SYDE Advanced Math 2, Solutions for Practice Problem Set 6

1: Let $F(x,y)=\begin{pmatrix} f(x,y)\\ g(x,y)\end{pmatrix}$ where $f(x,y)=x^2+y^2-5$ and $g(x,y)=x^3+y^3-2$.

(a) Sketch the curves $f(x,y)=0$ and $g(x,y)=0$ on the same grid, and use your picture to approximate the coordinates of the points of intersection of the two curves $\big($that is the points (x,y) such that $F(x,y)=\binom{0}{0}\big)$.

Solution: We have $f(x,y)=0$ when $x^2+y^2=5$, that is when (x,y) lie on the circle centred at $(0,0)$ of radius 5. We have $g(x,y)=0$ when $y^3=2-x^3$, that is when $y=\sqrt[3]{2-x^3}$, and we can sketch this curve by plotting points (with the help of a calculator). We sketch the two curves below. From the sketch it appears that there are two points of intersection with approximate coordinates $(x,y)=(1.7,-0.4),(-1.4,1.7)$.

(b) Find a more accurate approximation for the coordinates of the lowest point of intersection as follows: Starting with $a_0=\binom{x_0}{y_0}=\binom{2}{-1}$, carry out two iterations of Newton's method. Calculate the first iteration to find $a_1=\binom{x_1}{y_1}$ by hand, expressing x_1 and y_1 as fractions, then carry out the second iteration to find $a_2=\binom{x_2}{y_2}$ with the help a calculator (or computer).

Solution: We have $F\binom{x}{y}=\begin{pmatrix} x^2+y^2-5\\ x^3+y^3-2\end{pmatrix}$, $DF\binom{x}{y}=\begin{pmatrix} 2x & 2y\\ 3x^2 & 3y^2\end{pmatrix}$, and $DF\binom{x}{y}^{-1}=\frac{1}{6xy^2-6x^2y}\begin{pmatrix} 3y^2 & -2y\\ 3x^2 & 2x\end{pmatrix}$. Starting with $\binom{x_0}{y_0}=\binom{2}{-1}$, the first iteraration of Newton's metod gives

$$\binom{x_1}{y_1}=\binom{2}{-1}-DF\binom{2}{-1}^{-1}F\binom{2}{-1}=\begin{pmatrix}2\\-1\end{pmatrix}-\frac{1}{36}\begin{pmatrix}3 & 2\\ -12 & 4\end{pmatrix}\begin{pmatrix}0\\5\end{pmatrix}=\begin{pmatrix}2\\-1\end{pmatrix}-\frac{1}{18}\begin{pmatrix}5\\10\end{pmatrix}=\frac{1}{18}\begin{pmatrix}31\\-28\end{pmatrix}.$$

Using a calculator, we have

$$\begin{pmatrix}x_2\\y_2\end{pmatrix}=\begin{pmatrix}\frac{31}{18}\\-\frac{14}{9}\end{pmatrix}-\frac{1}{6(\frac{31}{18})(\frac{14}{9})^2+6(\frac{31}{18})^2(\frac{14}{9})}\begin{pmatrix}3(\frac{14}{9})^2 & 2(\frac{14}{9})\\ -3(\frac{31}{18})^2 & 2(\frac{31}{18})\end{pmatrix}\begin{pmatrix}(\frac{31}{18})^2+(\frac{14}{9})^2-5\\ (\frac{31}{18})^3-(\frac{14}{9})^3-2\end{pmatrix}=\begin{pmatrix}\frac{28112}{16461}\\-\frac{86087}{59472}\end{pmatrix}.$$

2: (a) Show that when $n = 2$, the open Newton-Cotes rule is given by

$$I_2^o = \tfrac{b-a}{3}\big(2f(x_0) - f(x_1) + 2f(x_2)\big) \quad , \text{ where } \; x_k = a + (k+1)\,\tfrac{b-a}{4}.$$

