SYDE Advanced Math 2, Solutions to the Midterm Test

1: Consider the ODE given by $x^2y'' - 2xy' + 2y = x + 2$ for $y = y(x)$ with $x > 0$.

(a) Use variation of parameters to solve the given ODE given that $y = y_1(x) = x$ and $y = y_2(x) = x^2$ are solutions to the associated homogeneous ODE $x^2y'' - 2xy' + 2y = 0$.

Solution: We can write the DE in the form $y'' - \frac{2}{x}y' + \frac{2}{x^2}y = \frac{x+2}{x^2}$. Using variation of parameters, we let $y = y_p = y_1u + y_2v = xu + x^2v$ where $u = u(t)$ and $v = v(t)$ satisfy $xu' + x^2v' = 0$ (1) and $u' + 2xv' = \frac{x+2}{x^2}$ (2). Multiply (1) by $\frac{2}{x}$ and subtract (2) to get $u' = -\frac{x+2}{x^2}$. Put this in (1) to get $-\frac{x+2}{x} + x^2v' = 0$ so $v' = \frac{x+2}{x^3}$. To get $u' = -\frac{x+2}{x^2} = -\frac{1}{x} - \frac{2}{x^2}$ we choose $u = -\ln x + \frac{2}{x}$, and to get $v' = \frac{x+2}{x^3} = \frac{1}{x^2} + \frac{2}{x^3}$ we choose $v = -\frac{1}{x} - \frac{1}{x^2}$. A particular solution to the given ODE is given by $y_p = xu + x^2v = -x\ln x + 2 - x - 1 = 1 - x - x\ln x$, and the general solution is given by $y = ax + bx^2 + 1 - x - x\ln x$, or equivalently by $y = bx^2 + cx + 1 - x\ln x$ (where $c = a - 1$).

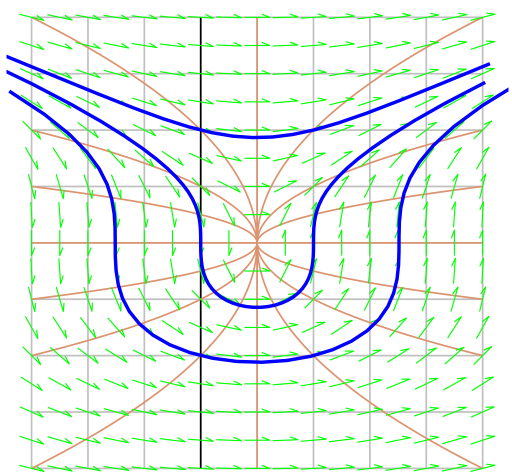
(b) Solve the IVP given by $x^2y'' - 2xy' + 2y = x + 2$ with $y(1) = 2$ and $y'(1) = 3$.

Solution: For $y = bx^2 + cx + 1 - x\ln x$ we have $y' = 2bx + c + \ln x - 1$. To get $y(1) = 2$ we need $2 = b + c + 1$ so that $b + c = 1$ (1), and to get $y'(1) = 3$ we need $3 = 2b + c - 1$ so that $2b + c = 4$ (1). Subtract (1) from (2) to get $b = 3$ and put this in (1) to get $c = -2$. Thus the solution is $y = 3x^2 - 2x + 1 - x\ln x$.

2: Consider the pair of ODEs given by $x' = y^2$ and $y' = x - 1$.

(a) Sketch a phase portrait for this ODE: include the isoclines $x' = 0$, $y' = 0$ and $\frac{y'}{x'} = c$ for $c = \pm\frac{1}{4}, \pm 1, \pm 4$, and the direction field, and the solution curves through $(0, b)$ for $b = 0, \pm 2$.

Solution: We have $x' = 0$ when $y^2 = 0$, that is when $y = 0$ (this is the x-axis), we have $y' = 0$ when $x = 1$ (this is a vertical line), and we have $\frac{y'}{x'} = c$ when $y' = cx'$, that is when $x - 1 = cy^2$, that is $x = 1 + cy^2$ (this is a parabola, opening horizontally, with vertex at $(1, 0)$). The isoclines are shown below in tan, the direction field in green, and the solution curves in blue (we remark that if the phase portrait were sketched by hand, most of the direction field arrows would be placed along the isoclines).



