# Part 4. PDEs

**4.1 Definition:** Recall that a **partial differential equation**, or a PDE, is an equation which involves a function, of two or more variables, and some of its derivatives, and a **solution** to a PDE is a function which makes the equation hold (for all $x$ in the domain). The order of the PDE is the highest of the orders of the derivatives which appear in the PDE. A **first order linear** PDE for the function $u = u(x, y)$ can be written in the form

$$a(x,y)\tfrac{\partial u}{\partial x}(x,y) + b(x,y)\tfrac{\partial u}{\partial y}(x,y) = c(x,y)$$

for some continuous functions $a, b, c : U \subseteq \mathbb{R}^2 \to \mathbb{R}$. The above PDE is called **homogeneous** when $c = 0$ (that is when $c$ is the zero function given by $c(x, y) = 0$ for all $(x, y) \in U$). A **second order linear** PDE for $u = u(x, y)$ can be written in the form

$$a\tfrac{\partial^2 u}{\partial x^2} + b\tfrac{\partial^2 u}{\partial x \partial y} + c\tfrac{\partial^2}{\partial y^2} + d\tfrac{\partial u}{\partial x} + e\tfrac{\partial u}{\partial y} + f = 0$$

where $a, b, c, d, e, f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ are continuous. The above PDE is called **homogenous** when $f = 0$. As with linear homogeneous ODEs, a linear combination of solutions is also a solution. For a PDE involving a function $u = u(x, y)$ defined in a domain $U \subseteq \mathbb{R}^2$, a **boundary condition** is given by specifying the value of $u(x, y)$ at all points $(x, y)$ on the boundary of $U$. For a PDE involving a function $u = u(x, t)$ defined for $x$ in an interval and for $t$ in an interval containing $t = 0$, an **initial condition** is given by specifying the value of $u(x, 0)$ for all $x$.

**4.2 Exercise:** Solve the first order linear PDE given by $\frac{\partial u}{\partial t} = x^2 t$ for $u = u(x, t)$ satisfying the initial condition $u(x, 0) = x^2$ for all $x$.

**4.3 Exercise:** Solve the non-linear PDE given by $x\frac{\partial u}{\partial x} + yu\frac{\partial u}{\partial y} = -xy$ for $u = u(x, y)$ and for $xy \geq 1$, given that $u(x, y) = 5$ along the curve $xy = 1$.

**4.4 Note:** Consider a pair of first order liner PDEs given by $\frac{\partial u}{\partial x} = f(x, y)$ and $\frac{\partial u}{\partial y} = g(x, y)$. In order for a twice continuously differentiable solution $u = u(x, y)$ to exist we must have $\frac{\partial f}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial g}{\partial y}$. When this condition is satisfied, a solution $u = u(x, y)$ does exist, and it is unique up to adding a constant, (provided the domain has no holes). The solution with $u(a, b) = c$ can be found using **Euler's method**. First approximate the curve which follows the graph of the solution in the $x$-direction, starting at $(a, b, c)$. To do this, choose a small step size $\Delta x$, start with $x_{0,0} = a$, $y_{0,0} = b$, $u_{0,0} = c$ then, having found $x_{k,0}, y_{k,0}, u_{k,0}$ let $x_{k+1,0} = x_{k,0} + \Delta x$, let $y_{k+1,0} = y_{k,0}$ (so $y_{k,0} = b$ for all $k$), and let $u_{k+1,0} = u_{k,0} + f(x_{k,0}, y_{k,0})\Delta x$. Then, for each value of $k$, approximate the curve which follows the graph of the solution in the $y$-direction, starting at $(x_{k,0}, y_{k,0}, u_{k,0})$. To do this, choose a small step size $\Delta y$, and having found $x_{k,\ell}$, $y_{k,\ell}$ and $u_{k,\ell}$, let $x_{k,\ell+1} = x_{k,\ell}$ (so that $x_{k,\ell} = x_{k,0}$ for all $\ell$), let $y_{k,\ell+1} = y_{k,\ell} + \Delta y$, and let $u_{k,\ell+1} = u_{k,\ell} + g(x_{k,\ell}, y_{k,\ell})\Delta y$.

