SYDE Advanced Math 2, Solutions to Assignment 6

1: (a) Use the method of separation of variables to find a solution $u = u(x, y)$ to the PDE $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2(x+y)u$ with $u(1, 0) = 4$ and $u(0, 1) = 1$.

Solution: Let $u(x, y) = X(x)Y(y)$. Then $\frac{\partial u}{\partial x} = X'Y$ and $\frac{\partial u}{\partial y} = XY'$. Putting this in the PDE gives $X'Y + XY' = 2(x + y)XY$. Dividing by XY gives $\frac{X'}{X} + \frac{Y'}{Y} = 2x + 2y$, that is $\frac{X'(x)}{X(x)} - 2x = 2y - \frac{Y'(y)}{Y(y)}$. Since the left side depends only on x and the right side depends only on y, in order to be equal for all $x$and y both sides must be constant, say $\frac{X'}{X} - 2x = k = 2y - \frac{Y'}{Y}$. The DE $\frac{X'}{X} - 2x = k$ is linear, since we can write it as $X' - (2x + k)X = 0$, an integrating factor is given by $\lambda = e^{-x^2 - kx)}$ and the solution is given by $X = a\,e^{x^2+kx}$. Similarly, the DE $k = 2y - \frac{Y'}{Y}$ is linear as it can be written as $Y' - (2y - k)Y = 0$ and the solution is $Y = b\,e^{y^2-y}$. Thus we obtain the solution $u(x, y) = X(x)Y(y) = c\,e^{x^2+kx}e^{y^2-ky} = c\,e^{x^2+y^2+k(x-y)}$ to the original PDE (where $c = ab$). To get $u(1, 0) = 4$ we need $ce^{1+k} = 4$, that is $ce\,e^k = 4$ (1) and to get $u(0, 1) = 1$ we need $ce^{1-k} = 1$, that is $ce\,e^{-k} = 1$ (2). Dividing equation (1) by equastion (2) gives $e^{2k} = 4$ so that $k = \ln 2$, and putting this in equation (1) gives $c = \frac{2}{e}$. Thus we obtain the solution

$$u = c\,e^{x^2+y^2+k(x-y)} = \frac{2}{e}\,e^{x^2+y^2+(x-y)\ln 2} = \frac{2}{e}\,e^{x^2+y^2}\,2^{x-y} = 2^{x-y+1}e^{x^2+y^2-1}.$$

(b) Solve the PDE given by $\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = y$ for $u = u(x, y)$ with $u(x, y) = 1$ on the line $x + y = 1$ by making a change of variables, letting $r = x$ and $s = y - 2x$.

Solution: Let $r = x$ and $s = y - 2x$ and note that $x = r$ and $y = s + 2x = s + 2r$. By the Chain Rule, we have

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial .x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} = \frac{\partial u}{\partial r} - 2\frac{\partial u}{\partial s}\\
\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} = \frac{\partial u}{\partial s}\\
\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} &= \big(\frac{\partial u}{\partial r} - 2\frac{\partial u}{\partial s}\big) + 2\big(\frac{\partial u}{\partial s}\big) = \frac{\partial u}{\partial r},
\end{aligned}$$

so the PDE becomes $\frac{\partial u}{\partial r} = s + 2r$. This gives $u = \int s+2r\;dr = sr + r^2 + k(s)$. The line $x + y = 1$ in the new coordinates r, s becomes $r + (s + 2r) = 1$, that is $3r + s = 1$, so to have $u = 1$ on the line $x + y = 1$ we need $1 = sr + r^2 + k(s)$ whenever $r = \frac{1-s}{3}$, which implies $k(s) = 1 - sr - r^2 = 1 - s\big(\frac{1-s}{3}\big) - \big(\frac{1-s}{3}\big)^2 = \frac{1}{9}(8 - s + s^2)$. Thus the solution is given by

$$\begin{aligned}
u &= sr + r^2 + k(s) = sr + r^2 + \tfrac{1}{9}(8 - s + s^2)\\
&= \tfrac{1}{9}(9sr + 9r^2 + 8 - s + 2s^2)\\
&= \tfrac{1}{9}\big(9(y - 2x)x + 9x^2 + 8 - (y - 2x) + 2(y - 2x)^2\big)\\
&= \tfrac{1}{9}\big(- x^2 + xy + 2y^2 + 2x - y + 8\big).
\end{aligned}$$