Solution: For the interval $[0,4]$ we take $x_0 = 1$, $x_1 = 2$ and $x_2 = 3$, and we take $g_0(x) = \frac{(x-2)(x-3)}{(1-2)(1-3)} = \frac{1}{2}(x-2)(x-3)$, $g_1(x) = \frac{(x-1)(x-3)}{(2-1)(2-3)} = -(x-1)(x-3)$ and $g_2(x) = \frac{(x-1)(x-2)}{(3-1)(3-2)} = \frac{1}{2}(x-1)(x-2)$, and we take

$$\begin{aligned}
w_0 &= \int_0^4 g_0(x)\,dx = \tfrac{1}{2}\int_0^4 x^2 - 5x + 6 \; dx = \tfrac{1}{2}\Big[\tfrac{1}{3}x^3 - \tfrac{5}{2}x^2 + 6x\Big]_0^4 = 2\big(\tfrac{16}{4} - 10 + 6\big) = \tfrac{8}{3},\\
w_1 &= \int_0^4 g_1(x)\,dx = -\int_0^4 x^2 - 4x + 3 \; dx = -\Big[\tfrac{1}{3}x^3 - 2x^2 + 3\Big]_0^4 = -4\big(\tfrac{16}{3} - 8 + 3\big) = -\tfrac{4}{3} \text{ , and}\\
w_2 &= \int_0^4 g_2(x)\,dx = \tfrac{1}{2}\int_0^4 x^2 - 3x + 2 \; dx = \tfrac{1}{2}\Big[\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 + 2x\Big]_0^4 = 2\big(\tfrac{16}{3} - 6 + 2\big) = \tfrac{8}{3}.
\end{aligned}$$

For the interval $[a,b]$, we scale horizontally by $\frac{b-a}{4}$ and take $x_0 = a + \frac{1}{4}(b-a) = \frac{3a+b}{4}$, $x_1 = a + \frac{1}{2}(b-a) = \frac{a+b}{2}$ and $x_2 = a + \frac{3}{4}(b-a) = \frac{a+3b}{4}$, and we take $w_0 = w_2 = \frac{8}{3}\frac{b-a}{4} = \frac{2(b-a)}{3}$ and $w_1 = -\frac{4}{3}\frac{b-a}{4} = -\frac{b-a}{3}$, and we make the approximation

$$\int_a^b f(x)\,dx \cong I_2^o(f) = \tfrac{b-a}{3}\Big(2f\big(\tfrac{3a+b}{4}\big) - f\big(\tfrac{a+b}{2}\big) + 2f\big(\tfrac{a+3b}{4}\big)\Big).$$

(b) Show that when $n = 4$, the closed Newton-Cotes rule is given by

$$I_4^c = \tfrac{b-a}{90}\big(7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)\big) \quad , \text{ where } \; x_k = a + k\,\tfrac{b-a}{4}.$$

Solution: We could solve this using the same method used in Part (a), but we choose to present a different solution. For the interval $[-2,2]$ we take $x_0 = -2$, $x_1 = -1$, $x_2 = 0$, $x_3 = 1$ and $x_4 = 2$, and then we choose w_0, w_1, w_2, w_3, w_4 so that $\int_0^4 p(x)\,dx = \sum_{k=0}^{w} w_k p(x_k)$ for each of the polynomials $p(x) \in \{1, x, x^2, x^3, x^4\}$. Taking $p(x) = 1$ gives the requirement $w_0 + w_1 + w_2 + w_3 + w_4 = \int_{-2}^2 1\,dx = 4$ (1). Taking $p(x) = x$ gives $-2w_0 - w_1 + w_3 + 2w_4 = \int_{-2}^2 x\,dx = 0$ (2). Taking $p(x) = x^2$ gives $4w_0 + w_1 + w_3 + 4w_4 = \int_{-2}^2 x^2\,dx = \frac{16}{3}$ (3). Taking $p(x) = x^3$ gives $-8w_0 - w_1 + w_3 + 8w_4 = \int_{-2}^2 x^3\,dx = 0$ (4). Taking $p(x) = x^4$ gives the requirement $16w_0 + w_1 + w_3 + 16w_4 = \int_{-2}^2 x^4\,dx = \frac{64}{5}$ (5). We solve these 5 equations using linear algebra:

