdu$$

Integration by parts is one of the methods used for asymptotic analysis of integrals. It analyses the integrand near the boundaries, so this method is useful for asymptotic analysis when the major contribution is near the boundaries of $[a, b]$.

2.4 Laplace Type integrals

Integrals of the form

$$I(k) = \int_a^b f(t) e^{-k\phi(t)} dt$$

are said to be in the general form of Laplace integrals. When $\phi(t) = t$, the integral is simply the ordinary type of Laplace integral (i.e. Laplace transform).

There are three different ways to asymptotically analyse this type of integral.

1. Integration by parts:

   As we have seen before, integration by parts is a useful method when the major contributions are near the boundaries. Here, the exponential function is dominant. So, the major contribution to this integral is near the maximum of the exponent, as $k \to \infty$. Because of the minus sign in the exponent, we have to find the asymptotic behavior of the integral near the minimum of $\phi(t)$. If $\phi(t)$ is a monotonic function, the minimum occurs at one of the boundaries. Another point that we should pay attention to is that the function $f(t)$ should be sufficiently smooth at the boundary that minimises $\phi(t)$, because we need the value of this function and some of its derivatives at this boundary point. Hence, the integration by parts works in this case. The truncated integration by parts yields

   $$I(k) \sim \sum_{n=0}^{N} \frac{(-1)^n}{(ik)^{n+1}} \left[f^{(n)}(b)e^{ikb} - f^{(n)}(a)e^{ika} \right], \quad k \to \infty$$

   In the above formula, the major contribution happens at either a or b, hence we only need include one of these terms.

2. Watson's Lemma:

   If $f(t)$ is not sufficiently smooth at the boundary of interest, integration by parts no longer works. In this case, Watson suggests that we use the asymptotic power series expansion of $f(t)$. Consider the minimum of $\phi(t)$ to be at $t = a$, and a singularity at this point for f. Thus, the major contribution would be:

   $$I(k) = \int_a^{a+\epsilon} f(t)e^{-k\phi(t)}dt$$

   Now, we expand $f(t)$ near a:

   $$f(t) \sim \sum_{n=0}^{\infty} a_n t^{\alpha+\beta n}, t \to a$$

   Where, $\alpha > -1, \beta > 0$, and ϵ is the radius of validity of the series for $f(t)$. If the end point b is finite, then we should have $|f(t)| \leq A$, where A is a constant, for $t > 0$. When b is infinite, $|f(t)| \leq Me^{Ct}$ for $t > 0$, M constant.

3. Laplace Method:

   If $\phi(t)$ is not monotonic, the major contribution no longer happens at the boundaries. Thus, we have to find the asymptotic behaviour of the integral near the minimum of ϕ, which is in the interior of the interval. Let $c \in (a, b)$ be the value that minimises ϕ, so $\phi'(c) = 0$ and $\phi''(c) > 0$. The integral becomes

   $$\int_{c-\epsilon}^{c+\epsilon} f(c) \exp\left\{ k \left(\phi(c) + \frac{(t-c)^2}{2} \phi\prime\prime(c) \right) \right\} dt.$$

Here, we used the Taylor expansion for ϕ near c. Using the change of variable, $\tau = \sqrt{\frac{k\phi''}{2}}(t-c)$, we have

$$I(k) \sim e^{-k\phi(c)} f(c) \int_{c-\epsilon}^{c+\epsilon} \exp\left\{-k\frac{(t-c)^2}{2}\phi\prime\prime(c)\right\} dt = \frac{e^{-k\phi(c)} f(c)}{\sqrt{k\phi''/2}} \int_{+\epsilon\sqrt{\frac{k\phi''}{2}}}^{-\epsilon\sqrt{\frac{k\phi''}{2}}} e^{-\tau^2} d\tau$$

Now, this integral is familiar; as $k \to \infty$ the integral is $\sqrt{\pi}$. So,

$$I(k) = \int_a^{a+\epsilon} f(t) e^{-k\phi(t)} dt \sim e^{-k\phi(c)} f(c) \sqrt{\frac{2\pi}{k\phi\prime\prime(c)}}$$

3 The Method of Steepest Descent

Now, the goal is to find the asymptotic behaviour of the following integral:

$$I(k) = \int_C f(z) e^{k\phi(z)} dz, \quad k \to \infty$$

where, $f(z)$ and $\phi(z)$ are complex analytic functions. The general idea is, by using the analytic character of the functions, to reform the contour C to another contour C' (Cauchy theorem) in which the imaginary part of the exponent is constant. Hence, the integral would have the form of a Laplace integral, then we use the rigorous Laplace method. Write

$$\phi(z) = u(x,y) + iv(x,y), \; z = x + iy$$

Supposing, $\text{Im}\{\phi\} = v$ is constant in contour C', then,

$$I(k) = \int_C f(z) e^{k\phi(z)} dz = e^{ikv} \int_{C'} f(z) e^{ku} dz$$

In order to choose the contour C', we usually choose the path of steepest descent passing through z_0 in which $\phi'(z_0) = 0$ (saddle point). We later will show how to find this path. After choosing the path, we have to find where the major contributions come from. The dominant contributions will happen at critical points i.e. where $\phi'(z) = 0$, singular points, and end points. Hence, we have to analyse the integral of the Laplace form at these points.

3.1 Steepest Path

Let $\phi(z) = u(x,y) + iv(x,y)$, with $z = x + iy$; then the paths passing through the point $z = z_0$ (where $v(x,y) = v(x_0,y_0)$) are the paths where the imaginary part of ϕ is constant. The direction of descent is a direction away from z_0 in which u is decreasing; when this decrease is maximal, the path is called the path of steepest descent. Similarly, the direction of ascent is a direction away from z_0 in which u is increasing; when this increase is maximal, the path is called the path of steepest ascent. From calculus we know that if $u(z_0)$ and $\nabla u \neq 0$, then $-\nabla u$ is the steepest ath decreasing away from $u(z_0)$. It is easily shown that the curves defined by $v(x,y) = v(x_0,y_0)$ are curves of steepest descent or ascent. If we consider $\delta\phi$ as the change of the function ϕ from the point z_0, then

$$\delta\phi = \phi(z) - \phi(z_0) = \delta u + i\delta v \to |\delta u| \leq |\delta\phi|.$$

Equality occurs when δu is maximal, so, $\delta v = 0 \to v(x,y) = v(x_0,y_0)$. This, in fact, shows why we need the steepest path.

3.2 The Saddle point

We say the point $z = z_0$ is saddle point of order N for the complex function ϕ if:

$$\left.\frac{d^m\phi}{dz^m}\right|_{z=z_0} = 0, \quad m = 1, 2, ..., N, \qquad \frac{d^{N+1}\phi}{dz^{N+1}} \neq 0$$

3.3 How to Find Steepest Paths

If z_0 is a saddle point of order N, then we can write:

$$\phi(z) - \phi(z_0) \sim \frac{(z-z_0)^{N+1}}{(N+1)!}\left.\frac{d^{N+1}\phi}{dz^{N+1}}\right|_{z=z_0}$$

Taking $\left.\frac{d^{N+1}\phi}{dz^{N+1}}\right|_{z=z_0} = ae^{i\alpha}$ and $z - z_0 = \rho e^{i\theta}$, then

$$\phi(z) - \phi(z_0) \sim \frac{\rho^{N+1}e^{i(N+1)\theta}}{(N+1)!} \times ae^{i(N+1)\alpha} = [\cos(\alpha + (N+1)\theta) + i\sin(\alpha + (N+1)\theta)] \times \frac{\rho^{N+1}a}{(N+1)!}$$

Steepest direction:
$\mathrm{Im}\{\phi(z) - \phi(z_0)\} = 0 \;\rightarrow\; \sin(\alpha + (N+1)\theta) \rightarrow\; \alpha + (N+1)\theta = m\pi \rightarrow \theta = -\frac{\alpha}{N+1} + m\frac{\pi}{N+1},\; m = 0, 1, ..., N$

Steepest descent directions: $Re\{\phi(z) - \phi(z_0)\} < 0 \rightarrow \cos(\alpha + (N+1)\theta) < 0 \rightarrow \theta_{sd} = -\frac{\alpha}{N+1} + (2m+1)\frac{\pi}{N+1}$.

Steepest ascent directions: $Re\{\phi(z) - \phi(z_0)\} < 0 \rightarrow \cos(\alpha + (N+1)\theta) < 0 \rightarrow \theta_{sa} = -\frac{\alpha}{N+1} + (2m)\frac{\pi}{N+1}$.

3.4 Laplace Method for Complex Contours

After we have found the steepest descent path, we are faced with a Laplace type integral. As before, for evaluating the asymptotic behaviour of this type of integral, we should consider the points at which the integrand has the major contribution i.e. the critical points. Depending on the type of integrand or critical points , we can then use integration by parts, Watson's lemma, or Laplace method to find the asymptotic behaviour of the integral. As an example, we want to employ the Laplace method for an integral with one saddle point in the portion of path that gives the major contribution. Assuming that f is of order $(z-z_0)^{\beta-1}$ as $z \rightarrow z_0$ and z_0 is a saddle point of order N for ϕ, we can write:

$$f \sim f_0(z-z_0)^{\beta-1},\; Re\{\beta\} > 0 \qquad\qquad \phi(z) - \phi(z_0) \sim \frac{(z-z_0)^{N+1}}{(N+1)!}\phi^{(N+1)}(z_0),$$

Now, we deform C to C', a path of steepest descent going away from the saddle point. We can make the change of variable $-t = \phi(z) - \phi(z_0),\; dt = -\phi'(z)dz$, where t is real and positive. Then,

$$-t = \frac{(z-z_0)^{N+1}}{(N+1)!}\phi^{(N+1)}(z_0).$$

So, by substituting into the main integral, we have:

$$I(k) \sim e^{ik\phi(z_0)}\int_0^\infty \left(-\frac{f(z)}{\phi'(z)}\right)e^{-kt}dt$$

Here, we just consider one portion of the path C' that gives the major contribution, and we change the upper limit of integration to ∞ because the major contribution happens near the origin, according to Watson's lemma.

