Part 3. Power Series and Fourier Series for ODEs

3.1 Definition: A function $f : U \subseteq \mathbb{R} \to \mathbb{R}$ is **analytic** at $a \in U$ when there exists $R > 0$ with $(a-R, a+R) \subseteq U$ such that $f(x)$ is equal to the sum of its Taylor series centred at a for all $|x-a| < R$, that is when

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

for all x with $|x-a| < R$, where $c_n = \frac{f^{(n)}(a)}{n!}$.

3.2 Example: The functions e^x, $\sin x$, $\cos x$, $\sinh x$, $\cosh x$, are all analytic at 0 with

$$e^x = \sum_{n=0}^{\infty} \tfrac{1}{n!} x^n = 1 + x + \tfrac{1}{2!}x^2 + \tfrac{1}{3!}x^3 + \tfrac{1}{4!}x^4 + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} \tfrac{(-1)^n}{(2n+1)!} x^{2n+1} = x = \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \tfrac{1}{7!}x^7 + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} \tfrac{(-1)^n}{(2n)!} x^{2n} = 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 = \tfrac{1}{6!}x^6 + \cdots$$

$$\sinh x = \sum_{n=0}^{\infty} \tfrac{1}{(2n+1)!} x^{2n+1} = x + \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 + \cdots$$

$$\cosh x = \sum_{n=0}^{\infty} \tfrac{1}{(2n)!} x^{2n} = 1 + \tfrac{1}{2}x^2 + \tfrac{1}{4!}x^4 + \cdots$$

for all $x \in \mathbb{R}$, and the functions $\frac{1}{1-x}$ and $(1+x)^p$ where $p \in \mathbb{R}$, are analytic at 0 with

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

$$(1+x)^p = \sum_{n=0}^{\infty} \tbinom{p}{n} x^n = 1 + px + \tfrac{p(p-1)}{2!}x^2 + \tfrac{p(p-1)(p-2)}{3!}x^3 + \cdots$$

for all $x \in \mathbb{R}$ with $|x| < 1$, where $\binom{p}{0} = 1$ and $\binom{p}{n} = \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}$.

3.3 Example: Analytic functions can be added, subtracted, multiplied, divided, composed, differentiated and integrated as if they were polynomials. For example, for $|x| < 1$ we have

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \cdots$$

$$\arctan x = \sum_{n=-0}^{\infty} \tfrac{(-1)^n}{(2n+1)} x^{2n+1} = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \cdots$$

3.4 Theorem: *If the functions $r(x), p_0(x), p_1(x), \cdots, p_{n-1}(x)$ are all analytic at $a \in U$ and are all equal to the sum of their Taylor series for $|x-a| < R$ where $R > 0$, then for all $b_0, b_1, b_2, \cdots, b_{n-1} \in \mathbb{R}$, the unique solution $y = y(x)$ to the IVP given by*

$$y^{(n)} = p_{n-1}y^{(n-1)} + \cdots + p_1 y' + p_0 = r \ \text{, with}$$
$$y(a) = b_0, y'(a) = b_1, y''(a) = b_2, \cdots, y^{(n-1)}(a) = b_{n-1}$$

is also analytic at a and equal to the sum of its Taylor series converges for $|x-a| < R$.

Proof: We omit the proof.

3.5 Exercise: Solve the first order ODE $y' = 2y$ using power series (centred at 0).

3.6 Exercise: Find the Taylor polynomial of degree 5, centred at 0, for the solution to the IVP given by $y' + e^{2x}y = 3x$ with $y(0) = 1$.

3.7 Exercise: Use power series (centred at 0) to solve the ODE $(1+x^2)y'' + 3xy' + y = 0$. Find an explicit, closed form, formula for one solution. For an optional challenge, find an explicit, closed form, formula for two independent solutions.

3.8 Exercise: A number of differential equations, which are named after various mathematician's, involve a parameter $k \in \mathbb{R}$, and admit polynomial solutions when the parameter is a positive integer. Solve some of the following ODEs and determine the polynomial solutions (which are named after the same mathematician).

Hermite's Equation:	$y'' - 2xy' + 2ky = 0$
Chebyshev's Equation:	$(1 - x^2)y'' - xy' + k^2y = 0$
Legendre's Equation:	$(1 - x^2)y'' - 2xy' + k(k + 1)y = 0$

Frobenius' Method

3.9 Exercise: Solve the **Cauchy-Euler Equation**, which is given by $x^2y''+kxy'+\ell y = 0$ for $x > 0$, where $k, \ell \in \mathbb{R}$, by looking for a solution of the form $y(x) = x^r$ or, alternatively, by making the substitution $t = \ln x$.