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 4\\ -2 & -1 & 0 & 1 & 2 & 0\\ 4 & 1 & 0 & 1 & 4 & \frac{16}{3}\\ -8 & -1 & 0 & 1 & 8 & 0\\ 16 & 1 & 0 & 1 & 16 & \frac{64}{5} \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 4\\ 0 & 1 & 2 & 3 & 4 & 8\\ 0 & 3 & 4 & 3 & 0 & \frac{32}{3}\\ 0 & 7 & 8 & 9 & 16 & 32\\ 0 & 15 & 16 & 15 & 0 & \frac{256}{5} \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 0 & -1 & -2 & -3 & -4\\ 0 & 1 & 2 & 3 & 4 & 8\\ 0 & 0 & 2 & 6 & 12 & \frac{40}{3}\\ 0 & 0 & 6 & 12 & 12 & 24\\ 0 & 0 & 14 & 30 & 60 & \frac{344}{5} \end{array}\right)$$

$$\sim \left(\begin{array}{ccccc|c} 1 & 0 & -1 & -2 & -3 & -4\\ 0 & 1 & 2 & 3 & 4 & 8\\ 0 & 0 & 1 & 272 & & 4\\ 0 & 0 & 1 & 3 & 6 & \frac{20}{3}\\ 0 & 0 & 7 & 15 & 30 & \frac{172}{5} \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & 0\\ 0 & 1 & 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 2 & 2 & 4\\ 0 & 0 & 0 & 1 & 4 & \frac{8}{3}\\ 0 & 0 & 0 & 6 & 12 & \frac{184}{15} \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0 & 4 & 0\\ 0 & 0 & 1 & 0 & -6 & 4\\ 0 & 0 & 0 & 1 & 4 & \frac{8}{3}\\ 0 & 0 & 0 & 0 & 1 & \frac{14}{15} \end{array}\right)$$

$$\sim \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & \frac{14}{45}\\ 0 & 1 & 0 & 0 & 0 & \frac{64}{45}\\ 0 & 0 & 1 & 0 & 0 & \frac{24}{45}\\ 0 & 0 & 0 & 1 & 0 & \frac{64}{45}\\ 0 & 0 & 0 & 0 & 1 & \frac{14}{45} \end{array}\right)$$

Thus for the interval $[-2,2]$ we take $w_0 = w_4 = \frac{14}{45}$ and $w_1 = w_3 = \frac{64}{45}$ and $w_2 = \frac{24}{45}$. For the interval $[a,b]$, we shift, and scale horizontally by $\frac{(b-a)}{4}$ and take $x_0 = a$, $x_1 = \frac{3a+b}{4}$, $x_2 = \frac{a+b}{2}$, $x_3 = \frac{a+3b}{4}$ and $x_4 = b$, and we take $w_0 = w_4 = \frac{7}{90}(b-a)$, $w_1 = w_3 = \frac{32}{90}(b-a)$ and $w_2 = \frac{12}{90}(b-a)$. Thus we make the approximation

$$\int_a^b f(x)\,dx \cong I_4^c(f) = \tfrac{b-a}{90}\Big(7f(a) + 32f\big(\tfrac{3a+b}{4}\big) + 12f\big(\tfrac{a+b}{2}\big) + 32f\big(\tfrac{a+3b}{4}\big) + 7f(b)\Big).$$

3: (a) Show that when $n = 3$, the Gaussian quadrature rule on the interval $[-1, 1]$ is given by

$$\int_{-1}^{1} f(x)\,dx \cong I_3^g = \tfrac{5}{9}\, f\big(-\tfrac{\sqrt{3}}{\sqrt{5}}\big) + \tfrac{8}{9}\, f(0) + \tfrac{5}{9}\, f\big(\tfrac{\sqrt{3}}{\sqrt{5}}\big).$$