Now, we find the leading behaviour of the term $-\frac{f(z)}{\phi'(z)}$:

$$\frac{f(z)}{\phi'(z)} \sim \frac{f_0(z-z_0)^{\beta-1}}{(z-z_0)^N \phi^{N+1}(z_0)/N!} = N! \frac{f_0(z-z_0)^{\beta-N-1}}{\phi^{N+1}(z_0)} = N! \frac{f_0(\rho e^{i\theta})^{\beta-N-1}}{(ae^{i\alpha})^{N+1}}$$

Plugging this term into the above integral (along with some algebra) gives us:

$$I(k) \sim \frac{f_0[(N+1)!]^{\beta/(N+1)} e^{i\beta\theta}}{N+1} \frac{e^{k\phi(z_0)}\Gamma(\frac{\beta}{N+1})}{[ka]^{\beta/(N+1)}}$$

Example: Find the asymptotic evaluation of the integral $\int_0^1 \log(t) e^{ikt} dt$ as $k \to \infty$.

Here we have: $\phi(z) = iz = i(x+iy) = -y + ix$. So, $\phi'(z) = i$, and we have no saddle point. Actually the dominant contribution is the endpoint, and the steepest paths (i.e. $\mathrm{Im}\{\phi\} = const$) are given by $x = const$. Thus, if $y > 0$ or $y < 0$, we are on steepest descent or steepest ascent path, respectively. We have to choose the path of steepest descent so that it passes through end points. Therefore $x = 0$, $y > 0$ and $x = 1$, $y > 0$ are the paths of steepest descent going through the endpoints. Notice that $\mathrm{Im}\{\phi(0)\} = \mathrm{Im}\{\phi(1)\}$; so there is no continuous contour joining $t = 0$ and $t = 1$ on which $\mathrm{Im}\{\phi\}$ is constant. Hence, we have to connect these two portions of the steepest descent path. We connect these two contours by another contour at infinity which, by Jordan's Lemma, will not give any major contribution (the integral over this contour ~ 0). Then, using Cauchy's theorem, we deform the contour in $[0, 1]$. Note that, since $t = 0$ is an integrable singularity, we shall allow the contour to pass through the origin. Thus, we can write:

$$I(k) = \int_{C_1 \cup C_2 \cup C_3} \log z e^{ikz} dz = i \int_0^R \log(ir) e^{-kr} dr + \int_0^1 \log(x+iR) e^{ikx-kR} dx - ie^{ik} \int_0^R \log(1+ir) e^{-kr} dr$$

where, according to Figure 1, $R \to \infty$. Hence the second integral vanishes and we get

$$I(k) = i \int_0^\infty \log(ir) e^{-kr} dr - ie^{ik} \int_0^\infty \log(1+ir) e^{-kr} dr$$

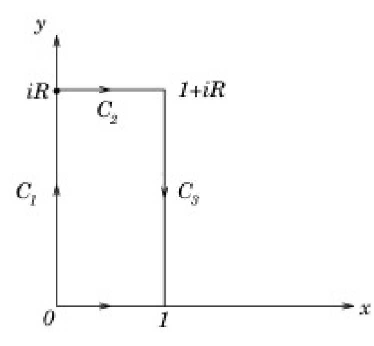


Figure 1: Contour of integration

Letting $s = kr$, we get

$$I_1(k) = \frac{i}{k} \int_0^\infty \left(\log\left(\frac{i}{k}\right) + \log(s) \right) e^{-s} ds$$

Using the results from the familiar integral $\int_0^\infty \log(s) e^{-s} ds$,

$$I_1(k) = -\frac{i \log k}{k} - \frac{(i\gamma + \frac{\pi}{2})}{k}$$

where, $\gamma = 0.577216...$ is the Euler constant. For the second integral, we use Watson's lemma. We know that

$$\log(1 + ir) = -\sum_{n=1}^\infty \frac{(-ir)^n}{n}.$$

So, as $k \to \infty$, the complete expansion for the second integral is $ie^{ik} \sum\limits_{n=1}^\infty \frac{(-i)^n (n-1)!}{k^{n+1}}$. Finally, we have

$$I(k) \sim -\frac{i \log k}{k} - \frac{(i\gamma + \pi/2)}{k} + ie^{ik} \sum_{n=1}^\infty \frac{(-i)^n (n-1)!}{k^{n+1}}, \quad k \to \infty$$

4 Small-amplitude limit of the Korteweg-deVries equation

The Korteweg-deVries (KdV) equation is a model for waves on shallow water surfaces. Its non-dimensional form is given by

$$V_t + 6VV_x + V_{xxx} = 0 \tag{4.1}$$

Here we will be investigating the small-amplitude limit of this equation. This can be obtained by setting $V = \epsilon u$ in the above equation and taking the limit as $\epsilon \to 0$. We will thus be working with the equation

$$u_t + u_{xxx} = 0, \quad -\infty < x < \infty, \quad t > 0. \tag{4.2}$$

with initial values

$$u(x, 0) = u_0(x). \tag{4.3}$$

Since there are no external energy sources we may assume that $u \to 0$ sufficiently rapidly as $|x| \to \infty$.

4.1 Fourier Transform solution

Equation (4.2) can be solved using Fourier transforms. Taking $\tilde{u}(k, t)$ to be the Fourier Transform of u in x, we have

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{u}(k, t) e^{ikx} dk. \tag{4.4}$$

Upon substitution of this form of u into the PDE (4.2) we obtain

$$\frac{\partial}{\partial t} \tilde{u}(k, t) + (ik)^3 \tilde{u}(k, t) = 0 \tag{4.5}$$

which can be readily solved to give

$$\tilde{u}(k, t) = \tilde{u}(k, 0) e^{ik^3 t}. \tag{4.6}$$

Using the initial condition we have

$$u_0(x) = u(x, 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{u}(k, 0) e^{ikx} dk \tag{4.7}$$

and so we see

$$\tilde{u}(k, 0) = \tilde{u}_0(k)$$

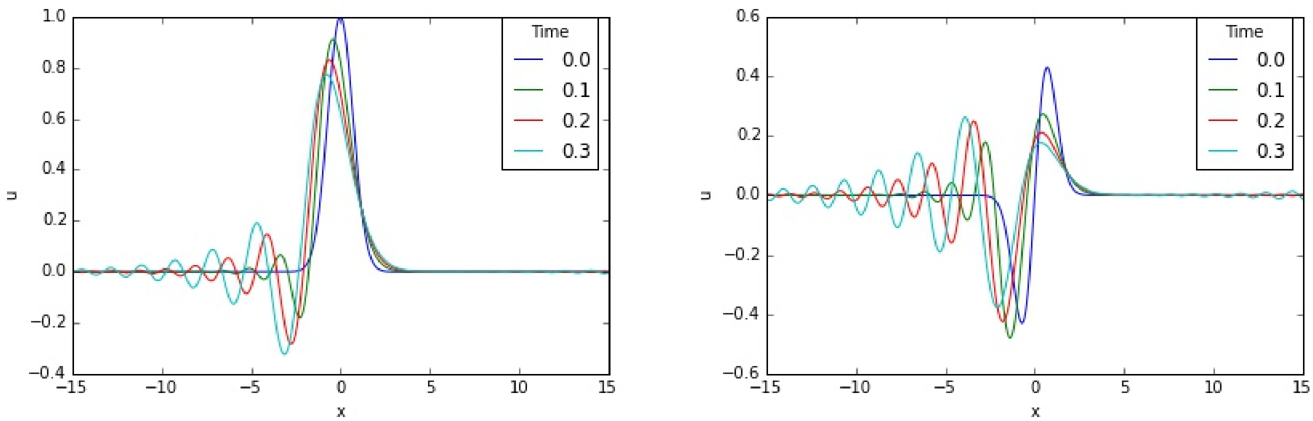
where $\tilde{u}_0(k)$ is the Fourier transform of $u_0(x)$ in x. Thus our solution to (4.2) in integral form is given by

$$\begin{aligned} u(x, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{u}_0(k) e^{ikx + ik^3 t} dk \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{u}_0(k) e^{t\phi(k)} dk, \qquad \phi(k) = i\left(k^3 + \frac{kx}{t}\right) \end{aligned} \tag{4.8}$$

This now has the required form for use of the method of steepest descent, assuming that $\tilde{u}_0(k)$ can be continued analytically off the real k axis.

4.2 Solution curves for various initial conditions

Before we embark on the method of steepest descents to find the asymptotic behaviour as $t \to \infty$, let us observe some solution curves for different initial conditions. It is interesting to see the way in which the energy dissipates for various initial waves. These were constructed using Fast Fourier Transform methods on equation (4.8), using the Python programming language.



(a) Initial wave $u_0(x) = e^{-x^2}$

(b) Initial wave $u_0(x) = xe^{-x^2}$

Figure 2: Solution curves for fixed t.

4.3 Applying the Method of Steepest Descent

Extending (4.8) onto the complex plane, we have the integral

$$I(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{u}_0(z) e^{t\phi(z)} dz \tag{4.9}$$

where

$$\phi(z) = i\left(z^3 + \frac{zx}{t}\right) \tag{4.10}$$

First, we look for the saddle points z_0 of ϕ. These are points that satisfy $\phi'(z_0) = 0$. Differentiating, we have

$$\phi'(z) = i\left(3z^2 + \frac{x}{t}\right). \tag{4.11}$$

The nature of the saddle points will depend upon the sign of x/t, so we will consider the three cases

(a) $\frac{x}{t} < 0$. This gives two real saddle points $z_\pm = \pm\left|\frac{x}{3t}\right|^{1/2}$.