2: (a) Use separation of variables and Fourier series to solve the wave equation $\frac{\partial^2 u}{\partial t^2} = \frac{1}{4}\frac{\partial^2 u}{\partial x^2}$ for $u = u(x,t)$ with $0 \le x \le 2$ and $t \in \mathbb{R}$ satisfying the fixed ends condition $u(0,t) = u(2,t) = 0$ for all $t \in \mathbb{R}$ and the initial conditions $u(x,0) = (\sin \pi x)(1 + \cos \pi x)$ and $\frac{\partial u}{\partial t}(x,0) = 0$ for all $0 \le x \le 2$.

Solution: Let $f(x) = (\sin \pi x)(1+\cos \pi x) = \sin \pi x + \sin \pi x \cos \pi x = \sin \pi x + \frac{1}{2}\sin 2\pi x$. We know (see Example 4.10) that the solution $u = u(x,t)$ to the wave equation with $c = \frac{1}{2}$ and $\ell = 2$ with fixed ends satisfying $u(x,0) = f(x)$ and $\frac{\partial u}{\partial t}(x,0) = 0$ is given by

$$u(x,t) = \sum_{n=0}^{\infty} a_n \cos\left(\tfrac{cn\pi}{\ell}t\right)\sin\left(\tfrac{n\pi}{\ell}x\right) = \sum_{n=0}^{\infty} a_n \cos\left(\tfrac{n\pi}{4}t\right)\sin\left(\tfrac{n\pi}{2}x\right)$$

where the a_n are the Fourier coefficients of the odd function of period $2\ell = 4$ equal to $f(x) = \sin \pi x + \frac{1}{2}\sin 2\pi x$ for $0 \le x \le 2$. By inspection, the Fourier coefficients of $f(x)$ are given by $a_2 = 1$ and $a_4 = \frac{1}{2}$ and $a_n = 0$ for all $n \ne 2, 4$, so the solution is

$$u(x,t) = \cos\left(\tfrac{\pi}{2}\,t\right)\sin(\pi x) + \tfrac{1}{2}\cos(\pi t)\sin(2\pi x).$$

(b) Find a constant c and function $g(x)$ such that $u(x,t) = g(x+ct) + g(x-ct)$ for all x, t (and show that this is the case).

Solution: Let $c = \frac{1}{2}$ and $g(x) = \frac{1}{2}f(x) = \frac{1}{2}\sin(\pi x) + \frac{1}{4}\sin(2\pi x)$. Then

$$\begin{aligned} g(x+ct) &= g\big(x + \tfrac{t}{2}\big) = \tfrac{1}{2}\sin\big(\pi\big(x+\tfrac{t}{2}\big)\big) + \tfrac{1}{4}\sin\big(2\pi\big(x+\tfrac{c}{2}\big)\big) \\ &= \tfrac{1}{2}\sin\big(\pi x + \tfrac{\pi t}{2}\big) + \tfrac{1}{4}\sin\big(2\pi x + \pi t\big) \\ &= \tfrac{1}{2}\big(\sin(\pi x)\cos\big(\tfrac{\pi t}{2}\big) + \cos(\pi x)\sin\big(\tfrac{\pi t}{2}\big)\big) + \tfrac{1}{4}\big(\sin(2\pi x)\cos(\pi t) + \cos(2\pi x)\sin(\pi t)\big) \end{aligned}$$