Solution: For the interval $[-1, 1]$, we need to choose x_1, x_2, x_3 and w_1, w_2, w_3 with $-1 \le x_1 < x_2 < x_3 \le 1$ such that for $p(x) \in \{1, x, x^2, x^3, x^4, x^5\}$ we have $\int_{-1}^{1} p(x)\,dx = \sum_{k=1}^{3} w_k p(x_k)$. Taking $p(x) = 1$ gives the requirement $w_1+w_2+w_3 = \int_{-1}^{1} 1\,dx = 2$ (1). Taking $p(x) = x^2$ gives $w_1x_1^2+w_2x_2^2+w_3x_3^2 = \int_{-1}^{1} x^2\,dx = \frac{2}{3}$ (2). Taking $p(x) = x^4$ gives the requirement $w_1x_1^4 + w_2x_2^4 + w_3x_3^4 = \int_{-1}^{1} x^4\,dx = \frac{2}{5}$ (3). Using symmetry, we can assume that $-1 < x_1 < 0$, $x_2 = 0$ and $0 < x_3 < 1$ with $x_1 = -x_3$, and that $w_1 = w_3$. With these assumptions, the above three equations become $w_2 + 2w_3 = 2$ (1), $2w_3x_3^2 = \frac{2}{3}$ (2) and $2w_3x_3^4 = \frac{2}{3}$ (3). Divide both sides of (3) by the corresponding sides of (2) to get $x_3^2 = \frac{3}{5}$ so that $x_3 = \frac{\sqrt{3}}{\sqrt{5}}$. Then (2) gives $w_3 = \frac{1}{3x_3^2} = \frac{5}{9}$ and (1) gives $w_2 = 2 - w_3 = 2 - \frac{5}{9} = \frac{8}{9}$. Thus we have $x_1 = -\frac{\sqrt{3}}{\sqrt{5}}$, $x_2 = 0$ and $x_3 = \frac{\sqrt{3}}{\sqrt{5}}$ and $w_1 = w_3 = \frac{5}{9}$ and $w_2 = \frac{8}{9}$, as required.

(b) Find the Gaussian quadrature rule, for $n = 3$, on the interval $[a, b]$.

Solution: For the interval $[a, b]$, we shift and scale horizontally by $\frac{b-a}{2}$ to get $x_1 = \frac{a+b}{2} - \frac{\sqrt{3}(b-a)}{2\sqrt{5}}$, $x_2 = \frac{a+b}{2}$ and $x_3 = \frac{a+b}{2} + \frac{\sqrt{3}(b-a)}{2\sqrt{5}}$, and $w_1 = w_3 = \frac{5}{18}(b-a)$ and $w_2 = \frac{4}{9}(b-a)$. Thus we obtain the quadrature rule

$$\int_a^b f(x)\,dx \cong I_3^g(f) = \tfrac{b-a}{18}\Big(5f\big(\tfrac{a+b}{2} - \tfrac{\sqrt{3}(b-a)}{2\sqrt{5}}\big) + 8f\big(\tfrac{a+b}{2}\big) + 5f\big(\tfrac{a+b}{2} + \tfrac{\sqrt{3}(b-a)}{2\sqrt{5}}\big)\Big).$$

4: Approximate the value of $\ln 2 = \int_1^2 \frac{1}{x}\,dx$ using the open Newton-Cotes rule for $n = 2$, using the closed Newton-Cotes rule for $n = 4$, and using the Gaussian quadrature rule for $n = 3$.

Solution: For the function $f(x) = \frac{1}{x}$ on the interval $[1, 2]$, we have

$$I_2^o(f) = \tfrac{1}{3}\Big(2f\big(\tfrac{5}{4}\big) - f\big(\tfrac{3}{2}\big) + 2f\big(\tfrac{7}{4}\big)\Big) = \tfrac{1}{3}\big(\tfrac{8}{5} - \tfrac{2}{3} + \tfrac{8}{7}\big) = \tfrac{218}{315},$$

$$I_4^c(f) = \tfrac{1}{90}\Big(7f(1) + 32f\big(\tfrac{5}{4}\big) + 12f\big(\tfrac{3}{2}\big) + 32f\big(\tfrac{7}{4}\big) + 7f(2)\Big) = \tfrac{1}{90}\big(7 + \tfrac{128}{5} + 8 + \tfrac{128}{7} + \tfrac{7}{2}\big) = \tfrac{4367}{6300}$$