(b) $\frac{x}{t} > 0$. This gives two pure-imaginary saddle points $z_\pm = \pm i\left(\frac{x}{3t}\right)^{1/2}$.

(c) $\frac{x}{t} \to 0$. Now we have the single saddle point $z = 0$ of higher order which requires a different approach as we shall see.

Recall the equation for the direction of steepest descent θ at a saddle point z_0 of order $n-1$.

$$\theta = -\frac{\alpha}{n} + (2m+1)\frac{\pi}{n}, \quad m = 0, 1, ..., n-1 \tag{4.12}$$

where

$$\left.\frac{d^n\phi}{dz^n}\right|_{z=z_0} = ae^{i\alpha}. \tag{4.13}$$

4.3.1 Case (a)

To compute the order of the real saddle points we must compute further derivatives of ϕ. We see that

$$\phi''(z_\pm) = \pm 6i\left|\frac{x}{3t}\right|^{\frac{1}{2}} = 6\left|\frac{x}{3t}\right|^{\frac{1}{2}} e^{\pm i\frac{\pi}{2}} \tag{4.14}$$

is non-zero and so these are order one saddle points. Thus our equation for the direction of steepest descent is

$$\theta = -\frac{\alpha}{2} + (2m+1)\frac{\pi}{2}, \quad m = 0, 1 \tag{4.15}$$

where α is seen to be $\pm\frac{\pi}{2}$ from (4.14).

So at z_+ we have steepest descent directions $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$. At z_- we have steepest descent directions $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$. This is illustrated in Figure 3.

The next stage is to deform the contour $(-\infty < z < \infty)$ onto the steepest descent contour going through both of the saddle points. From the earlier theory, we know that the contours of steepest descent are given by $\text{Im}\phi(z) = const$. In our case then, we have

$$\text{Im}\phi(z) = \text{Im}\phi(z_\pm). \tag{4.16}$$

This gives a hyperbola as the steepest descent contour. Consider $z = |z|e^{i\varphi}$, $\varphi = \pi/4, 3\pi/4$, for large $|z|$. Then $\phi(z)$ will be dominated by the cubic term and we will have

$$\phi(z) \approx iz^3 = i|z|^3 e^{3i\varphi} = i|z|^3(\pm 1 + i) = |z|^3(-1 \pm i). \tag{4.17}$$

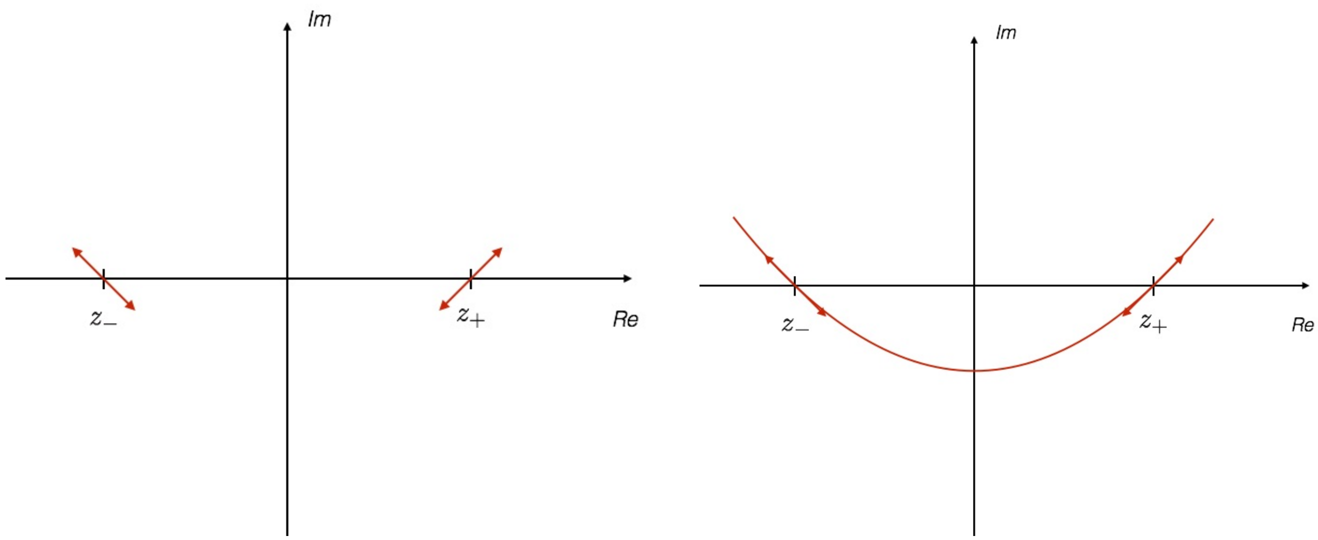
Then clearly for large $|z|$, the exponential $e^{t\phi(z)}$ will decay rapidly. Thus we can deform our original contour onto the steepest descent contour going through both of the saddle points without any problem.

To find the asymptotic behaviour of $I(t)$, it remains to sum up the contributions to the integral from each saddle point. The equation for the contributions is outlined in the theory earlier and in this case is given by

$$I_{z_0}(t) \sim \tilde{u_0}(z_0)e^{i\theta}\frac{e^{t\phi(z_0)}\Gamma(\frac{1}{2})}{(t|\phi''(z_0)|)^{\frac{1}{2}}} \tag{4.18}$$

Substituting values in, we obtain the two contributions

$$\frac{\tilde{u}_0\left(\left|\frac{x}{3t}\right|^{\frac{1}{2}}\right)e^{-2it\left|\frac{x}{3t}\right|^{\frac{3}{2}}+\frac{i\pi}{4}}}{2\sqrt{\pi t}\left|\frac{x}{3t}\right|^{\frac{1}{4}}} \quad \text{and} \quad \frac{\tilde{u}_0\left(-\left|\frac{x}{3t}\right|^{\frac{1}{2}}\right)e^{2it\left|\frac{x}{3t}\right|^{\frac{3}{2}}-\frac{i\pi}{4}}}{2\sqrt{\pi t}\left|\frac{x}{3t}\right|^{\frac{1}{4}}}. \tag{4.19}$$



(a) Steepest descent directions (b) Deformation of the contour

Figure 3: Saddle point illustrations for the case where $\frac{x}{t} < 0$.

These contributions form a conjugate pair. Write

$$\tilde{u}_0\left(\left|\frac{x}{3t}\right|^{\frac{1}{2}}\right) = \rho\left(\left|\frac{x}{t}\right|\right) e^{i\psi\left(\frac{x}{t}\right)} \tag{4.20}$$

then adding the contributions gives

$$u(x,t) \sim \frac{\rho\left(\frac{x}{t}\right)}{\sqrt{\pi t}\left|\frac{x}{3t}\right|^{\frac{1}{4}}} \cos\left(2t\left|\frac{x}{3t}\right|^{\frac{3}{2}} - \frac{\pi}{4} - \psi\left(\frac{x}{t}\right)\right) \quad \text{as} \quad t \to \infty, \quad \frac{x}{t} < 0. \tag{4.21}$$

4.3.2 Case (b)

We now consider the case where $\frac{x}{t} > 0$. This gives two complex roots for $\phi'(z) = 0$ given by

$$z_{\pm} = i\sqrt{\frac{x}{3t}}. \tag{4.22}$$

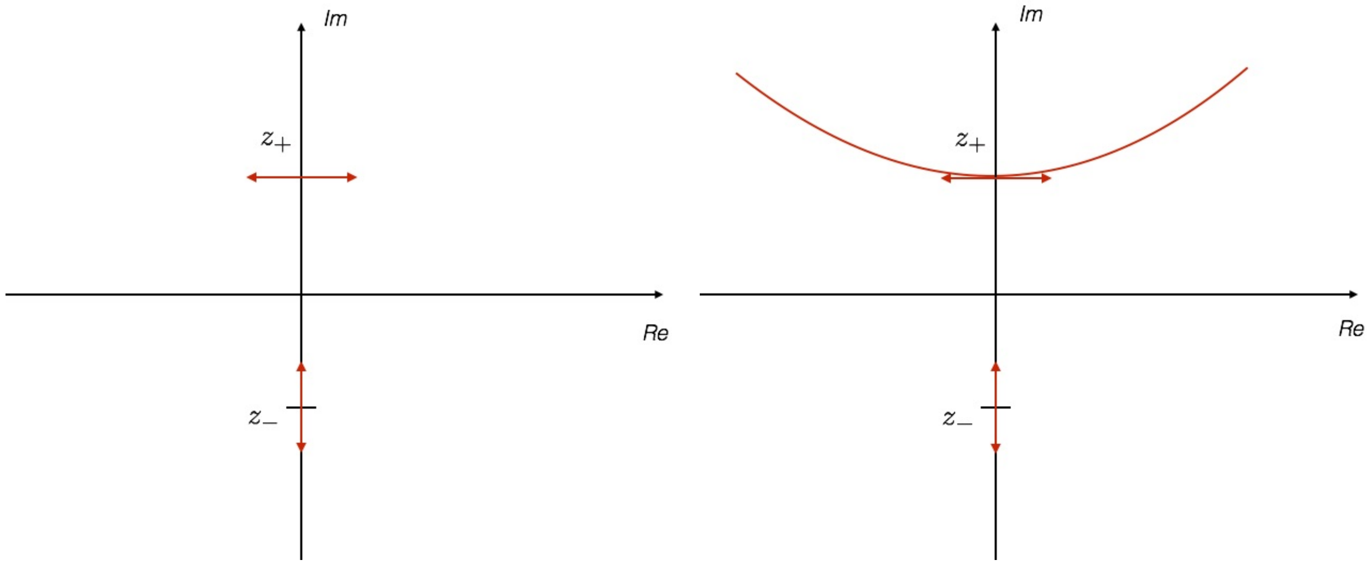
We then compute

$$\phi''(z_{\pm}) = \mp 6\sqrt{\frac{x}{3t}} \tag{4.23}$$

which in the form of (4.13) has $a = 6\sqrt{\frac{x}{3t}}$ and $\alpha = \pi, 0$. Note also that since $\phi''(z_{\pm}) \neq 0$, these saddle points are of first order. The direction of steepest descent at these saddle points can then be computed using (4.12). For z_+ we get $\theta = 0, \pi$. For z_- we get $\theta = \pi/2, 3\pi/2$. This is illustrated in Figure 4.