The **direction field** (or the **distribution**) for this pair of PDEs is constructed as follows: at each point $(x, y, u) \in \mathbb{R}^3$ (with $(x, y)$ in the domain of $f$ and $g$) we place a small square angled so that the slope in the $x$-direction is $f(x, y)$ and the slope in the $y$-direction is $g(x, y)$. The graph of any solution $u = u(x, y)$ is tangent to each of the small squares that it passes through.

**4.5 Definition:** Given $a, b, c, d, e, f \in \mathbb{R}$, the curve in $\mathbb{R}^2$ given by the equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

is either a hyperbola, a parabola, or an ellipse, depending on whether the discriminant $b^2 - 4ac$ is positive, zero, or negative. For this reason, the second order linear PDE given by

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + f = 0$$

is called **hyperbolic**, **parabolic**, or **elliptic**, when the discriminant $b^2 - 4ac$ is positive, zero, or negative, When the coefficients $a, b, \cdots, f$ are functions of $x$ and $y$, the PDE can be hyperbolic at some points, and parabolic or elliptic at other points. When the coefficients are constant real numbers, the discriminant is also a constant real number, so the PDE is of the same type at all points.

## The Wave, Heat and Laplace Equations

**4.6 Definition:** We shall concentrate on the following three second order linear homogenous PDEs with constant coefficients (any second order linear homogeneous PDE with constant coefficients can be converted to one of these three PDEs by changing variables):

The (1-dimensional) **wave equation**, given by $\frac{\partial u^2}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ for $u = u(x, t)$ is hyperbolic.

The (1-dimensional) **heat equation**, given by $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ for $u = u(x, t)$ is parabolic.

The (2-dimensional) **Laplace equation** given by $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ for $u = u(x, y)$ is elliptic.

These three PDEs have higher dimensional versions involving additional space variables:

The wave equation for $u = u(x, t)$ is given by $\frac{\partial^2 u}{\partial t^2} = c^2 \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2}$.

The heat equation for $u = u(x, t)$ is given by $\frac{\partial u}{\partial t} = c^2 \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2}$.

Laplace's equation for $u = u(x)$ is given by $\sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2} = 0$.

**4.7 Exercise:** A string lies along the $x$-axis with $0 \le x \le \ell$ and is held taught at both ends. Points along the string are displaced vertically so that, at time $t = 0$, the string follows the shape of the curve $u(0, x) = f(x)$ with $f(0) = f(\ell) = 0$. The string is then released and allowed to vibrate (with the endpoints fixed), with all points along the string moving up or down, pulled by the tension in the string. Give an argument, using principles of physics, to explain why the position $u = u(x, t)$ of points along the string satisfies the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ with $u(x, 0) = f(x)$ and with $u(0, t) = u(\ell, t) = 0$.

**4.8 Exercise:** A rod lies along the $x$-axis with $0 \le x \le \ell$. Initially, the rod is heated to varying temperatures along its length so that, at time $t = 0$, the temperature at position $x$ is given by $u(x, 0) = f(x)$. Starting at time $t = 0$, heat is no longer applied to the rod except at the two endpoints which are held at constant temperatures with the temperature at $x = 0$ always equal to $a = f(0)$ and the temperature at $x = \ell$ always equal to $b = f(\ell)$. Give an argument, using principles of physics, to explain why the temperature $u(x, t)$ at points along the rod satisfies the heat equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ with $u(x, 0) = f(x)$ and with $u(0, t) = a$ and $u(\ell, t) = b$.

**4.9 Remark:** Note that for a function $u = u(x, t) = u(x) = f(x)$ with $x \in \mathbb{R}^n$, that is for a function which is constant in $t$, we have $\frac{\partial u}{\partial t} = 0$. So Laplace's equation is equivalent to the heat equation for a function $u = u(x)$, which constant in $t$.

**4.10 Example:** Solve the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ for $u = u(x,t)$ with $0 \leq x \leq \ell$ and $t \in \mathbb{R}$ satisfying the fixed endpoint boundary conditions $u(0,t) = 0$ and $u(\ell,t) = 0$ for all $t \in \mathbb{R}$ and the initial conditions $u(x,0) = f(x)$ and $\frac{\partial u}{\partial t} = g(x)$ for all $0 \leq x \leq \ell$.