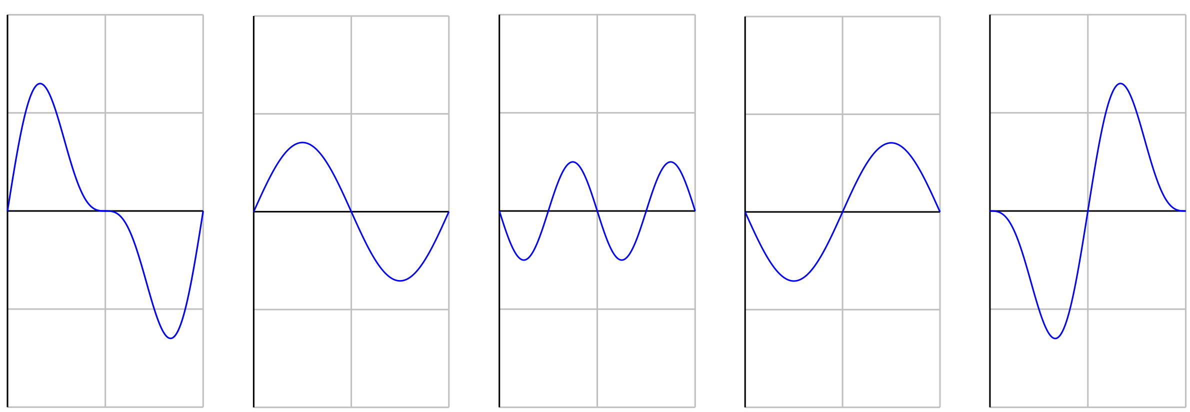
and similarly

$$\begin{aligned} g(x-ct) &= g\big(x - \tfrac{t}{2}\big) = \tfrac{1}{2}\sin\big(\pi x - \tfrac{\pi t}{2}\big) + \tfrac{1}{4}\sin\big(2\pi x - \pi t\big) \\ &= \tfrac{1}{2}\big(\sin(\pi x)\cos\big(\tfrac{\pi t}{2}\big) - \cos(\pi x)\sin\big(\tfrac{\pi t}{2}\big)\big) + \tfrac{1}{4}\big(\sin(2\pi x)\cos(\pi t) - \cos(2\pi x)\sin(\pi t)\big) \end{aligned}$$

and so $g(x+ct) + g(x-ct) = \sin(\pi x)\cos\big(\frac{\pi t}{2}\big) + \frac{1}{2}\sin(2\pi x)\cos(\pi t) = u(x,t)$.

(c) By plotting points, accurately sketch the graphs $u = u(x,t)$ (in the xu-plane) for $t = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$.

Solution: We have $u(x,0) = \sin(\pi x) + \frac{1}{2}\sin(2\pi x)$, and $u\big(x, \frac{1}{2}\big) = \frac{\sqrt{2}}{2}\sin(\pi x)$, and $u(x,1) = -\frac{1}{2}\sin(2\pi t)$, and $u\big(x, \frac{3}{2}\big) = -\frac{\sqrt{2}}{2}\sin(\pi x)$, and $u(x,2) = -\sin(\pi x) + \frac{1}{2}\sin(2\pi x)$. The graphs are shown below:



3: (a) Solve the heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ for $u = u(x,t)$ with $0 \leq x \leq \pi$ and $t \geq 0$ satisfying the insulated ends condition $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi,t) = 0$ for all $t \geq 0$ and the initial condition $u(x,0) = x^2$ for all $0 \leq x \leq \pi$.

Solution: We know (Exercise 4.14) that the solution to the heat equation with $c = 1$ and $\ell = \pi$ with insulated ends satisfying $u(x,0) = f(x) = x^2$ is given by

$$u(x,t) = \sum_{n=0}^{\infty} a_n e^{-n^2 t} \cos nx$$

where the constants a_n are the Fourier coefficients of the even 2π-periodic function which is equal to $f(x) = x^2$ for $0 \leq x \leq \pi$. The coefficients are