Writing z as $z = z_R + iz_I$ for $z_R, z_I \in \mathbb{R}$, we find that

$$\text{Im}\phi(z) = z_R\left(z_R^2 - 3z_I^2 + \frac{x}{t}\right). \tag{4.24}$$



(a) Steepest descent directions (b) Deformation of the contour

Figure 4: Saddle point illustrations for the case where $\frac{x}{t} > 0$

One can also calculate

$$\phi(z_\pm) = \mp 2 \left(\frac{x}{3t}\right)^{\frac{3}{2}} \tag{4.25}$$

which we see is real. The contours of steepest descent that go through the saddle points $z_\pm$ satify

$$\text{Im}\phi(z) = \text{Im}\phi(z_\pm) \tag{4.26}$$

$$\Rightarrow \quad z_R \left(z_R^2 - 3z_I^2 + \frac{x}{t}\right) = 0. \tag{4.27}$$

At z_+, it is the hyperbola $z_R^2 - 3z_I^2 + x/t = 0$ that is consistent with the direction of steepest descent. We can deform the original contour onto the upper half plane part of this hyperbola, since the exponential $e^{t\phi}$ decays exponentially in this region. The exponential grows in the lower half plane, so we cannot use the path going through z_-.
Using equation (4.18) to find the contribution of z_+ to the integral we get

$$u(x,t) \sim \frac{\tilde{u}_0 \left(i \left(\frac{x}{3t}\right)^{\frac{1}{2}}\right) e^{-2t\left(\frac{x}{3t}\right)^{\frac{3}{2}}}}{2\sqrt{\pi t}\left(\frac{3x}{t}\right)^{\frac{1}{4}}}, \quad t \to \infty, \quad \frac{x}{t} > 0. \tag{4.28}$$

4.3.3 Case (c)

In the case where $\frac{x}{t} \to 0$ our asymptotic expressions for u derived thus far break down due to the occurrence of $\frac{x}{t}$ in the denominator. The way around this is to introduce the similarity variables

$$\xi = k(3t)^{\frac{1}{2}}, \quad \eta = \frac{x}{(3t)^{\frac{1}{2}}}. \tag{4.29}$$

The expression for u in (4.8) becomes

$$u(x,t) = \frac{1}{2\pi(3t)^{\frac{1}{3}}} \int_{-\infty}^{\infty} \tilde{u}_0 \left(\frac{\xi}{(3t)^{\frac{1}{3}}} \right) e^{i\xi\eta + \frac{i\xi^3}{3}} d\xi \tag{4.30}$$

For large t we may Taylor expand $\tilde{u}_0$ near $\xi = 0$ to give the asymptotic expression

$$u(x,t) \sim \frac{1}{2\pi(3t)^{\frac{1}{3}}} \int_{-\infty}^{\infty} e^{i\xi\eta + \frac{i\xi^3}{3}} \left(\tilde{u}_0(0) + \frac{\xi}{(3t)^{\frac{1}{3}}} \tilde{u}_0'(0) + \dots \right) d\xi \quad \text{as} \quad t \to \infty. \tag{4.31}$$

Recall the Airy function $Ai(\eta)$ is the solution to the differential equation

$$A_{\eta\eta} - \eta A = 0 \tag{4.32}$$

that satisfies $A(\eta) \to 0$ as $\eta \to \infty$. One can verify that it can be written in integral form as

$$Ai(\eta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi\eta + \frac{i\xi^3}{3}} d\xi \tag{4.33}$$

which is what we observe in (4.31). Note that each derivative of $Ai(\eta)$ includes an extra factor of $i\xi$, which accommodates the higher order terms in (4.31). Thus we can write

$$u(x,t) \sim (3t)^{-\frac{1}{3}} \tilde{u}_0(0) Ai(\eta) - i(3t)^{-\frac{2}{3}} \tilde{u}_0'(0) Ai'(\eta) \quad \text{as} \quad t \to \infty. \tag{4.34}$$

4.3.4 Asymptotic Matching of solutions

We should check that our asymptotic solution for u in the region where $\frac{x}{t} \to 0$ (the inner region) matches smoothly with the solution in the two outer regions. For this, we will need the asymptotic behaviour of the Airy function, given by

$$Ai(\eta) \sim \frac{1}{2\sqrt{\pi}\eta^{\frac{1}{4}}} e^{-\frac{2}{3}\eta^{\frac{3}{2}}} \quad \text{as} \quad \eta \to \infty \tag{4.35}$$

$$Ai(\eta) \sim \frac{1}{\sqrt{\pi}|\eta|^{\frac{1}{4}}} \cos\left(\frac{2}{3}|\eta|^{\frac{3}{2}} - \frac{\pi}{4} \right) \quad \text{as} \quad \eta \to -\infty. \tag{4.36}$$

The leading order solution for u in the inner region is given by

$$u(x,t) \sim (3t)^{-\frac{1}{3}} \tilde{u}_0(0) Ai(\eta) \quad \text{as} \quad t \to \infty \tag{4.37}$$

Taking the limit as $\eta \to \infty$ gives

$$u(x,t) \sim \frac{(3t)^{-\frac{1}{3}} \tilde{u}_0(0)}{2\sqrt{\pi}\eta^{\frac{1}{4}}} e^{-\frac{2}{3}\eta^{\frac{3}{2}}} \quad \text{as} \quad t \to \infty \tag{4.38}$$

$$= \frac{\tilde{u}_0(0)}{2\sqrt{\pi t}} \left(\frac{t}{3x} \right)^{\frac{1}{4}} e^{-2t\left(\frac{x}{3t}\right)^{3/2}} \tag{4.39}$$

which is in accordance with the right outer solution (4.28) upon Taylor expanding $\tilde{u}_0$ and keeping the leading order term.

Taking the limit as $\eta \to -\infty$ gives

$$u(x,t) \sim \frac{(3t)^{-\frac{1}{3}}\tilde{u}_0(0)}{\sqrt{\pi}|\eta|^{\frac{1}{4}}}\cos\left(\frac{2}{3}|\eta|^{\frac{3}{2}} - \frac{\pi}{4}\right) \quad \text{as} \quad t \to \infty \tag{4.40}$$

$$= \frac{\tilde{u}_0(0)}{\sqrt{\pi t}\left|\frac{x}{3t}\right|^{\frac{1}{4}}}\cos\left(2t\left|\frac{x}{3t}\right|^{\frac{3}{2}} - \frac{\pi}{4}\right) \tag{4.41}$$

which agrees nicely with the leading order solution in the left outer region (4.21).

5 Non-linear Steepest Descent and the KdV equation

Abstract

The method of nonlinear steepest descent was developed and applied to the canonical Korteweg-de Vries (KdV) and modified KdV (mKdV) equations by Deift and Zhou in [2, 3], and further expounded by Kamvassis in [5]. Grunert and Teschl then applied the method to the KdV equation in [4]; in this section of the presentation, we will closely follow Grunert's approach in [4] to find the asymptotic behaviour of the solution in the soliton region $x > \beta t$ for some positive constant $\beta > 0$. Since $q(-x,-t)$ is a solution of the KdV equation if and only if $q(x,t)$ is, we only investigate the behaviour of the solution for large positive t.

5.1 Background

First we gather the necessary information from Scattering Theory. As outlined in [4, 6, 11], we start with KdV equation, given as

$$\partial_t q - 6q\partial_x q + \partial_x^3 q = 0$$

Consider the connection between the KdV equation and the Schrödinger operator $H(t) = -\partial_x^2 + q$ defined on H^2 (order-2 Sobolev space), where the potential $q = q(x,t)$ is a solution of the KdV equation. The corresponding eigenvalue problem is given by

$$-\frac{\partial^2 \Phi}{\partial x^2} + q(x,*)\Phi = \lambda^2 \Phi, \;\; \lambda = a + ib \tag{5.1}$$

with the assumption that the solution decays quickly ($\|(1+|x|)q(x,*)\|_{L^1} < \infty$). Whenever $b > 0$, there exists a unique pair of (eigenvalue-dependent) analytic solutions $\psi_\pm(\lambda,x,*)$ called *Jost solutions*, implicitly defined by

$$\psi_\pm(\lambda,x,*) = e^{\pm i\lambda x} \pm \frac{1}{\lambda}\int_x^{\pm\infty} \sin(\lambda(x-y))q(y,t)\psi_\pm(\lambda,x,*)\,dy$$

and which behave asymptotically like $e^{\pm i\lambda x}$ as $x \to +\infty$ [6]. If $\lambda \in \mathbb{R}$, then $\overline{\psi_\pm}$ is also a pair of solutions of (5.1), and is linearly independent to $\psi_\pm$, since the Wronskian determinant of the pair (taking functions with corresponding signs) is $2i\mathrm{Re}(\lambda)$. We then have the relation

$$\psi_- = \underbrace{\frac{1}{2i\mathrm{Re}(\lambda)}\left[\psi_+,\psi_-\right]}_{f(a)} \cdot \overline{\psi_+} + \underbrace{\frac{1}{2i\mathrm{Re}(\lambda)}\left[\psi_-,\overline{\psi_+}\right]}_{g(a)} \cdot \psi_+$$