Solution: We use a method known as **separation of variables**. We begin by looking for solutions of the special form $u(x,t) = X(x)T(t)$ which satisfy the boundary conditions. When $u(x,t) = X(x)T(t)$, the wave equation becomes $X(x)T''(t) = c^2 X''(x)T(t)$ which we can write as $\frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)}$. The left side depends only on $x$ (and is constant in $t$), and the right side depends only on $t$ (and is constant in $x$), so in order for the two sides to be equal for all $x$ and $t$, they must be constant, so for some $k \in \mathbb{R}$ we must have

$$\frac{X''}{X} = k = \frac{1}{c^2}\frac{T''}{T}.$$

Also note that when $u(x,t) = X(x)T(t)$, the boundary conditions become $X(0)T(t) = 0$ and $X(\ell)T(t) = 0$ for all $t$. If $T(t) = 0$ for all $t$, then $u(x,t) = X(x)T(t) = 0$ for all $x, t$, so in order to obtain a nonzero solution, we must have $X(0) = 0$ and $X(\ell) = 0$.

Let us solve the DE $\frac{X''}{X} = k$ with the boundary conditions $X(0) = 0$ and $X(\ell) = 0$. We can write the DE as $X'' - kX = 0$. If $k = 0$ then the solution is $X(x) = ax + b$ and $X(0) = 0 \implies b = 0$ and $X(\ell) = 0 \implies a\ell = 0 \implies a = 0$, so we only obtain the zero solution. If $k > 0$, say $k = \sigma^2$ with $\sigma > 0$, then the DE becomes $X'' - \sigma^2 X = 0$ which has solution $X = ae^{\sigma x} + be^{-\sigma x}$ and $X(0) = 0 \implies a + b = 0 \implies b = -a \implies X = a(e^{\sigma x} + e^{-\sigma x})$, and $X(\ell) = 0 \implies a(e^{\sigma \ell} + e^{-\sigma \ell}) = 0 \implies a = 0$, so again we only obtain the zero solution. Thus to obtain a nonzero solution, we must have $k < 0$, say $k = -\sigma^2$ where $\sigma > 0$. The DE becomes $X'' + \sigma^2 X = 0$ which has solution $X = c\cos\sigma t + d\sin\sigma t$. The condition $X(0) = 0$ gives $b = 0$ so that $X = d\sin\sigma t$, then the condition $X(\ell) = 0$ gives $d\sin\sigma\ell = 0$. If $d = 0$ we obtain the trivial solution and if $\sin\sigma\ell = 0$ then we must have $\sigma\ell = n\pi$ for some $n \in \mathbb{Z}$. Thus to obtain a nonzero solution to the DE which satisfies the boundary conditions we must have $k = -\sigma^2 = -\left(\frac{n\pi}{\ell}\right)^2$ for some $n \in \mathbb{Z}^+$ and, in this case, the solution is

$$X = X_n(x) = d_n \sin\left(\tfrac{n\pi}{\ell}\,x\right).$$

When $k = -\left(\frac{n\pi}{\ell}\right)^2$, the second DE $\frac{1}{c^2}\frac{T''(t)}{T(t)} = k$ becomes $T''(t) + \left(\frac{cn\pi}{\ell}\right)^2 T(t) = 0$, which has solution $T = T_n(t) = a_n\cos(\frac{cn\pi}{\ell}\,t) + b_n\sin(\frac{cn\pi}{\ell}\,t)$. Thus, for each $n \in \mathbb{Z}^+$, the function

$$u_n(x,t) = X_n(x)T_n(t) = \left(a_n\cos\left(\tfrac{cn\pi}{\ell}t\right) + b_n\sin\left(\tfrac{cn\pi}{\ell}t\right)\right)\sin\left(\tfrac{n\pi}{\ell}x\right)$$

is a solution to the wave equation with $u(0,t) = 0$ and $u(\ell,t) = 0$. Finally, we let