$$\begin{aligned} I_3^g(f) &= \tfrac{1}{18}\Big(5f\big(\tfrac{3}{2} - \tfrac{\sqrt{3}}{2\sqrt{5}}\big) + 8f\big(\tfrac{3}{2}\big) + 5f\big(\tfrac{3}{2} + \tfrac{\sqrt{3}}{2\sqrt{5}}\big)\Big) = \tfrac{1}{18}\Big(5f\big(\tfrac{3\sqrt{5}-3}{2\sqrt{5}}\big) + 8f\big(\tfrac{3}{2}\big) + 5f\big(\tfrac{3\sqrt{5}+\sqrt{3}}{2\sqrt{5}}\big)\Big) \\ &= \tfrac{1}{18}\big(\tfrac{10\sqrt{5}}{3\sqrt{5}-\sqrt{3}} + \tfrac{16}{3} + \tfrac{10\sqrt{5}}{3\sqrt{5}+\sqrt{3}}\big) = \tfrac{1}{18}\big(\tfrac{10\sqrt{5}(3\sqrt{5}+\sqrt{3})}{42} + \tfrac{16}{3} + \tfrac{10\sqrt{5}(3\sqrt{5}-\sqrt{3})}{42}\big) \\ &= \tfrac{1}{18}\big(\tfrac{16}{3} + \tfrac{300}{42}\big) = \tfrac{1}{18} \cdot \tfrac{16\cdot 7+150}{21} = \tfrac{131}{189}. \end{aligned}$$

5: The **third order Taylor method** for approximating the solution to $y' = f(x,y)$ with $y(x_0) = y_0$ is performed by choosing a step size $h = \Delta x$, starting with (x_0, y_0) and then, after having found (x_k, y_k), letting $x_{k+1} = x_k + h$ and letting y_{k+1} be given by

$$y_{k+1} = y(x_k) + y'(x_k)h + \tfrac{1}{2}y''(x_k)h^2 + \tfrac{1}{6}y'''(x_k)h^3$$

where $y = y(x)$ is the solution to the given DE with $y(x_k) = y_k$.

(a) Recall that when $y = y(x)$ is a solution to the DE $y' = f(x,y)$ we have $y' = f$ and $y'' = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\, f$. Show that we also have

$$y''' = \frac{\partial^2 f}{\partial x^2} + 2\,\frac{\partial^2 f}{\partial x \partial y}\, f + \frac{\partial^2 f}{\partial y^2}\, f^2 + \frac{\partial f}{\partial x}\,\frac{\partial f}{\partial y} + \big(\tfrac{\partial f}{\partial y}\big)^2 f.$$

Solution: Recall that when $y = y(x)$ is a solution to the DE we have $y'(x) = f(x, y(x))$ and hence we have $y''(x) = \frac{d}{dt} f(x,y(x)) = \frac{\partial f}{\partial x}(x,y(x)) + \frac{\partial f}{\partial y}(x,y(x))y'(x) = \frac{d}{dt} f(x,y(x)) = \frac{\partial f}{\partial x}(x,y(x)) + \frac{\partial f}{\partial y}(x,y(x))f(x,y(x))$. Use the chain rule again to get