Tanaka [11] interprets $f(a)$ (respectively, $g(a)$) as the boundary value of the analytic function $f(\lambda)$ (resp. $g(\lambda)$). Also, the roots of f are finite, simple and purely imaginary[1] (since $\psi_\pm(\lambda,x,*) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ at

[1] By the Spectral Theorem for Self-Adjoint Differential Operators, the spectrum of (5.1) splits into a continuous part ($\lambda > 0$) and a discrete part ($\lambda < 0$). Denote the discrete eigenvalues as $i\lambda_1, i\lambda_2, \ldots, i\lambda_n$, and their corresponding *bound states* $\psi_n(x,t) \in L^2(\mathbb{R})$. The reflection and transmission coefficients $R(\lambda)$ and $T(\lambda)$ are recovered from the continuous spectrum by looking at the asymptotic behavior of the eigenfunctions:

$$\psi \sim \begin{cases} T(\lambda)\, e^{i\lambda x} & x \to +\infty \\ R(\lambda)\, e^{-i\lambda x} & x \to -\infty \end{cases}$$

the roots of f); say that they are $\{i\lambda_j\}_{1\leq j\leq n}$.

The *reflection coefficient* is given by the function $R = \dfrac{g}{f}$, and the *norming constants* are defined as

$$\gamma_j(t) = \frac{1}{\|\psi_+(i\lambda_j,*,t)\|_2} = \frac{1}{\int_{\mathbb{R}} |\psi_+(i\lambda_j,*,t,)|^2\,dx}$$

The *scattering data* is the set $\{R(a),\lambda_j,\gamma_j(*)\}_j$, with its time evolution given by the relations[2]

$$R(k,t) = R(\lambda_j)e^{8ik^3t} \qquad \gamma_j(t) = \gamma_j e^{4\lambda_j^3 t} \qquad \gamma_j = \gamma_j(0),\ R(\lambda_j) = R(\lambda_j,0)$$

We also have the relation

$$T(k)\psi_\pm(k,x,t) = \overline{\psi_\pm(k,x,t)} + R_\pm(k,t)\psi_\pm(k,x,t)$$

where $R_\pm$ denotes the left and right reflection coefficients. The asymptotic behaviour of the Jost solutions with respect to k is given in [2, 4] as

$$\psi_\pm \sim \left(1 + Q_\pm(x,t)\frac{1}{2ik} + \mathcal{O}\left(\frac{1}{k^2}\right)\right)e^{\pm ikx} \text{ as } k\to+\infty,\ Q_\pm = \mp\int_x^{\pm\infty} q(y,t)\,dy \tag{5.2}$$

Lemma 1 (Properties of the Reflection and Transmission Coefficients [4])

1. *$T(k)\in C^1(\mathbb{C}\setminus\mathbb{R})$ is meromorphic on $\mathbb{H}^+$ with simple poles $\{i\lambda_j\}_{1\leq j\leq n}$*
2. $\operatorname{Res}[T(k),i\lambda_j] = \dfrac{\psi_+}{\psi_-}(i\lambda_j,x,*)\gamma_j^2(*)$
3. *$T(k)\overline{R_+(k,t)} = -\overline{T(k)}R_-(k,t)$ and $|T(k)|^2 + |R_\pm(k,t)|^2 = 1$*

5.2 Posing the Riemann-Hilbert Problem

We start with the following definitions.

Definition 1 (Sectionally Analytic Function)
A sectionally analytic function is a function Φ defined by a Cauchy-type integral

$$\Phi(z) = \frac{1}{2\pi i}\oint_\Gamma \frac{\phi(s)}{s-z}\,ds$$

where $\phi(s)$ is Hölder continuous on the compact curve Γ. Notice that $\Phi(z) \sim \left(\frac{1}{2\pi i}\oint_\Gamma \phi(s)\,ds\right)\frac{1}{z}$ as $|z|\to\infty$.

■

Definition 2 (Riemann–Hilbert problem [7, 8])
Let Γ be an oriented contour (that is, an oriented non-intersecting Hölder-continuous union of smooth curves) $\Gamma\subset\mathbb{C}$, and a smooth jump matrix $V:\Gamma\to \mathrm{M}_{2\times2}(\mathbb{C})$. The problem is that of finding a matrix-valued function $\Upsilon:\mathbb{C}\setminus\Gamma\to\mathrm{M}_{2\times2}(\mathbb{C})$ partitioning the plane into the interior and exterior regions D_+ and D_- respectively. Υ is (sectionally) analytic on $\mathbb{C}\setminus\Gamma = D_+\cup D_-$, with $\Upsilon_+(z) = \Upsilon_-(z)V(z)$ for all $z\in\Gamma$, and uniqueness of solution guaranteed by the normalisation $\Upsilon(\infty) = I_{2\times2}$. Here, $\Upsilon_\pm(z)$ denotes $\lim_{z\to\Gamma^\pm}\Upsilon(z)$, and $\Upsilon(\infty) = \lim_{|z|\to\infty}\Upsilon(z)$, where $\Upsilon_\pm$ must exist and be continuous. ■

[2] Tanaka [11] gives the evolution of the norming constant as $\gamma_j(x,t) = \gamma_j e^{8\lambda_j^3 t}$.

In the previous section, we dealt with the linear variant of the method of steepest decent. Here, we move to the nonlinear variant. The main difference between the methods is that the linear method calls for the deformation of the contour of integration onto the path of stationary phase, while the nonlinear method calls for deformation of the Riemann-Hilbert (hereafter written RH) problem (by conjugating the jump matrix) into a form that is exactly soluble [5, 9][3].

Here, we offer the first formulation of our KdV RH problem:

Vector Riemann-Hilbert Problem 1 ([4, 6])
Find a $\Upsilon(k) = \Upsilon(k,x,t)$ meromorphic on $\mathbb{C} \setminus \mathbb{R}$ with simple poles at $\pm i\lambda_j$ so that

1. $\Upsilon^+(k) = \Upsilon^-(k) \begin{bmatrix} 1-|R(k)|^2 & -\overline{R(k)}e^{-\phi(k,t)} \\ R(\lambda)e^{\phi(k,t)} & 1 \end{bmatrix}$ *(jump condition)*

2. $\operatorname{Res}[\Upsilon(k), i\lambda_j] = \gamma_j^2 e^{\phi(i\lambda_j,t)} \lim_{k\to i\lambda_j} \Upsilon(k) \begin{bmatrix} 0 & 0 \\ i & 0 \end{bmatrix}$ *(pole condition)*

3. $\Upsilon(-k) = \Upsilon(k) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ *(symmetry condition)*

4. $\Upsilon(+i\infty) = \begin{bmatrix} 1 & 1 \end{bmatrix}$ *(normalisation)*

where the phase of the solution is given by $\phi(k,t) = 2ikx + 8ik^3t$, with $\phi(k) = \dfrac{\phi(k,t)}{t}$. ■

It follows from simple calculations that

$$\Upsilon(k,x,t) = \begin{cases} \begin{pmatrix} T(k)\psi_-(k,x,t)e^{ikx} & \psi_+(k,x,t)e^{-ikx} \end{pmatrix} & k \in \mathbb{H}^+ \\ \\ \begin{pmatrix} \psi_+(-k,x,t)e^{ikx} & T(-k)\psi_-(-k,x,t)e^{-ikx} \end{pmatrix} & k \in \mathbb{H}^- \end{cases}$$

is a solution to the given RH problem, if the orientation of the contour matches that of $\mathbb{R}$. Now, (5.2) gives us that

$$T(k)\psi_+\psi_-(k,x,t) = 1 + \frac{q(x,t)}{2k^2} + \mathcal{O}\left(\frac{1}{k^4}\right) \text{ as } k \to +\infty \tag{5.3}$$

and we have the asymptotic behaviour of Υ in k

$$\Upsilon(k,x,t) = \begin{pmatrix} 1 & 1 \end{pmatrix} + Q_+(x,t)\frac{1}{2ik}\begin{pmatrix} -1 & 1 \end{pmatrix} + \mathcal{O}\left(\frac{1}{k^2}\right) \tag{5.4}$$

Grunert and Teschl now follow Deift and Zhou ([3]) in rewriting the pole condition of Problem 1 as a jump condition in order to transform the problem to a holomorphic RH problem, rather than working with the current meromorphic form. Deift and Zhou (later Kamvassis) do this by compatibly redefining the behaviour of Υ on a pairwise-disjoint collection of small ε-neighbourhoods around each pole (and their complex conjugates).

[3]Palais [9] demonstrates the existence of a conjugation matrix that achieves this (assuming that the RH problem depends only on one variable) by decomposing the original (k-regular) contour into a sum of two analytic functions, where one is a multiple of a fixed contour. By constructing a sequence of RH problems (in the dependent variable) converging in the operator norm on H^k, we can find the leading behaviour of the solution up to some small order.