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty}\left(a_n\cos\left(\tfrac{cn\pi}{\ell}t\right) + b_n\sin\left(\tfrac{cn\pi}{\ell}t\right)\right)\sin\left(\tfrac{n\pi}{\ell}x\right).$$

In order to get $u(x,0) = f(x)$ we need $\sum_{n=1}^{\infty} a_n\sin\left(\frac{n\pi}{\ell}x\right) = f(x)$, so the $a_n$ must be the Fourier coefficients of the odd $2\ell$-periodic function which is equal to $f(x)$ for $0 \leq x \leq \ell$. Also, we have $\frac{\partial u}{\partial t}(x,t) = \sum\limits_{n=1}^{\infty}\left(-\frac{cn\pi}{\ell}a_n\sin\left(\frac{cn\pi}{\ell}t\right) + \frac{cn\pi}{\ell}b_n\cos\left(\frac{cn\pi}{\ell}t\right)\right)\sin\left(\frac{n\pi}{\ell}x\right)$, so in order to get $\frac{\partial u}{\partial t}(x,0) = g(x)$ we need $\sum_{n=1}^{\infty}\frac{cn\pi}{\ell}b_n\sin\left(\frac{n\pi}{\ell}x\right) = g(x)$ so the constants $\frac{cn\pi}{\ell}b_n$ must be the Fourier coefficients of the odd $2\pi$-periodic function which is equal to $g(x)$ for $0 \leq x \leq \pi$. Using our formulas for Fourier coefficients, $a_n$ and $b_n$ are given by

$$a_n = \tfrac{2}{\ell}\int_0^\ell f(x)\sin\left(\tfrac{n\pi}{\ell}x\right)\,dx \quad , \text{ and } \quad b_n = \tfrac{2}{cn\pi}\int_0^\ell g(x)\sin\tfrac{n\pi}{\ell}x\,dx\,.$$

**4.11 Example:** Solve $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ for $u = u(x,t)$ with $x \in \mathbb{R}$ and $t \in \mathbb{R}$ satisfying $u(x,0) = f(x)$ and $\frac{\partial u}{\partial t}(x,0) = g(x)$ for all $x \in \mathbb{R}$.

Solution: Let $p, q : \mathbb{R} \to \mathbb{R}$ be any two twice differentiable functions. Note that $p(x+ct)$ and $q(x-ct)$ are solutions to the wave equation: for example, when $u(x,t) = q(x-ct)$ we have $\frac{\partial u}{\partial x} = q'(x-ct)$ and $\frac{\partial^2 u}{\partial t^2} = q''(x-ct)$, and $\frac{\partial u}{\partial t} = -c\,q'(x-ct)$ and $\frac{\partial^2 u}{\partial t^2} = c^2 q''(x-ct)$. Thus the function

$$u(x,t) = p(x+ct) + q(x-ct)$$

is a solution to the wave equation. To get $u(x,0) = f(x)$, we need $p(x) + q(x) = f(x)$. To get $\frac{\partial u}{\partial t}(x,0) = g(x)$ we need $c\,p'(x) - c\,q'(x) = g(x)$, that is $p'(x) - q'(x) = \frac{1}{c} g(x)$, and integrating both sides gives $p(x) - q(x) = \frac{1}{c}\int_{u=0}^{x} g(u)\,du + k$ for some constant $k$. Solving the two equations $p(x)+q(x) = f(x)$ and $p(x) - q(x) = \frac{1}{c}\int_{u=0}^{x} g(u)\,du + k$ gives $p(x) = \frac{1}{2}\big(f(x) + \frac{1}{c}\int_{u=0}^{x} g(u)\,du + k\big)$ and $q(x) = \frac{1}{2}\big(f(x) - \int_{u=0}^{x} g(u)\,du - k\big)$, so we obtain

$$\begin{aligned} u(x,t) &= p(x+ct) + q(x-ct) \\ &= \tfrac{1}{2}\Big(f(x+ct) + \int_0^{x+ct} g(u)\,du + k\Big) + \tfrac{1}{2}\Big(f(x-ct) - \int_0^{x-ct} g(u)\,du - k\Big) \\ &= \tfrac{1}{2}\big(f(x+ct) + f(x-ct)\big) + \tfrac{1}{2}\int_{x-ct}^{x+ct} g(u)\,du\,. \end{aligned}$$