3.10 Definition: For the second order homogeneous linear ODE $y'' + p(x)y' + q(x)y = 0$, we say that the point $a \in \mathbb{R}$ is an **ordinary point** of the ODE when $p(x)$ and $q(x)$ are both analytic at a, and otherwise we say that a is a **singular point** of the ODE. For a singular point $a \in \mathbb{R}$, we say that a is a **regular singular point** of the ODE when $(x - a)p(x)$ and $(x - a)^2q(x)$ are both analytic at a, and otherwise we say that a is an **irregular singular point** of the ODE.

3.11 Theorem: *(Frobenius) If $(x - a)p(x)$ and $(x - a)^2q(x)$ are both analytic at a, then the homogeneous linear ODE $y'' + p(x)y' + q(x) = 0$ has at least one solution of the form $y = y(x) = x^r f(x)$ for some $r \in \mathbb{R}$ and some function $f(x)$ which is analytic at a.*

Proof: We omit the proof.

3.12 Note: To solve an ODE $y'' + p(x)y' + q(x) = 0$, as in the above theorem, we can try $y = x^r f(x)$ with $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$. This method is known as **Frobenius' method**.

3.13 Exercise: Use Frobenius' method to solve the ODE $2x^2y'' - xy' + (1 + x)y = 0$. Find explicit, closed form formulas for two independent solutions.

3.14 Exercise: Use Frobenius' method to solve the ODE $xy'' + 2y' + xy = 0$. Find explicit, closed form formulas for two independent solutions.

3.15 Exercise: Solve **Laguerre's Equation**, given by $xy''+(1-x)y'+ky = 0$ with $k\in\mathbb{R}$, and find the polynomial solutions when $k\in\mathbb{Z}^+$ (which are called Laguerre polynomials).

3.16 Exercise: Solve **Bessel's Equation** $x^2y'' + xy' + (x^2 - k^2) = 0$, where $k\in\mathbb{R}$.

Fourier Series

3.17 Definition: A real **trigonometric polynomial** is a 2π-periodic function $f:\mathbb{R}\to\mathbb{R}$ of the form

$$f(x)=a_0+\sum_{n=1}^{m}\big(a_n\cos nx+b_n\sin nx\big)$$

where $m\in\mathbb{Z}^+$ and $a_n,b_n\in\mathbb{R}$, and if $a_m\neq 0$ or $b_m\neq 0$, we say that $f(x)$ is of **degree** m. A real **trigonometric series** is a series of the form

$$a_0+\sum_{n=1}^{\infty}\big(a_n\cos nx+b_n\sin nx\big)$$

where $a_n,b_n\in\mathbb{R}$; this is the series whose m^{th} partial sum is the trigonometric polynomial

$$s_m(x)=a_0+\sum_{n=1}^{m}\big(a_n\cos nx+b_n\sin nx\big).$$

We say that the series **converges** at x when its sequence of partial sums converges at x, and when the series converges at every x, its sum is the 2π-periodic function $f:\mathbb{R}\to\mathbb{R}$ given by $f(x)=\lim_{m\to\infty}s_m(x)$, and we write

$$f(x)=a_0+\sum_{n=1}^{\infty}\big(a_n\cos nx+b_n\sin nx\big).$$

When $f:\mathbb{R}\to\mathbb{R}$ is a trigonometric polynomial, we can determine the coefficients a_n and b_n with the help of the following formulas, where n,m are non-negative integers:

$$\int_{-\pi}^{\pi}1=2\pi\ ,\ \int_{-\pi}^{\pi}\cos^2 nx\,dx=\pi\ ,\ \int_{-\pi}^{\pi}\sin^2 nx\,dx=\pi$$

$$\int_{-\pi}^{\pi}\cos nx\,dx=0\ ,\ \int_{-\pi}^{\pi}\sin nx\,dx=0\ ,\ \text{ and }\int_{-\pi}^{\pi}\cos nx\,\sin mx\,dx=0$$

and if $n\neq m$ then

$$\int_{-\pi}^{\pi}\cos nx\,\cos mx\,dx=0\text{ and }\int_{-\pi}^{\pi}\sin nx\,\sin mx\,dx=0.$$

Using these formulas, it is not hard to show that when $f(x)$ is a trigonometric polynomial, the coefficients are given by

$$a_0=\tfrac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,dx\ ,\ a_n=\tfrac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx\ ,\text{ and }b_n=\tfrac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx.$$

