SYDE Advanced Math 2, Practice Problem Set 6

1: Let $F(x,y) = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$ where $u(x,y) = x^2 + y^2 - 5$ and $v(x,y) = x^3 + y^3 - 2$.

(a) Sketch the curves $f(x,y) = 0$ and $g(x,y) = 0$ on the same grid, and use your picture to approximate the coordinates of the points of intersection of the two curves $\big($that is the points (x,y) such that $F(x,y) = \binom{0}{0}\big)$.

(b) Find a more accurate approximation for the coordinates of the lowest point of intersection as follows: Starting with $a_0 = \binom{x_0}{y_0} = \binom{2}{-1}$, carry out two iterations of Newton's method. Calculate the first iteration to find $a_1 = \binom{x_1}{y_1}$ by hand, expressing x_1 and y_1 as fractions, then carry out the second iteration to find $a_2 = \binom{x_2}{y_2}$ with the help a calculator (or computer).

2: (a) Show that when $n = 2$, the open Newton-Cotes rule is given by

$$I_2^o = \tfrac{b-a}{3}\big(2f(x_0) - f(x_1) + 2f(x_2)\big) \quad \text{, where} \quad x_k = a + (k+1)\,\tfrac{b-a}{4}.$$

(b) Show that when $n = 4$, the closed Newton-Cotes rule is given by

$$I_4^c = \tfrac{b-a}{90}\big(7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)\big) \quad \text{, where} \quad x_k = a + k\,\tfrac{b-a}{4}.$$

3: (a) Show that when $n = 3$, the Gaussian quadrature rule on the interval $[-1,1]$ is given by

$$\int_{-1}^{1} f(x)\,dx \cong I_3^g = \tfrac{5}{9}\,f\big(-\tfrac{\sqrt{3}}{\sqrt{5}}\big) + \tfrac{8}{9}\,f(0) + \tfrac{5}{9}\,f\big(\tfrac{\sqrt{3}}{\sqrt{5}}\big).$$

(b) Find the Gaussian quadrature rule, for $n = 3$, on the interval $[a,b]$.

4: Approximate the value of $\ln 2 = \int_1^2 \frac{1}{2}\,dx$ using the open Newton-Cotes rule for $n = 2$, using the closed Newton-Cotes rule for $n = 4$, and using the Gaussian quadrature rule for $n = 3$.

5: The **third order Taylor method** for approximating the solution to $y' = f(x,y)$ with $y(x_0) = y_0$ is performed by choosing a step size $h = \Delta x$, starting with (x_0, y_0) and then, after having found (x_k, y_k), letting $x_{k+1} = x_k + h$ and letting y_{k+1} be given by

$$y_{k+1} = y(x_k) + y'(x_k)h + \tfrac{1}{2}y''(x_k)h^2 + \tfrac{1}{6}y'''(x_k)h^3$$

where $y = y(x)$ is the solution to the given DE with $y(x_k) = y_k$.

(a) Recall that when $y = y(x)$ is a solution to the DE $y' = f(x,y)$ we have $y' = f$ and $y'' = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\,f$. Show that we also have

$$y''' = \tfrac{\partial^2 f}{\partial x^2} + 2\,\tfrac{\partial^2 f}{\partial x \partial y}\,f + \tfrac{\partial^2 f}{\partial y^2}\,f^2 + \tfrac{\partial f}{\partial x}\,\tfrac{\partial f}{\partial y} + \big(\tfrac{\partial f}{\partial y}\big)^2\,f.$$

(b) Apply the third-order Taylor method using the step size $h = \frac{1}{2}$ to approximate the value of $y(2)$ when $y = y(x)$ is the solution to the IVP given by $y' = 1 + \frac{y}{x}$ with $y(1) = 1$.

6: Consider the IVP given by $y' = \frac{x}{y}$ with $y(0) = 1$. Find the exact solution $y = y(x)$ and the exact value of $y(1)$. Then, approximate the value of $y(1)$ several times: use Euler's method with step size $h = \frac{1}{4}$, use the second-order Taylor method with $h = \frac{1}{2}$, use Heun's method with $h = \frac{1}{2}$, and use RK4 with $h = 1$.