SYDE Advanced Math 2, Practice Problem Set 5

1: The following ODEs are examples of Sturm-Liouville boundary value problems involving a parameter k. In each case, non-zero solutions only occur for certain values of k, which are called *eigenvalues* and the corresponding solutions are called *eigenfunctions*.

(a) Find the possible values of $k \in \mathbb{R}$ and the non-zero solutions the ODE $u'' = ku$ for $u = u(x)$ satisfying the boundary conditions $u'(0) = 0$ and $u(1) = 0$.

(b) Find the possible values of $k \in \mathbb{R}$ and the non-zero solutions to the ODE $x^2u'' + xu' + ku = 0$ satisfying the boundary conditions $u(1) = 0$ and $u(4) = 0$.

2: Solve the wave equation $\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x}^2$ for $u = u(x,t)$ with $0 \leq x \leq 4$ and $t \geq 0$, satisfying the fixed endpoint condition $u(0,t) = u(4,t) = 0$ for all $t \geq 0$ and the initial conditions $u(x,0) = 0$ and $\frac{\partial u}{\partial t}(x,0) = 2\sin\frac{\pi x}{4}$ for $0 \leq x \leq 4$.

3: Solve the heat equation $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$ for $u = u(x,t)$ with $0 \leq x \leq \ell$ and $t \geq 0$ satisfying the fixed endpoint temperature condition $u(0,t) = 0$ and $u(\ell,t) = 0$ for all $t \geq 0$ and the initial condition $u(x,0) = f(x)$ for all $0 \leq x \leq \ell$ where $f(x)$ is given by $f(x) = 0$ for $0 \leq x < \frac{1}{4}\ell$, $f(x) = 1$ for $\frac{1}{4} < x < \frac{3\ell}{4}$ and $f(x) = 0$ for $\frac{3\ell}{4} < x \leq \ell$ (with $f\left(\frac{\ell}{4}\right) = f\left(\frac{3\ell}{4}\right) = \frac{1}{2}$ so that $f(x)$ is equal to the sum of its Fourier series).

4: Solve the heat equation $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$ for $u = u(x,t)$ with $0 \leq x \leq \ell$ and $t \geq 0$ satisfying the insulated ends condition $\frac{\partial u}{\partial x}(0,t) = 0$ and $\frac{\partial u}{\partial x}(\ell,t) = 0$ for all $t \geq 0$ and the initial condition $u(x,0) = f(x)$ for all $0 \leq x \leq \ell$ where $f(x)$ is given by $f(x) = 1$ for $0 < x < \frac{2\ell}{3}$ and $f(x) = 3$ for $\frac{2\ell}{3} < x < \ell$ (with $f(0) = f(\frac{2\ell}{3}) = f(1) = 2$).

5: Solve Dirichlet's problem, that is solve Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, for $u = u(x,y)$ on the square $0 \leq x \leq 1$, $0 \leq y \leq 1$ satisfying the boundary conditions $u(x,0) = x$ and $u(x,1) = x$ for $0 \leq x \leq 1$, and $u(0,y) = \sin\pi y$ and $u(1,y) = 1 - \sin\pi y$ for $0 \leq y \leq 1$.

6: Consider Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

(a) Change to polar coordinates by letting $x = r\cos\theta$ and $y = r\sin\theta$. Use the Chain Rule to calculate $\frac{\partial u}{\partial r}$ and $\frac{\partial^2 u}{\partial r^2}$, and $\frac{\partial u}{\partial \theta}$ and $\frac{\partial^2 u}{\partial \theta^2}$, and hence show that Laplace's equation, for $u = u(r,\theta) = u(x(r,\theta), y(r,\theta))$, becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

(b) Find a solution $u = u(x,y)$ to Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ in the annulus given by $1 \leq x^2 + y^2 \leq 2$ satisfying the boundary conditions $u(x,y) = 6$ when $x^2 + y^2 = 1$ and $u(x,y) = 10$ when $x^2 + y^2 = 2$.