This is achieved by applying a series of well-chosen conjugation matrices; interestingly, Kamvassis [5] states that each possible choice of redefinition corresponds to some analytic interpolating function of the norming constants. Grunert proposes the fix

$$\Upsilon(k) = \Upsilon(\pm k) \begin{bmatrix} 1 & 0 \\ -\dfrac{\gamma_j^2 e^{\phi(i\lambda_j,t)}}{\pm k \mp i\lambda_j} i & 1 \end{bmatrix} \text{ if } k \in B_\varepsilon(\pm i\lambda_j)$$

which preserves the symmetry condition of Problem 1. Rewritten as a jump condition on the ε-neighbourhood of the poles (thereby replacing the pole condition of Problem 1), this becomes

$$\Upsilon_+(k) = \Upsilon_-(k) \begin{bmatrix} 1 & 0 \\ -\dfrac{\gamma_j^2 e^{\psi(i\lambda_j,t)}}{\pm k \mp i\lambda_j} i & 1 \end{bmatrix} \text{ if } k \in B_\varepsilon(\pm i\lambda_j)$$

Here, $\partial B_\varepsilon(-i\lambda_j)$ and $\partial B_\varepsilon(i\lambda_j)$ have clockwise and counterclockwise orientations respectively.

Further, suppose the reflection coefficient of our RH problem is identically zero; this eliminates the jumps all along $\mathbb{R}$, and our symmetry condition demands that the solution be of form $\begin{pmatrix} f(k) & f(-k) \end{pmatrix}$. Given that the only possible singularity of $f(k)$ occurs at $i\lambda$, then f has the form $f(k) = A + B\frac{k}{k-i\lambda}$, with $A, B \in \mathbb{C}$. Since these constants are completely determined by the RH criteria, then we can write the explicit solution.

Lemma 2 (Initial Solution)
If the RH problem has a unique eigenvalue λ and an identically zero reflection coefficient, then we have the unique solution to Problem 1

$$m_0(k) = \begin{pmatrix} f(k) \\ f(-k) \end{pmatrix}^T, \text{ where } f(k) = \frac{2\lambda}{2\lambda + \gamma^2 e^{\phi(i\lambda,t)}} + \frac{\gamma^2 e^{\phi(i\lambda,t)}}{2\lambda + \gamma^2 e^{\phi(i\lambda,t)}} \frac{k + i\lambda}{k - i\lambda}$$

Comparison with the series expansion (5.4) *gives* $Q_+(x,t) = \dfrac{4\lambda\gamma^2 e^{\phi(i\lambda,t)}}{2\lambda + \gamma^2 e^{\phi(i\lambda,t)}}$.

■

5.3 Deforming Our Riemann-Hilbert Problem

In [2, 3, 4, 5], the vector RH problem is deformed such that all jumps are exponentially close to $\mathcal{I}_{2\times 2}$ away from its stationary points. We follow that process here, using the assumption that the right reflection coefficient $R(k)$ can be analytically continued to any neighbourhood of $\mathbb{R}$ (this follows from our assumption that the solution vanishes exponentially). Grunert goes on to remove this assumption in [4, section 6].

Lemma 3 ([4])
Recall the jump matrix $V(k) = \begin{bmatrix} 1 - |R(k)|^2 & -\overline{R(k)} e^{-\phi(k,t)} \\ R(\lambda) e^{\phi(k,t)} & 1 \end{bmatrix}$ *from Problem 1. Let $\tilde{\Sigma}$ and Σ be two contours in $\mathbb{C}$ such that $\tilde{\Sigma} \subseteq \Sigma$, with a sectionally analytic function $d : \mathbb{C} \setminus \tilde{\Sigma} \to \mathbb{C}$, and a matrix $D(k)$ of the form* $\begin{bmatrix} \frac{1}{d(k)} & 0 \\ 0 & d(k) \end{bmatrix}$. *If we have $\tilde{\Upsilon}(k) = \Upsilon(k) D(k)$, then the jump matrix of the deformed problem is given by*

$$\tilde{V}(k) = D_-^{-1}(k) V(k) D_+(k)$$

Further, if $d(-k) = \frac{1}{d(k)}$ and $d(+i\infty) = 1$, then the transformation satisfies both the symmetry and normalisation criteria of Problem 1.

■

Specifically, we have

$$\tilde{V}(k) = \overbrace{\underset{\left(k \in \Sigma \setminus \tilde{\Sigma}\right)}{\begin{bmatrix} v_{11} & v_{12}d^2(k) \\ \dfrac{v_{21}}{d^2(k)} & v_{22} \end{bmatrix}} \qquad \underset{(k \in \Sigma)}{\begin{bmatrix} \dfrac{d_-}{d_+}v_{11} & d_+d_-v_{12} \\ \dfrac{v_{21}}{d_+d_-} & \dfrac{d_+}{d_-}v_{22} \end{bmatrix}}}$$

Now, we (following [3]) remove the poles by considering two cases:

1. $\mathrm{Re}(\phi(i\lambda_j)) < 0$. Then $\phi(i\lambda_j) \to \mathcal{I}_{2\times 2}$ exponentially, and we are done.
2. $\mathrm{Re}(\phi(i\lambda_j)) \geq 0$. In this case, we follow Deift and Zhou [3] and Kamvassis [5] in conjugating the jump matrix to one whose off-diagonal entries vanish exponentially.

Lemma 4 ([4])
The RH problem $\Upsilon_+(k) = \Upsilon_-(k)V(k)$ with

$$\tilde{V}(k) = \overbrace{\underset{k \in \partial B_\varepsilon(i\lambda)}{\begin{bmatrix} 1 & 0 \\ -\dfrac{i\gamma^2}{k - \lambda i} & 1 \end{bmatrix}} \qquad \underset{k \in \partial B_\varepsilon(-i\lambda)}{\begin{bmatrix} 1 & -\dfrac{i\gamma^2}{k + i\lambda} \\ 0 & 1 \end{bmatrix}}}$$

is equivalent to the RH problem $\tilde{\Upsilon}_+(k) = \tilde{\Upsilon}_-(k)\tilde{V}(k)$ by Lemma 3, with

$$D(k) = \begin{bmatrix} \left(\dfrac{k+i\lambda}{k-i\lambda}\right)^{-1} & 0 \\ 0 & \dfrac{k+i\lambda}{k-i\lambda} \end{bmatrix} \qquad \tilde{V}(k) = \begin{cases} \begin{bmatrix} 1 & -\dfrac{(k+i\lambda)^2}{i\gamma^2(k-i\lambda)} \\ 0 & 1 \end{bmatrix} & k \in B_\varepsilon(i\lambda) \\ \begin{bmatrix} 1 & 0 \\ \dfrac{(k-i\lambda)^2}{i\gamma^2(k+i\lambda)} & 1 \end{bmatrix} & k \in B_\varepsilon(-i\lambda) \end{cases}$$

Notice that D is analytic outside the ε-balls around the poles.

■

Given that all the jumps along $\mathbb{R}$ are oscillatory, we further deform the problem into a region where the oscillations vanish. Recalling the given phase $\phi(k,t) = 2ikx + 8ik^3t$, we see that the points of stationary phase are given by $k_0 = \pm\sqrt{-\frac{x}{12t}}$. We also take the value $\lambda_0 = \sqrt{\frac{x}{4t}}$ ($\lambda_0 \in \mathbb{R}$ such that $\phi(i\lambda_0) = 0$); if $\frac{x}{t} < 0$, then we let λ_0 vanish for convenience.

Factorise $V(k)$ as follows:

$$V(k) = \begin{cases} \underbrace{\begin{bmatrix} 1 & \overline{R(k)}e^{\phi(k,t)} \\ 0 & 1 \end{bmatrix}^{-1}}_{b_-(k)} \underbrace{\begin{bmatrix} 1 & 0 \\ R(k)e^{\phi(k,t)} & 1 \end{bmatrix}}_{b_+(k)} & |k| > \mathrm{Re}(\lambda_0) \\ \underbrace{\begin{bmatrix} 1 & 0 \\ -\frac{R(k)e^{\phi(k,t)}}{1-|R(k)|} & \end{bmatrix}^{-1}}_{B_-(k)} \underbrace{\begin{bmatrix} 1-|R(k)|^2 & 0 \\ 0 & \frac{1}{1-|R(k)|} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & -\frac{\overline{R(k)}e^{-\phi(k,t)}}{1-|R(k)|} \\ 0 & 1 \end{bmatrix}}_{B_+(k)} & |k| < \mathrm{Re}(\lambda_0) \end{cases}$$

Now we eliminate the terms introduced by matrix A in the above factorisation, and conjugate the jumps near the eigenvalues, by defining the *partial transmission coefficient* with respect to k_0 as

$$T(k,k_0) = \begin{cases} \prod_{\lambda_j > \lambda_0} \left(\frac{k+i\lambda_j}{k-i\lambda_j} \right) & \frac{x}{t} > 0 \\ \prod_{j=1}^{n} \left(\frac{k+i\lambda_j}{k-i\lambda_j} \right) \exp\left(\frac{1}{2\pi i} \int_{-k_0}^{k_0} \frac{\ln |T(\xi)|^2}{\xi - k} \, d\xi \right) & \frac{x}{t} < 0 \end{cases}$$

meromorphic for $k \in \mathbb{C} \backslash \Sigma(\lambda_0)$, with $k \in \mathbb{C} \backslash \Sigma(k_0)$ and $\Sigma(k_0) = \partial B_{\mathrm{Re}(k_0)}$ (with left-to-right orientation). Taking an $\mathcal{O}\left(\frac{1}{k^2}\right)$ expansion of the coefficient around $k = 0$, we get that

$$T_1(k_0) = \begin{cases} \sum_{\lambda_j > \lambda_0} 2\lambda_j & \frac{x}{t} > 0 \\ \sum_{j=1}^{n} 2\lambda_j + \frac{1}{2\pi} \int_{-k_0}^{k_0} \ln\left(|T(\xi)|^2\right) d\xi & \frac{x}{t} < 0 \end{cases}$$

for

$$T(k,k_0) = 1 + T_1(k_0) i \frac{1}{k} + \mathcal{O}\left(\frac{1}{k^2}\right) \tag{5.5}$$