This solution to the wave equation is known as **d'Alembert's solution**

**4.12 Remark:** The solutions $u(x,t)$ to the wave equation that we found in Example 4.10, satisfying the fixed endpoint conditions $u(0,t) = 0$ and $u(\ell,t) = 0$, are called **standing waves**, and the solutions $p(x+ct)$ and $q(x-ct)$ described in Example 4.10 are called **travelling waves**. It is interesting to note that a standing wave is equal to the sum of two travelling waves, moving in opposite direction. For example, for $p(x) = \frac{1}{2}\sin\big(\frac{n\pi}{\ell}x\big)$, we have

$$\begin{aligned} p(x+ct) + p(x-ct) &= \tfrac{1}{2}\Big(\sin\big(\tfrac{n\pi}{\ell}x + \tfrac{cn\pi}{\ell}t\big) + \sin\big(\tfrac{n\pi}{\ell}x - \tfrac{cn\pi}{\ell}t\big)\Big) \\ &= \tfrac{1}{2}\Big(\sin\big(\tfrac{n\pi}{\ell}x\big)\cos\big(\tfrac{cn\pi}{\ell}t\big) + \cos\big(\tfrac{n\pi}{\ell}x\big)\sin\big(\tfrac{cn\pi}{\ell}t\big) \\ &\qquad + \sin\big(\tfrac{n\pi}{\ell}x\big)\cos\big(\tfrac{cn\pi}{\ell}t\big) - \cos\big(\tfrac{n\pi}{\ell}x\big)\sin\big(\tfrac{cn\pi}{\ell}t\big)\Big) \\ &= \sin\big(\tfrac{n\pi}{\ell}x\big)\cos\big(\tfrac{cn\pi}{\ell}t\big). \end{aligned}$$

**4.13 Exercise:** Solve the heat equation $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$ for $u = u(x,t)$ with $0 \le x \le \ell$ and $t \ge 0$ satisfying the fixed endpoints temperature conditions $u(0,t) = 0$ and $u(\ell,t) = 0$ for all $t \ge 0$ and the initial condition $u(x,0) = f(x)$ for $0 \le x \le \ell$. You should find that the solution is given by $u(x,t) = \sum_{n=1}^{\infty} b_n e^{-(cn\pi/\ell)^2 t}\sin\big(\frac{n\pi}{\ell}x\big)$ where the $b_n$ are the Fourier coefficients of the *odd* $2\ell$-periodic function which is equal to $f(x)$ for $0 \le x \le \ell$.

**4.14 Exercise:** Solve the heat equation $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$ for $u = u(x,t)$ with $0 \le x \le \ell$ and $t \ge 0$ satisfying the insulated ends boundary conditions $u(0,t) = 0$ and $u(\ell,t) = 0$ for all $t \ge 0$ and the initial condition $u(x,0) = f(x)$ for $0 \le x \le \ell$. You should find that the solution is given by $u(x,t) = \sum_{n=0}^{\infty} a_n e^{-(cn\pi/\ell)^2 t}\cos\big(\frac{n\pi}{\ell}x\big)$ where the $a_n$ are the Fourier coefficients of the *even* $2\ell$-periodic function which is equal to $f(x)$ for $0 \le x \le \ell$.

**4.15 Example:** Solve Laplace's equation $\frac{\partial^2 u}{\partial x}^2 + \frac{\partial^2 v}{\partial y^2} = 0$ for $u = u(x,t)$ with $0 \leq x \leq 1$ and $0 \leq y \leq 1$ satisfying the boundary conditions $u(x,0) = 1 - x + f_1(x)$, $u(x,1) = x + f_2(x)$, $u(0,y) = 1 - y + f_3(y)$ and $u(1,y) = y + f_4(y)$, where the functions $f_k$ are continuous with $f_k(0) = 0$ and $f_k(1) = 1$.