(b) Find a conserved quantity $H(x, y)$ for this pair of ODEs, and use it to find $f(2)$ and $f(4)$ where $y = f(x)$ is the solution curve through $(0, 0)$.

Solution: We solve the DE $\frac{dy}{dx} = \frac{y'}{x'} = \frac{x-1}{y^2}$. The DE is separable as we can write it as $y^2dy = (x - 1)dx$. Integrate both sides to get $\frac{1}{3}y^3 = \frac{1}{2}x^2 - x + c$, that is $\frac{1}{3}y^3 - \frac{1}{2}x^2 + x = c$. Thus $H(x, y) = \frac{1}{3}y^3 - \frac{1}{2}x^2 - x$ is a conserved quantity. We have $H(0, 0) = 0$, so every point (x, y) on the solution curve through $(0, 0)$ satisfies $0 = H(x, y) = \frac{1}{3}y^3 - \frac{1}{2}x^2 + x$, that is $\frac{1}{3}y^3 = \frac{1}{2}x^2 - x$. Thus the solution curve has equation

$$y = f(x) = \sqrt[3]{\tfrac{3}{2}x^2 - 3x}.$$

In particular, we have $f(2) = \sqrt[3]{6 - 6} = 0$ and $f(4) = \sqrt[3]{24 - 12} = \sqrt[3]{12}$.

3: Consider the pair of ODEs given by $x' = y - x - 2$ and $y' = x^2 - y$.

(a) Find all the equilibrium points.

Solution: Write $F(x,y) = \big(y - x - 2\,,\, x^2 - y\big)$. A point (x,y) is an equilibrium point when $F(x,y) = (0,0)$, that is when $y = x+2$ and $y = x^2$. This implies that $x^2 = x + 2$ so that $0 = x^2 - x - 2 = (x-2)(x+1)$ and hence $x \in \{-1, 2\}$. The equilibrium points are given by $x \in \{-1,2\}$ with $y = x+2$, so the points are $(-1,1)$ and $(2,4)$.

(b) For each equilibrium point, determine whether it is an attracting point, a repelling point, or a saddle point.

Solution: Let $A = DF = \left(\begin{smallmatrix} -1 & 1 \\ 2x & -1 \end{smallmatrix}\right)$. When $(x,y) = (-1,1)$ we have $A = \left(\begin{smallmatrix} -1 & 1 \\ -2 & -1 \end{smallmatrix}\right)$. The characteristic polynomial is $r^2 + 2r + 3$, so the eigenvalues are $r = \frac{-2 \pm \sqrt{4-12}}{2} = -1 \pm \sqrt{2}\,i$. Since the real part of these eigenvalues is equal to -1, which is negative, this equilibrium point is an attracting point.

When $(x,y) = (2,4)$ we have $A = \left(\begin{smallmatrix} -1 & 1 \\ 4 & -1 \end{smallmatrix}\right)$. The characteristic polynomial is $r^2 + 2r - 3 = (r-1)(r+3)$. The eigenvalues are 1 and -3, so this equilibrium point is a saddle point.

(c) Find the solution to the linearized system at each of the equilibrium points.

Solution: When $(x,y) = (-1,1)$ and $r = -1 + \sqrt{2}\,i$ we have $A - rI = \left(\begin{smallmatrix} -\sqrt{2}\,i & 1 \\ -2 & -\sqrt{2}\,i \end{smallmatrix}\right)$. A complex eigenvector is given by $\left(\begin{smallmatrix} 1 \\ \sqrt{2}\,i \end{smallmatrix}\right)$, a complex solution is given by $e^{(-1+\sqrt{2}\,i)t}\left(\begin{smallmatrix} 1 \\ \sqrt{2}\,i \end{smallmatrix}\right) = e^{-t}\big(\cos\sqrt{2}\,t + i\sin\sqrt{2}\,t\big)\big(\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) + i\left(\begin{smallmatrix} 0 \\ \sqrt{2} \end{smallmatrix}\right)\big)$, and the general real solution is

$$\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = a\,e^{-t}\Big(\cos\sqrt{2}\,t\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) - \sin\sqrt{2}\,t\left(\begin{smallmatrix} 0 \\ \sqrt{2} \end{smallmatrix}\right)\Big) + b\,e^{-t}\Big(\cos\sqrt{2}\,t\left(\begin{smallmatrix} 0 \\ \sqrt{2} \end{smallmatrix}\right) + \sin\sqrt{2}\,t\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\Big)$$