Given any integrable 2π-periodic function $f:\mathbb{R}\to\mathbb{R}$, or given any integrable function $f:[-\pi,\pi]\to\mathbb{R}$, we define the (real) **Fourier coefficients** of f to be

$$a_0(f)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,dx\ ,\ a_n(f)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx\ ,\text{ and }b_n(f)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx,$$

and the (real) **Fourier series** of f is the trigonometric series

$$a_0(f)+\sum_{n=1}^{\infty}\big(a_n(f)\cos nx+b_n(f)\sin nx\big)$$

and the m^{th} partial sums of the Fourier series are the trigonometric polynomials

$$s_m(f)(x)=a_0(f)+\sum_{n=1}^{m}\big(a_n(f)\cos nx+b_n(f)\sin nx\big).$$

3.18 Theorem: *(Convergence of Fourier Series) Let $f:\mathbb{R}\to\mathbb{R}$ be a 2π-periodic function, or let $f:[-\pi,\pi]\to\mathbb{R}$ be a function. If f is continuously differentiable, then at all points $x\in\mathbb{R}$, the Fourier series of f converges to $f(x)$, so we have*

$$f(x)=a_0(f)+\sum_{n=1}^{\infty}\big(a_n(f)\cos nx+b_n(f)\sin nx\big).$$

More generally, if f is piecewise continuously differentiable, and we have a partition of $[-\pi,\pi]$ given by $-\pi=p_0<p_1<\cdots<p_\ell=\pi$, and we have continuously differentiable functions $g_k:[p_{k-1},p_k]\to\mathbb{R}$ such that $f(x)=p_k(x)$ for all $x\in(p_{k-1},p_k)$, then when $x\in(p_{k-1},p_k)$, the Fourier series of f converges to $f(x)=g_k(x)$, and when $x=p_k$ with $1<k<\ell$, the Fourier series of f converges to the midpoint $\frac{1}{2}(g_{k-1}(p_k)+g_k(p_k))$, and when $x=\pm\pi$, the Fourier series of f converges to the midpoint $\frac{1}{2}(g_1(-\pi)+g_\ell(\pi))$.

3.19 Exercise: *Find the Fourier coefficients of the 2π-periodic function $f:\mathbb{R}\to\mathbb{R}$ with*

$$f(x)=\left\{\begin{array}{ll}\frac{\pi}{2}+x \text{ for } & -\pi\le x\le 0\\ \frac{\pi}{2}-x \text{ for } & 0\le x\le\pi\end{array}\right\}.$$

By evaluating the Fourier series at $x=0$, show that $\sum\limits_{k=1}^{\infty}\frac{1}{(2k+1)^2}=\frac{\pi^2}{8}$.

3.20 Definition: *Let L_2 denote the (infinite dimensional) vector space of integrable 2π-periodic functions $f:\mathbb{R}\to\mathbb{R}$, or equivalently integrable functions $f:[-\pi,\pi]\to\mathbb{R}$, which are* **square integrable**, *meaning that $\int_{-\pi}^{\pi}f(x)^2dx<\infty$, and where we consider two functions f and g to be equal when $\int_{-\pi}^{\pi}(f(x)-g(x))^2dx=0$ (so that, for example, two piecewise continuously differentiable functions are considered to be equal in L_2 when they differ at finitely many points). For $f,g\in L_2$, we define the* **inner-product** *of f and g to be*

$$\langle f,g\rangle=\int_{-\pi}^{\pi}f(x)g(x)\,dx.$$

Using the inner product, we define the **norm** *of $f\in L_2$ to be*

$$\|f\|=\sqrt{\langle f,f\rangle}=\Big(\int_{-\pi}^{\pi}f(x)^2\,dx\Big)^{1/2}$$

and we define the **distance** *between f and g to be*

$$d(f,g)=\|f-g\|,$$

and when $f,g\neq 0$ we define the **angle** *between f and g to be*

$$\theta(f,g)=\cos^{-1}\frac{\langle f,g\rangle}{\|f\|\,\|g\|}.$$

For $f,g\in L_2$, we say that f and g are **orthogonal** *when $\langle f,g\rangle=0$. A set of functions $S\subseteq L_2$ is called* **orthonormal** *when for all $f\in S$ we have $\|f\|=1$ and for all $f,g\in S$ with $f\neq g$ we have $\langle f,g\rangle=0$. Note that the infinite set*