Also, we have the following properties ([4]):

(a) $T_+(k,k_0) \equiv (1 - |R(k)|^2) \cdot T_-(k,k_0) \quad \forall k \in \Sigma(k_0)$

(b) $T(-k,k_0) \cdot T(k,k_0) \equiv 1 \quad \forall k \in \mathbb{C} \setminus \Sigma(k_0)$

(c) $T(-k,k_0) \equiv \overline{T(\bar{k},k_0)} \quad \forall k \in \mathbb{C}$

As in [3] and [4], we define $F(k) = \begin{bmatrix} \frac{1}{T(k,k_0)} & 0 \\ 0 & T(k,k_0) \end{bmatrix}$, and create the transformation

$$\overbrace{\begin{matrix} \begin{bmatrix} 1 & -\frac{k-i\lambda_j}{i\gamma_j^2 e^{\phi(i\lambda_j,t)}} \\ \frac{i\gamma_j^2 e^{\phi(i\lambda_j,t)}}{k-i\lambda_j} & 0 \end{bmatrix} \cdot F(k) & \begin{bmatrix} 0 & -\frac{i\gamma_j^2 e^{\phi(i\lambda_j,t)}}{k+i\lambda_j} \\ \frac{k+i\lambda_j}{i\gamma_j^2 e^{\phi(i\lambda_j,t)}} & 1 \end{bmatrix} \cdot F(k) & F(k) \\ k \in B_\varepsilon(i\lambda_j) \text{ and } \lambda_0 < \lambda_j & k \in B_\varepsilon(-i\lambda_j) \text{ and } \lambda_0 < \lambda_j & \text{else} \end{matrix}}^{D(k)=}$$

Simplifying, we have that

$$D(k) = \begin{cases} \begin{bmatrix} \frac{1}{T(k,k_0)} & -\frac{(k-i\lambda_j)T(k,k_0)}{i\gamma_j^2 e^{\phi(i\lambda_j,t)}} \\ \frac{i\gamma_j^2 e^{\phi(i\lambda_j,t)}}{(k-i\lambda_j)T(k,k_0)} & 0 \end{bmatrix} & k \in B_\varepsilon(i\lambda_j) \text{ and } \lambda_0 < \lambda_j \\ \begin{bmatrix} 0 & -\frac{i\gamma_j^2 e^{\phi(i\lambda_j,t)}T(k,k_0)}{k+i\lambda_j} \\ \frac{k+i\lambda_j}{i\gamma_j^2 e^{\phi(i\lambda_j,t)}T(k,k_0)} & T(k,k_0) \end{bmatrix} & k \in B_\varepsilon(-i\lambda_j) \text{ and } \lambda_0 < \lambda_j \\ \begin{bmatrix} \frac{1}{T(k,k_0)} & 0 \\ 0 & T(k,k_0) \end{bmatrix} & \text{else} \end{cases} \tag{5.6}$$

We transform the Riemann-Hilbert problem $\tilde{\Upsilon}(k) = \Upsilon(k)D(k)$, where

$$D(-k) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} D(k) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Using Lemmas 3 and 4, we can formulate the jump by

$$\tilde{V}(k) = \begin{cases} \left.\begin{array}{ll} \begin{bmatrix} 1 & -\dfrac{(k-i\lambda_j)T^2(k,k_0)}{i\lambda_j^2 e^{\phi(i\lambda_j,t)}} \\ 0 & 1 \end{bmatrix} & k \in \partial B_\varepsilon(i\lambda_j) \\ \\ \begin{bmatrix} 1 & 0 \\ -\dfrac{(k+i\lambda_j)T^2(k,k_0)}{i\gamma_j^2 e^{\phi(i\lambda_j,t)}} & 1 \end{bmatrix} & k \in \partial B_\varepsilon(-i\lambda_j) \end{array}\right\} & \lambda_0 < \lambda_j \\ \\ \left.\begin{array}{ll} \begin{bmatrix} 1 & 0 \\ -\dfrac{i\gamma_j^2 e^{\phi(i\lambda_j,t)}}{(k-i\lambda_j)T^2(k,k_0)} & 1 \end{bmatrix} & k \in \partial B_\varepsilon(i\lambda_j) \\ \\ \begin{bmatrix} 1 & -\dfrac{i\lambda_j^2 e^{\phi(i\lambda_j,t)}}{(k+i\lambda_j)T^2(k,k_0)} \\ 0 & 1 \end{bmatrix} & k \in \partial B_\varepsilon(-i\lambda_j) \end{array}\right\} & \lambda_0 > \lambda_j \end{cases}$$

Now, all the jumps near singularities will approach $\mathcal{I}_{2\times 2}$ as $t \to +\infty$ when $\lambda_0 \neq \lambda_j$. If $\lambda_j = \lambda_0$, then we retain a pole condition resembling that in Problem 1, but this time compensating for the introduction of the partial transmission coefficient to our new problem:

$$\operatorname{Res}\left[\tilde{\Upsilon}(k), i\lambda_j\right] = \frac{\gamma_j^2 e^{\phi(i\lambda_j,t)}}{T^2(k,k_0)} \lim_{k\to i\lambda_j} \tilde{\Upsilon}(k) \begin{bmatrix} 0 & 0 \\ i & 0 \end{bmatrix}$$

with the other residue given by the symmetry criterion. The jump along $\mathbb{R}$ is then

$$\tilde{V}(k) = \begin{cases} \underbrace{\begin{bmatrix} 1 & \dfrac{R(-k)e^{-\phi(k,t)}}{T^2(k,k_0)} \\ 0 & 1 \end{bmatrix}^{-1}}_{\tilde{b}_-(k)} \underbrace{\begin{bmatrix} 1 & 0 \\ \dfrac{R(k)e^{\psi(k,t)}}{T^2(k,k_0)} & 0 \end{bmatrix}}_{\tilde{b}_+(k)} & k \in \mathbb{R} \setminus \Sigma(k_0) \\ \\ \underbrace{\begin{bmatrix} 1 & 0 \\ -\dfrac{T_-(k,k_0)R(k)e^{\psi(k,t)}}{T_-(k,k_0)} & 1 \end{bmatrix}^{-1}}_{\tilde{B}_-(k)} \underbrace{\begin{bmatrix} 1 & -\dfrac{T_+(k,k_0)R(-k)e^{-\psi(k,t)}}{T_+(-k,k_0)} \\ 0 & 1 \end{bmatrix}}_{\tilde{B}_+(k)} & k \in \Sigma(k_0) \end{cases}$$

Finally, we deform the jump along $\mathbb{R}$ in a way that allows all the oscillatory terms to vanish. Grunert considers two cases:

(a) If $\dfrac{x}{t} > 0$, we take $\varepsilon > 0$ sufficiently small that

   i. $\Sigma_\pm = \{k \in \mathbb{C} : \mathrm{Im} = \pm\varepsilon\} \subseteq \{k : \pm\mathrm{Re}(\phi(k)) < 0\}$

ii. $B_\varepsilon(\pm i\lambda_j)$ lies outside of the strip $|\mathrm{Im}(z)| < \varepsilon$ for all $1 \leq j \leq n$

Finally, redefine the jump by $\hat{\Upsilon}(k) = \begin{Bmatrix} \tilde{\Upsilon}(k)\tilde{b}_+(k)^{-1} & 0 < \mathrm{Im}(k) < \varepsilon \\ \tilde{\Upsilon}(k)\tilde{b}_-(k)^{-1} & -\varepsilon < \mathrm{Im}(k) < 0 \\ \tilde{\Upsilon}(k) & \textit{else} \end{Bmatrix}$ so that it becomes $\hat{V}(k) = \begin{Bmatrix} \tilde{b}_+(k)^{-1} & k \in \Sigma_+ \\ \tilde{b}_-(k)^{-1} & k \in \Sigma_- \end{Bmatrix}$, vanishing along $\mathbb{R}$, and we are done.

(b) If $\frac{x}{t} < 0$, then Grunert deforms the contour $\Sigma_\pm$ into $\Sigma^1_\pm \cup \Sigma^2_\pm$, where

- $\Sigma^1_\pm$ holds the value $\pm\varepsilon$ with positive orientation everywhere on $|\mathrm{Re}\ \phi(k)| > k_0$, except in a neighbourhood of $\pm k_0$, where it vanishes smoothly
- $\Sigma^2_\pm$ holds the value $\pm\varepsilon$ with positive orientation everywhere on $|\mathrm{Re}\ \phi(k)| < k_0$, except in a neighbourhood of $\pm k_0$, where it vanishes smoothly
- The contours $\Sigma^{1,2}_\pm$ vanish at the same rate
- $B_\varepsilon(\pm i\lambda_j)$ lies outside of the region between the contours for all $1 \leq j \leq n$

We then redefine the problem as

$$\hat{\Upsilon}(k) = \begin{cases} \tilde{\Upsilon}(k)\tilde{b}_\pm(k)^{-1} & k \text{ between } \mathbb{R} \text{ and } \Sigma^1_\pm \\ \tilde{\Upsilon}(k)\tilde{B}_\pm(k)^{-1} & k \text{ between } \mathbb{R} \text{ and } \Sigma^2_\pm \\ \tilde{\Upsilon}(k) & \text{else} \end{cases}$$

Again, the jump along $\mathbb{R}$ vanishes, and we are left with the jump along the contour given by

$$\tilde{V}(x) = \begin{cases} \tilde{b}_\pm(k)^{\pm 1} & k \in \Sigma^1_\pm \\ \tilde{B}_\pm(k)^{\pm 1} & k \in \Sigma^2_\pm \end{cases}$$

Now, all jumps along $\Sigma_\pm \setminus \{\pm k_0\}$ vanish, and we are done.