In the case that $(x,y) = (2,4)$, when $r = 1$ we have $A - rI = \left(\begin{smallmatrix} -2 & 1 \\ 4 & -2 \end{smallmatrix}\right)$ so an eigenvalue is given by $\left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$, and when $r = -3$ we have $A - rI = \left(\begin{smallmatrix} 2 & 1 \\ 4 & 2 \end{smallmatrix}\right)$ so an eigenvector is given by $\left(\begin{smallmatrix} -1 \\ 2 \end{smallmatrix}\right)$, so the general solution is

$$\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = a\,e^{t}\left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right) + b\,e^{-3t}\left(\begin{smallmatrix} -1 \\ 2 \end{smallmatrix}\right).$$

4: Solve the ODE $(2 + x^2)y'' + 4xy' + 2y = 0$ using power series centred at 0. Find an explicit, closed form formula for the general solution.

Solution: Let $y = \sum_{n=0}^{\infty} c_n x^n$ so $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$. Put these in the DE to get

$$\begin{aligned} 0 &= \sum_{n\geq 2} 2n(n-1)c_n x^{n-2} + \sum_{n\geq 0} n(n-1)c_n x^n + \sum_{n\geq 0} 4nc_n x^n + \sum_{n\geq 0} 2c_n x^n \\ &= \sum_{m\geq 0} 2(m+2)(m+1)c_{m+2}x^m + \sum_{m\geq 0} \big(m(m-1) + 4m + 2\big)c_m x^m\,. \end{aligned}$$

All coefficients must vanish, so for all $m \geq 0$ we have

$$2(m+2)(m+1)c_{m+2} = -(m(m-1) + 4m + 2)c_m = -(m^2 + 3m + 2)c_m = -(m+2)(m+1)c_m\,,$$

that is

$$c_{m+2} = -\tfrac{1}{2}\,c_m.$$

If we take $c_0 = 1$ and $c_1 = 0$ then we get $c_n = 0$ for n odd and $c_2 = -\frac{1}{2}$, $c_4 = \frac{1}{2^2}$, $c_6 = -\frac{1}{2^3}$, and in general $c_{2k} = \frac{(-1)^k}{2^k}$, so we obtain the solution

$$y_1(x) = \sum_{k\geq 0} \tfrac{(-1)^k}{2^k} x^{2k} = 1 - \tfrac{1}{2}x^2 + \tfrac{1}{2^2}x^4 - \tfrac{1}{2^3}x^6 + \cdots = \tfrac{1}{1+\frac{1}{2}x^2} = \tfrac{2}{2+x^2}\,.$$

If we take $c_0 = 0$ and $c_1 = 1$ then we get $c_n = 0$ for n even and $c_3 = -\frac{1}{2}$, $c_5 = \frac{1}{2^2}$, $c_7 = -\frac{1}{2^3}$ and in general $c_{2k+1} = \frac{(-1)^k}{2^k}$, so we obtain the solution

$$y_2(x) = \sum_{k\geq 0} \tfrac{(-1)^k}{2^k} x^{2k+1} = x - \tfrac{1}{2}x^3 + \tfrac{1}{2^2}x^5 - \tfrac{1}{2^3}x^7 + \cdots = x\,y_1(x) = \tfrac{2x}{2+x^2}\,.$$

Thus the general solution to the DE is $y = y(x) = c_0\,\frac{2}{2+x^2} + c_1\,\frac{x}{2+x^2} = \frac{a+bx}{2+x^2}$.