$$\begin{aligned}
y'''(x) &= \tfrac{d}{dt}\Big(\tfrac{\partial f}{\partial x}(x,y(x)) + \tfrac{\partial f}{\partial y}(x,y(x))f(x,y(x))\Big) \\
&= \tfrac{d}{dt}\tfrac{\partial f}{\partial x}(x,y(x)) + \big(\tfrac{d}{dt}\tfrac{\partial f}{\partial y}(x,y(x))\big)f(x,y(x)) + \tfrac{\partial f}{\partial y}(x,y(x))\tfrac{d}{dt}f(x,y(x)) \\
&= \big(\tfrac{\partial^2 f}{\partial x^2}(x,y(x)) + \tfrac{\partial^2 f}{\partial x \partial y}(x,y(x))y'(x)\big) + \big(\tfrac{\partial^2 f}{\partial x \partial y}(x,y(x) + \tfrac{\partial^2 f}{\partial y^2}(x,y))y'(x)\big)f(x,y(x)) \\
&\qquad + \tfrac{\partial f}{\partial y}(x,y(x))\big(\tfrac{\partial f}{\partial x}(x,y(x)) + \tfrac{\partial f}{\partial y}(x,f(x))y'(x)\big) \\
&= \tfrac{\partial^2 f}{\partial x^2} + \tfrac{\partial^2 f}{\partial x \partial y}\, f + \tfrac{\partial f}{\partial y}\,\tfrac{\partial f}{\partial x} + \big(\tfrac{\partial^2 f}{\partial x \partial y} + \tfrac{\partial^2 f}{\partial y^2}\, f + \tfrac{\partial f}{\partial y}\,\tfrac{\partial f}{\partial x}\big)y' \\
&= \tfrac{\partial^2 f}{\partial x^2} + \tfrac{\partial^2 f}{\partial x \partial y}\, f + \tfrac{\partial f}{\partial y}\,\tfrac{\partial f}{\partial x} + \big(\tfrac{\partial^2 f}{\partial x \partial y} + \tfrac{\partial^2 f}{\partial y^2}\, f + \tfrac{\partial f}{\partial y}\,\tfrac{\partial f}{\partial x}\big)f \\
&= \tfrac{\partial^2 f}{\partial x^2} + 2\,\tfrac{\partial^2 f}{\partial x \partial y}\, f + \tfrac{\partial^2 f}{\partial y^2}\, f^2 + \tfrac{\partial f}{\partial x}\,\tfrac{\partial f}{\partial y} + \big(\tfrac{\partial f}{\partial y}\big)^2 f.
\end{aligned}$$

(b) Apply the third-order Taylor method using the step size $h = \frac{1}{2}$ to approximate the value of $y(2)$ when $y = y(x)$ is the solution to the IVP given by $y' = 1 + \frac{y}{x}$ with $y(1) = 1$.

Solution: For $y(x) = f(x, y(x))$ with $f(x,y) = 1 + \frac{y}{x}$, we have $\frac{\partial f}{\partial x} = -\frac{y}{x^2}$, $\frac{\partial f}{\partial y} = \frac{1}{x}$, $\frac{\partial^2 f}{\partial x^2} = \frac{2y}{x^3}$, $\frac{\partial^2 f}{\partial x \partial y} = -\frac{1}{x^2}$ and $\frac{\partial^2 f}{\partial y^2} = 0$ so that

$$\begin{aligned}
y'(x) &= f = 1 + \tfrac{y}{x} \\
y''(x) &= \tfrac{\partial f}{\partial x} + \tfrac{\partial f}{\partial y}\, f = -\tfrac{y}{x^2} + \tfrac{1}{x}\big(1 + \tfrac{y}{x}\big) = \tfrac{1}{x} \\
y'''(x) &= \tfrac{\partial^2 f}{\partial x^2} + 2\,\tfrac{\partial^2 f}{\partial x \partial y}\, f + \tfrac{\partial^2 f}{\partial y^2}\, f^2 + \tfrac{\partial f}{\partial x}\,\tfrac{\partial f}{\partial y} + \big(\tfrac{\partial f}{\partial y}\big)^2 f \\
&= \tfrac{2y}{x^3} + 2\big(-\tfrac{1}{x^2}\big)\big(1 + \tfrac{y}{x}\big) + 0 + \big(-\tfrac{y}{x^2}\big)\big(\tfrac{1}{x}\big) + \big(\tfrac{1}{x}\big)^2\big(1 + \tfrac{y}{x}\big) \\
&= \tfrac{2y}{x^3} - \tfrac{2}{x^2} - \tfrac{2y}{x^3} - \tfrac{y}{x^3} + \tfrac{1}{x^2} + \tfrac{y}{x^3} = -\tfrac{1}{x^2}.
\end{aligned}$$