Solution: First note that for any $a, b, c, d \in \mathbb{R}$, the function $v(x,y) = a + bx + cy + dxy$ is a solution to Laplace's equation and, given any values $p, q, r, s \in \mathbb{R}$ we can always choose $a, b, c, d$ so that $v(0,0) = p$, $v(0,1) = q$, $v(1,0) = r$ and $v(1,1) = s$. To get $v(0,0) = u(0,0) = 1$, $v(0,1) = u(0,1) = 0$, $v(1,0) = 0$ and $v(1,1) = 1$, we shall choose $v(x,y) = 1 - x - y + 2xy$ and note that $v(x,0) = 1 - x$, $v(x,1) = x$, $v(0,y) = 1 - y$ and $v(1,y) = y$. Note that a function $u = u(x,y)$ will satisfy Laplace's equation with the desired boundary conditions if and only if the function $w = w(x,y) = u(x,y) - v(x,y)$ satisfies Laplace's equation with the shifted boundary conditions $w(x,0) = f_1(x)$, $w(x,1) = f_2(x)$, $w(0,y) = f_3(y)$ and $w(1,y) = f_4(y)$. To find a solution $w = w(x,y)$, we shall solve 4 separate problems: we shall find 4 functions $w = w_k(x,y)$, each satisfying Laplace's equation, and each satisfying boundary conditions taking the value zero on 3 of the 4 sides. For example, we shall require that $w_3(x,y)$ satisfies the boundary conditions $w_3(x,0) = 0$, $w_3(x,1) = 0$, $w_3(0,y) = f_3(y)$ and $w_3(1,y) = 0$.

Let us find $w = w_3(x,y)$ satisfying Laplace's equation $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0$ and satisfying the boundary conditions $w_3(x,0) = 0$, $w_3(x,1) = 0$, $w_3(0,y) = f_3(y)$ and $w_3(1,y) = 0$. Let $w = X(x)Y(y)$. Laplace's equation becomes $X''Y + XY'' = 0$ which we can write as $\frac{X''}{X} = -\frac{Y''}{Y}$. For this to hold for all $x, y$, both sides must be constant, say $\frac{X''}{X} = k = -\frac{Y''}{Y}$. The boundary conditions become $X(x)Y(0) = 0$, $X(x)Y(1) = 0$, $X(0)Y(y) = f_3(y)$ and $X(1)Y(y) = 0$. For a nonzero solution, we need $Y(0) = 0$, $Y(1) = 0$ and $X(1) = 0$. First we consider the DE $\frac{Y''}{Y} = -k$ with $Y(0) = 0$ and $Y(1) = 0$. As with the wave equation, to have a nonzero solution we need $k$ to be of the form $k = (n\pi)^2$ and the solutions are given by $Y = Y_n = d_n \sin n\pi y$. In this case, the DE $\frac{X''}{X} = k$ becomes $X'' - (n\pi)^2 X = 0$ which has solutions $X = X_n = a_n\, e^{n\pi x} + b_n\, e^{-n\pi x}$. To get $X(1) = 0$ we need $a_n e^{n\pi} + b_n e^{-n\pi}$, that is $b_n = -a_n e^{2n\pi}$, so the solutions are given by $X = X_n = a_n(e^{n\pi x} - e^{2n\pi} e^{-n\pi x})$, which we can write as $X_n = c_n e^{n\pi} \sinh n\pi(1-x)$ with $c_n = -2a_n e^{n\pi}$. Thus for each $n \in \mathbb{Z}^+$ we have a solution $w = c_n \sinh(n\pi(1-x)) \sin(n\pi y)$. We add these and let

$$w = w_3(x,y) = \sum_{n=1}^{\infty} c_n \sinh(n\pi(1-x)) \sin(n\pi y).$$

In order to satisfy $w_3(0,y) = f_3(y)$, we need $\sum\limits_{n=1}^{\infty} c_n \sinh(n\pi) \sin(n\pi y) = f_3(y)$, we must choose the constants $c_n \sinh(n\pi)$ to be equal to the Fourier coefficients of the odd 2-periodic function which is equal to $f_3(y)$ for $0 \leq y \leq 1$, so we must choose

$$c_n = \frac{2}{\sinh(n\pi)} \int_{y=0}^{1} f_3(y) \sin(n\pi y)\, dy.$$

We leave it as an exercise to find formulas for the other 3 functions $w_k(x,y)$. Once these have been found, the solution to the given problem is $u = u(x,y) = v(x,y) + \sum_{k=1}^{4} w_k(x,y)$.