$$\left\{\tfrac{1}{\sqrt{2\pi}}\,,\ \tfrac{1}{\sqrt{\pi}}\cos nx\,,\ \tfrac{1}{\sqrt{\pi}}\sin nx\,\middle|\,n\in\mathbb{Z}^+\right\}$$

is an orthonomal set in L_2. If we fix $m\in\mathbb{Z}^+$, then the finite set

$$\left\{\tfrac{1}{\sqrt{2\pi}}\,,\ \tfrac{1}{\sqrt{\pi}}\cos nx\,,\ \tfrac{1}{\sqrt{\pi}}\sin nx\,\middle|\,1\le n\le m\right\}$$

is an orthonormal basis for the $(2m+1)$-dimensional space of trigonometric polynomials of degree at most m.

3.21 Theorem: *Let $f, g \in L_2$. Then*

(1) The m^{th} partial sum $s_m(f)$ of the Fourier series of f is the unique trigonometric polynomial of degree at most m which is nearest to f, that is $\|s_m(f) - f\| \leq \|p - f\|$ for every trigonometric polynomial p of degree at most m.

(2) We have $\lim\limits_{m\to\infty} \big\|s_m(f) - f\big\| = 0$.

(3) We have $\big\langle f, g \big\rangle = 2\pi\, a_0(f) a_0(g) + \pi \sum\limits_{n=1}^{\infty} \big(a_n(f)a_n(g) + b_n(f)b_n(g)\big)$.

(4) (Parseval's Identity) We have $\|f\|^2 = 2\pi\, a_0(f)^2 + \pi \sum\limits_{n=1}^{\infty} \big(a_n(f)^2 + b_n(f)^2\big)$.

Proof: We omit the proof. You will prove some of it in your next linear algebra course.

3.22 Exercise: Use Parseval's Identity on the function f from Exercise 3.19 to prove that $\sum\limits_{k=0}^{\infty} \frac{1}{(2k+1)^4} = \frac{\pi^4}{96}$, then use this result to calculate $\sum\limits_{n=1}^{\infty} \frac{1}{n^4}$.

3.23 Example: (Forced Damped Oscillations) Suppose an object of mass m is attached to a spring of spring-constant k and vibrates in a fluid of damping-constant c and let $x = x(t)$ be the displacement of the object from its equilibrium position at time t. Suppose, in addition, that the object is acted on by an external force $f(t)$. The total force $F(t)$ acting on the object consists of the force exerted by the spring, which is equal to $-kx(t)$, the resistive force exerted by the fluid, which is equal to $-cx'(t)$, and the external driving force, which is equal to $f(t)$. By Newton's Second Law of motion we have $F(t) = mx''(t)$ and so $x(t)$ satisfies the ODE

$$mx''(t) + cx'(t) + kx(t) = f(t).$$

3.24 Exercise: Use Fourier series to solve the above DE with $m = 1$, $c = 2$ and $k = 5$, and where $f(t)$ is the function from Exercise 3.19.

3.25 Definition: Often, we are interested in periodic functions whose period is not 2π. For an integrable 2ℓ-periodic function $f : \mathbb{R} \to \mathbb{R}$, or for an integrable function $f : [-\ell, \ell] \to \mathbb{R}$, the **Fourier coefficients** of f are given by

$$a_0(f) = \tfrac{1}{2\ell}\int_{-\ell}^{\ell} f(x)\, dx \ ,\ a_n(f) = \tfrac{1}{\ell}\int_{-\ell}^{\ell} f(x) \cos\left(\tfrac{\pi n}{\ell}\, x\right) dx \ ,\ b_n(f) = \tfrac{1}{\ell}\int_{-\ell}^{\ell} f(x) \sin\left(\tfrac{n\pi}{\ell}\, x\right) dx$$

and the **Fourier series** of f is the series

$$a_0(f) + \sum_{n=1}^{\infty} \left(a_n(f) \cos\left(\tfrac{n\pi}{\ell}\, x\right) + b_n(f) \sin\left(\tfrac{n\pi}{\ell}\, x\right)\right)$$

and the m^{th} partial sum of the Fourier series is given by

$$s_m(f)(x) = a_0(f) + \sum_{n=1}^{m} \left(a_n(f) \cos\left(\tfrac{n\pi}{\ell}\, x\right) + b_n(f) \sin\left(\tfrac{n\pi}{\ell}\, x\right)\right).$$

3.26 Exercise: Theorems 3.18 and 3.21 hold, mutatis mutandis, for Fourier series of 2ℓ-periodic functions. As an exercise, determine the modified version of Parseval's Identity.