We remark that it is easier to calculate $y'''(x)$ directly from $y''(x) = \frac{1}{x}$ without using the formula obtained in Part (a). Using the 3$^{\text{rd}}$-order Taylor method with $h = \frac{1}{2}$, we start with $(x_0, y_0) = (1, 1)$ then, having found (x_k, y_k) we let $x_{k+1} = x_k + h = x_k + \frac{1}{2}$ and

$$y_{k+1} = y_k + y'(x_k)h + \tfrac{1}{2}y''(x_k)h^2 + \tfrac{1}{6}y'''(x_k)h^3 = y_k + \tfrac{1}{2}\big(1 + \tfrac{y_k}{x_k}\big) + \tfrac{1}{8}\,\tfrac{1}{x_k} - \tfrac{1}{48}\,\tfrac{1}{x_k^2}.$$

We obtain

k	x_k	y_k	$1 + \frac{y_k}{x_k}$	$\frac{1}{x_k}$	$\frac{1}{x_k^2}$
0	1	1	2	1	1
1	$\frac{3}{2}$	$\frac{101}{48}$	$\frac{173}{72}$	$\frac{48}{101}$	$\frac{48^2}{101^2}$
2	2	y_2			

with $y(2) \cong y_2 = \frac{101}{48} + \frac{173}{144} + \frac{6}{101} - \frac{48}{101^2} = \frac{1234007}{367236} \cong 3.36.$

6: Consider the IVP given by $y'=\frac{x}{y}$ with $y(0)=1$. Find the exact solution $y=y(x)$ and the exact value of $y(1)$. Then, approximate the value of $y(1)$ several times: use Euler's method with step size $h=\frac{1}{4}$, use the second-order Taylor method with $h=\frac{1}{2}$, use Heun's method with $h=\frac{1}{2}$, and use RK4 with $h=1$.

Solution: Let $f(x,y)=\frac{x}{y}$. The given DE is separable as we can write it as $y\,dy=x\,dx$. Integrate both sides to get $\frac{1}{2}y^2=\frac{1}{2}x^2+c$, that is $y^2=x^2+2c$. To get $y(0)=1$ we need $2c=1$, so the solution is given by $y^2=x^2+1$. Since we want $y(0)=1>0$, we take $y=\sqrt{x^2+1}$. In particular, we have $y(1)=\sqrt{2}$.

To apply Euler's method with $h=\frac{1}{4}$, we start with $(x_0,y_0)=(0,1)$ then, having found (x_k,y_k), we let $x_{k+1}=x+k+h=x_k+\frac{1}{4}$ and we let $y_{k+1}=y_k+y'(x_k)h=y_k+\frac{1}{4}f(x_k,y_k)=y_k+\frac{1}{4}\frac{x_k}{y_k}$. We obtain

k	x_k	y_k	$\frac{x_k}{y_k}$
0	0	1	0
1	$\frac{1}{4}$	1	$\frac{1}{4}$
2	$\frac{1}{2}$	$\frac{17}{16}$	$\frac{8}{17}$
3	$\frac{3}{4}$	$\frac{321}{272}$	$\frac{204}{321}$
4	1	y_4	

with $y(1)\cong y_4=\frac{321}{272}+\frac{51}{321}=\frac{38971}{29104}\cong 1.3390$

To apply the 2nd-order Taylor method with $h=\frac{1}{2}$, we start with $(x_0,y_0)=(0,1)$ and, having found (x_k,y_k) we let $x_{k+1}=x_k+h=x_k+\frac{1}{2}$ and

$$\begin{aligned}y_{k+1}&=y_k+y'(x_k)h+\tfrac{1}{2}y''(x_k)h^2=y_k+\tfrac{1}{2}f(x_k,y_k)+\tfrac{1}{8}\big(\tfrac{\partial f}{\partial x}(x_k,y_k)+\tfrac{\partial f}{\partial y}(x_k,y_k)f(x_k,y_k)\big)\\&=y_k+\tfrac{1}{2}\tfrac{x_k}{y_k}+\tfrac{1}{8}\big(\tfrac{1}{y_k}-\tfrac{x_k}{y_k^2}\tfrac{x_k}{y_k}\big)=y_k+\tfrac{1}{2}\tfrac{x_k}{y_k}+\tfrac{1}{8}\tfrac{y_k^2-x_k^2}{y_k^3}.\end{aligned}$$

We obtain

k	x_k	y_k	$\frac{x_k}{y_k}$	$\frac{y_k^2-x_k^2}{y_k^3}$
0	0	1	0	1
1	$\frac{1}{2}$	$\frac{9}{8}$	$\frac{4}{9}$	$\frac{65\cdot 8}{729}$
2	1	y_2		

with $y(1)\cong y_2=\frac{9}{8}+\frac{2}{9}+\frac{65}{729}=\frac{8377}{5832}\cong 1.4364$.