Finally, we end by describing the asymptotic behaviour of the solution to the KdV equation when $\mathrm{sgn}(x) = \mathrm{sgn}(t)$.

Theorem 1 (Asymptotic Behaviour in the Soliton Range [4])
Assume $\int_{\mathbb{R}} (1+|x|)^{L+1}|q(x,0)|\,dx < \infty$ *for some integer* $L \geq 1$ *and take the velocity* $4\lambda_j^2 = \mathrm{Re}(\psi(i\lambda_j))$ *of the* j*th soliton. Let* $\varepsilon > 0$ *be sufficiently small that the collection* $\{B_\varepsilon(4\lambda_j^2)\}_{i\leq j\leq n} \subseteq \mathbb{R}^+$ *is pairwise disjoint.*

1. If $\frac{x}{t} \in B_\varepsilon(4\lambda_j^2)$ *for some* j*, then*

$$-Q_+(x,t) = \int_x^\infty q(y,t)\,dt = -4\left(\sum_{i=j+1}^{n} \lambda_i\right) - \frac{4\lambda_j\gamma_j^2(x,t)}{2\lambda_j + \gamma_j^2(x,t)} + \mathcal{O}\left(t^{-L}\right)$$

and so we have that $q(x,t) = \dfrac{-16\lambda_j^3\gamma_j^2(x,t)}{(2\lambda_j + \gamma_j^2(x,t))^2} + \mathcal{O}\left(t^{-L}\right)$*, where*

$$\gamma_j^2(x,t) = \gamma_j^2 e^{\phi(i\lambda_j,t)} \prod_{i=j+1}^{n} \left(\frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}\right)^2$$

2. Else, if $\frac{x}{t} \notin B_\varepsilon(4\lambda_j^2)$ *for all* j*, we have that*

$$-Q_+(x,t) = \int_x^\infty q(y,t)\,dy = -4\left(\sum_{\lambda_j>\lambda_0} \lambda_j\right) + \mathcal{O}\left(t^{-L}\right)$$

$\left(\textit{recall that } \lambda_0 = \sqrt{\frac{x}{4t}}\right)$*, and so* $q(x,t) = \mathcal{O}\left(t^{-L}\right)$*.*

Proof: ([4])
If k is sufficiently far from the $\mathbb{R}$, then $\hat{\Upsilon}(k) \equiv \tilde{\Upsilon}(k)$, so that we can use (5.4), with the matrix $D(k)$ as given in (5.6). Taking the truncated series expansion of the partial coefficient $T(k, k_0)$ as given in (5.5), we get the expansion

$$\hat{\Upsilon}(k) = \begin{pmatrix}1\\1\end{pmatrix}^T + \frac{Q_+(x,t) - 2T_1(k_0)}{2ki}\begin{pmatrix}-1\\1\end{pmatrix}^T + \mathcal{O}\left(\frac{1}{k^2}\right)$$

Recall that all jumps along the contour vanish exponentially, so we have the following:

Definition 3 (The Cauchy Transform [4, 9])
Let Σ *be a contour, and* $f \in \mathrm{Hom}_{L^2}(\Sigma, \mathbb{C})$*. The Cauchy transform is the analytic function* $L^2(\mathbb{C} \setminus \Sigma) \to L^2(\mathbb{C})$ *given by*

$$Cf(k) = \frac{1}{2\pi i}\int_\Sigma \frac{f(s)}{s-k}\,ds$$

with boundary values denoted by $C_\pm \in \mathrm{End}\left[L^2(\Sigma)\right]$ *such that* $C_+ \equiv \mathcal{I} + C_-$*. These operators can also be taken from the Plemelj-Sokhotsky formula* $C_\pm = \frac{1}{2}(iH \pm \mathcal{I})$*, where for* $k \in \Sigma$

$$Hf(k) = \frac{1}{\pi}\,\mathrm{PV}\!\int_\Sigma \frac{f(s)}{k-s}\,ds$$

is the Hilbert transform. ■

Definition 4 (The Cauchy Operator [4, 9])
If f *is a vector-valued function* $\Sigma \to \mathbb{C}^2$*, then the Cauchy operator is the integral operator* $C[f](k) = \frac{1}{2\pi i}\int_\Sigma \Xi_\lambda(s,k)\,ds$ *with the kernel*

$$\Xi_\lambda(s,k) = \mathrm{diag}\left[\frac{k+i\lambda}{(s-k)(s+i\lambda)}, \frac{k-i\lambda}{(s-k)(s-i\lambda)}\right]$$

Denote $C_\omega f = C_+(f\omega_-) + C_-(f\omega_+) \in \mathrm{End}\left[L^2(\Sigma)\right]$*.* ■

Lemma 5 ([4]) *Let* Σ *be a fixed contour, and choose* λ*,* $\gamma = \gamma^T$*, and* ν^T *depending on some parameter* $t \in \mathbb{R}$ *such that we satisfy the following criteria:*

1. Σ *is a finite collection of smooth oriented finite curves on* $\mathbb{C}$ *which self-intersects almost nowhere and only transversally*
2. $\pm i\lambda \notin \Sigma$ *and* $\exists\, y_0 \in \mathbb{R}^+$ *such that* $\mathrm{dist}\left(\Sigma, i\mathbb{R}_{\geq y_0}\right) > 0$
3. Σ *is invariant under the negation map, and is oriented so that all sequences converging to* Σ *also observe this negation*

4. The jump matrix ν is non-singular, with the factorisation

$$\nu = b_-^{-1} b_+ = (\mathcal{I} - \omega_-)^{-1} (\mathcal{I} + \omega_+)$$

$$\omega_\pm(-k) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \omega_\mp(k) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, k \in \Sigma$$

5. The jump satisfies $\|\omega\|_\infty = \|w_+\|_{L^\alpha(\Sigma)} + \|w_-\|_{L^\alpha(\Sigma)} < \infty$ for $\alpha \in \{2, \infty\}$

If $\|\omega^t\|_{L^\alpha} \leq \rho(t)$ for $\alpha \in \{2, \infty\}$ for some function $\rho(t) \to 0$ as $t \to \infty$, the operator $(\mathcal{I} - C_{\omega^t}) \in \mathrm{End}[L^2(\Sigma)]$ is invertible for sufficiently large t, and the solution $\Upsilon(k)$ of the Riemann-Hilbert problem satisfying the above criteria differs from the one soliton solution $m_0^t(k)$ only by $\mathcal{O}(\rho(t))$, with the error term dependent on $\mathrm{dist}\,(k, \Sigma \cup \{\pm i\lambda\})$. ■

Case 1: $\dfrac{x}{t} \in B_\varepsilon(4\lambda_j^2)$ or $k_0^2 \in B_\varepsilon(\lambda_j^2)$ for all $i \leq j \leq n$
Let us choose $\gamma^t = 0$ and $\omega^t = \hat{\omega}$. Now, $\hat{\omega}$ vanishes exponentially for large t, so that we can use Lemma 5 to say that all the solutions of the Riemann-Hilbert problems differ only by $\mathcal{O}\left(t^{-L}\right)$ for all $L \in \mathbb{N}^+$. Therefore, with reference to the two solutions $\hat{\Upsilon}(k)$ and $m_0(k)$, we must have $Q_+(x,t) - 2T_1(k_0) \equiv 0$. Hence,

$$Q_+(x,t) = 4T_1(k_0) = -4 \left(\sum_{\lambda_j > \lambda_0} \lambda_j \right) + \mathcal{O}\left(t^- L\right)$$

Case 2: $\dfrac{x}{t} \notin B_\varepsilon(4\lambda_j^2)$ or $k_0^2 \notin B_\varepsilon(\lambda_j^2)$ for some $i \leq j \leq n$.
Again, choose $\omega^t = \hat{\omega}$, but now $\gamma^t = \gamma_j(x,t)$, with

$$\gamma_j(x,t) \equiv \frac{\gamma_j e^{\frac{1}{2}\phi(i\lambda_j, t)}}{T\left(i\lambda_j, \frac{\lambda_j}{\sqrt{3}} i\right)} \equiv \gamma_j e^{\phi(i\lambda_j, t)} \prod_{i=j+1}^{n} \left(\frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j} \right)$$

Again, $\hat{\omega}$ vanishes exponentially, so we conclude that the solutions m_0 (Lemma 2) and $\hat{\Upsilon}$ are identical. This gives that

$$Q_+(x,t) \equiv 2T_1(k_0) + \frac{4\lambda\gamma^2 e^{\phi(i\lambda, t)}}{2\lambda + \gamma^2 e^{\phi(i\lambda, t)}} + \mathcal{O}\left(t^{-L}\right)$$

For k close to $\mathbb{R}$, we can follow the same argument, this time constructing the series expansion using (5.3) instead of (5.4), obtaining the above final terms. Writing

$$Q_+(x,t) \equiv -2 \left(2 \sum_{\lambda_j > \lambda_0} \lambda_j - \frac{2\lambda\gamma^2 e^{\phi(i\lambda, t)}}{2\lambda + \gamma^2 e^{\phi(i\lambda, t)}} \right) + \mathcal{O}\left(t^{-L}\right)$$

differentiation with respect to x and algebraic manipulation gives

$$q(x,t) \sim -2 \sum_{j=1}^{n} \left(\frac{\lambda_j^2}{\cosh^2\left[\lambda_j x - 4\lambda_j^2 - \tau(j)\right]} \right) \text{ with } \tau(j) = \frac{1}{2} \ln \left(\frac{\gamma_j^2}{2\lambda_j} \prod_{i=j+1}^{n} \left(\frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j} \right)^2 \right)$$

This shows that for large t, the solution will always split into a sum of one-solitons.

■

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