**4.16 Remark:** The problem of finding the solution $u = u(x,y)$ to Laplace's equation (or sometimes some other PDE) for all $(x,y)$ in region $U \subseteq \mathbb{R}^2$, which takes prescribed values on the boundary of $U$, is called the **Dirichlet problem**.

## Sturm-Liouville Problems

**4.17 Remark:** When we solved the wave, heat, and Laplace equations using separation of variables, we obtained a linear homogeneous DE in one of the variables, involving a constant $k$, together with boundary conditions, which were such that nonzero solutions only existed for certain specific values of $k$, and for each such $k$, the solution was unique up to multiplication by a constant. These boundary value problems are a special case of the following more general class of boundary value problems.

**4.18 Definition:** A **Sturm-Liouville problem** consists of a second order linear homogeneous ODE of the form

$$\tfrac{d}{dx}\big(p(x)y'(x)\big) + \big(q(x) + k\,r(x)\big)y(x) = 0$$

for $y = y(x)$ with $0 \leq x \leq \ell$, where $k \in \mathbb{R}$ and $p, q, r : [0, \ell] \to \mathbb{R}$ with $q$ and $r$ and $p'$ continuous, and with $p(x) > 0$ and $r(x) > 0$ for all $x \in [0, \ell]$, together with boundary conditions of the form $ay(0) + by'(0) = 0$ and $cy(\ell) + dy'(\ell) = 0$ where $a, b, c, d \in \mathbb{R}$ with $(a, b) \neq (0, 0)$ and $(c, d) \neq (0, 0)$.

**4.19 Remark:** Each Sturm-Liouville problem has the following special features. Nonzero solutions only exist for certain values of $k$, and these values of $k$ can be arranged to form a sequence $k_1 < k_2 < k_3 < \cdots$ with $\lim\limits_{n\to\infty} k_n = \infty$. For each of these values $k_n$, there is a corresponding solution $y = y_n(x)$, which is unique up to multiplying by a constant. The values $k_n$ are called the **eigenvalues** and the corresponding solutions $y_n$ are called the **eigenfunctions**. When $n, m \in \mathbb{Z}^+$ with $n \neq m$, we have $\int_0^\ell r(x)y_n(x)y_m(x)\,dx$. In other words, the set $\{y_1, y_2, y_3, \cdots\}$ is an orthogonal set using the inner product given by

$$\langle f, g\rangle = \int_0^\ell r(x)f(x)g(x)\,dx.$$

The eigenfunctions can be used in the same way as the trigonometric functions $\sin\frac{n\pi x}{\ell}$ and $\cos\frac{n\pi x}{\ell}$ are used to form Fourier series: given an integrable function $g : [0, \ell] \to \mathbb{R}$ we can form the series $\sum_{n=1}^\infty c_n y_n(x)$ with partial sums $s_m(g) = \sum_{n=1}^m c_n y_n(x)$, where $c_n = c_n(g) = \frac{\langle g, y_n\rangle}{\langle y_n, y_n\rangle}$ and then we will have $\lim\limits_{m\to\infty} \|s_m(g) - g\| = 0$, where $\|f\| = (\langle f, f\rangle)^{1/2}$.

**4.20 Example:** The boundary value problem given by $X''(x) + kX(x) = 0$ with $X(0) = 0$ and $X(\ell) = 0$ is a Sturm-Liouville problem for $X = X(x)$ with $0 \leq x \leq \ell$, as we can see by taking $p = 1$, $q = 0$ and $r = 1$, and $(a, b) = (c, d) = (1, 0)$ in Definition 4.18.

**4.21 Example:** Legendre's equation $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$ can be written as $((1 - x^2)y')' + n(n + 1)y = 0$, which is in the Sturm-Liouville form with $p(x) = 1 - x^2$, $q(x) = 0$, $r(x) = 1$, and with $k = n(n + 1)$. We remark that the Legendre polynomials are orthogonal using the inner product $\langle f, g\rangle = \int_{-1}^1 f(x)g(x)\,dx$.