To apply Heun's method with $h=\frac{1}{2}$, we start with $(x_0,y_0)=(0,1)$ and, having found (x_k,y_k) we let $x_{k+1}=x_k+h=x_k+\frac{1}{2}$ and $y_{k+1}=y_k+\frac{1}{2}(w_1+w_2)h=y_k+\frac{1}{4}(w_1+w_2)$ where $w_1=f(x_k,y_k)$ and $w_2=f\big((x_k,y_k)+(h,w_1,h)\big)$. At the first step we take $x_1=x_0+h=0+\frac{1}{2}=\frac{1}{2}$ and

$$\begin{aligned}m_1&=f(x_0,y_0)=f(0,1)=\tfrac{0}{1}=0\,,\\m_2&=f\big((x_0,y_0)+(h,m_1h)\big)=f\big((0,1)+(\tfrac{1}{2},0)\big)=f\big(\tfrac{1}{2},1\big)=\tfrac{1}{2}\ ,\text{ and}\\y_1&=y_0+\tfrac{1}{4}(m_1+m_2)=1+\tfrac{1}{4}\big(0+\tfrac{1}{2}\big)=\tfrac{9}{8}.\end{aligned}$$

At the second step, we take $x_2=x_1+h=\frac{1}{2}+\frac{1}{2}=1$ and

$$\begin{aligned}m_1&=f(x_1,y_1)=f\big(\tfrac{1}{2},\tfrac{9}{8}\big)=\tfrac{1}{2}\cdot\tfrac{8}{9}=\tfrac{4}{9}\,,\\m_2&=f\big((x_1,y_1)+(h,m_1h)\big)=f\big((\tfrac{1}{2},\tfrac{9}{8})+(\tfrac{1}{2},\tfrac{2}{9})\big)=f\big(1,\tfrac{97}{72}\big)=\tfrac{72}{97}\ ,\text{ and}\\y(1)\cong y_2&=y_1+\tfrac{1}{4}(m_1+m_2)=\tfrac{9}{8}+\big(\tfrac{1}{9}+\tfrac{18}{97}\big)=\tfrac{9929}{6984}\cong 1.4217\end{aligned}$$

To apply RK4 with $h=1$, we start with $(x_0,y_0)=(0,1)$ and let

$$\begin{aligned}x_1&=x_0+h=0+1=1\\m_1&=f(x_0,y_0)=f(0,1)=0\,,\\m_2&=f\big((x_0,y_0)+\tfrac{1}{2}(h,m_1h)\big)=f\big((0,1)+\tfrac{1}{2}(1,0)\big)=f\big(\tfrac{1}{2},1\big)=\tfrac{1}{2}\,,\\m_3&=f\big((x_0,y_0)+\tfrac{1}{2}(h,m_2h)\big)=f\big((0,1)+\tfrac{1}{2}(1,\tfrac{1}{2})\big)=f\big(\tfrac{1}{2},\tfrac{5}{4}\big)=\tfrac{2}{5}\,,\\m_4&=f\big((x_0,y_0)+(h,m_3h)\big)=f\big((0,1)+(1,\tfrac{2}{5})\big)=f\big(1,\tfrac{7}{5}\big)=\tfrac{5}{7}\ ,\text{ and}\\y(1)\cong y_1&=y_0+\tfrac{1}{6}\big(m_1+2m_2+2m_3+m_4\big)=1+\tfrac{1}{6}\big(0+1+\tfrac{4}{5}+\tfrac{5}{7}\big)=\tfrac{149}{105}\cong 1.4190.\end{aligned}$$