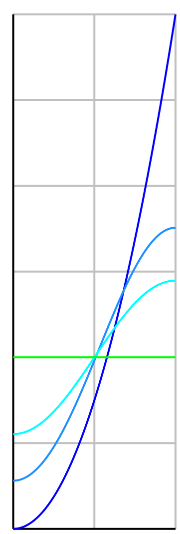
$$\begin{aligned} a_0 &= \tfrac{1}{\pi} \int_0^\pi x^2\,dx = \tfrac{1}{\pi}\Big[\tfrac{1}{3}x^2\Big]_0^\pi = \tfrac{\pi^2}{3}\ , \text{ and} \\ a_n &= \tfrac{2}{\pi} \int_0^\pi x^2 \cos nx\ dx = \tfrac{2}{\pi}\left(\Big[\tfrac{1}{n}x^2 \sin nx\Big]_0^\pi - \int_0^\pi \tfrac{2}{n} x \sin nx\ dx\right) \\ &= -\tfrac{2}{\pi}\left(- \Big[\tfrac{2}{n^2} x \cos nx\Big]_0^\pi + \int_0^\pi \tfrac{2}{n^2} \cos nx\ dx\right) \\ &= \tfrac{4}{\pi n^2}\Big[x \cos nx\Big]_0^\pi = \tfrac{4}{\pi n^2} \cdot \pi(-1)^n = \tfrac{4(-1)^n}{n^2}. \end{aligned}$$

Thus the solution is given by

$$\begin{aligned} u(x,t) &= \sum_{n=0}^{\infty} a_n e^{-n^2 t} \cos nx = \tfrac{\pi^2}{3} + \sum_{n=1}^{\infty} \tfrac{4(-1)^n}{n^2} e^{-n^2 t} \cos nx \\ &= \tfrac{\pi^2}{3} - 4\Big(e^{-t} \cos x - \tfrac{1}{4} e^{-4t} \cos 2x + \tfrac{1}{9} e^{-9t} \cos 3x - \tfrac{1}{16} e^{-16t} \cos 4x + \cdots\Big). \end{aligned}$$

(b) Give a fairly accurately sketch of the graphs of $u = u(x,t)$ (in the xu-plane) for $t = 0, \frac{1}{2}, 1, 10$.

Solution: When $t = 0$ we have $u(x,0) = x^2$, when $t = \frac{1}{2}$ we have $u\big(x, \frac{1}{2}\big) \cong \frac{\pi^2}{3} - 4e^{-1/2} \cos x + \frac{4}{9} e^{-2} \cos 2x$, when $t = 1$ we have $u(x,1) \cong \frac{\pi^2}{3} - 4e^{-1} \cos x$, and when $t = 10$ we have $u(x,10) \cong \frac{\pi^2}{3}$. These approximations are sketched below (in dark blue, lighter blue, cyan, and green):



4: (a) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(z) = z^2$ and let $v, w : \mathbb{R}^2 \to \mathbb{R}$ be the real and imaginary parts of f so that $f(x+iy) = v(x,y) + i\,w(x,y)$. Show that v and w both satisfy Laplace's equation.

Solution: We have $f(x+iy) = (x+iy)^2 = (x^2-y^2) + i(2xy)$ and so $v(x,y) = x^2-y^2$ and $w(x,y) = 2xy$. We have $\frac{\partial v}{\partial x} = 2x$ and $\frac{\partial v}{\partial y} = -2y$ so that $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 2-2 = 0$, and we have $\frac{\partial w}{\partial x} = y$ and $\frac{\partial w}{\partial y} = x$ so that $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0+0 = 0$.

(b) Solve Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ for $u = u(x,y)$ on the square $0 \le x \le 1$, $0 \le y \le 1$ satisfying the boundary conditions $u(x,0) = x^2$, $u(x,1) = x-1$, $u(0,y) = -y^2$ and $u(1,y) = 1-y^2$. Hint: use $v(x,y)$ from Part (a) and notice that $u(x,1) \ne v(x,1) = x^2-1$.

Solution: Note that for the function $v = v(x,y) = x^2-y^2$ from Part (a) we have $v(x,0) = x^2$, $v(x,1) = x^2-1$, $v(0,y) = -y^2$ and $v(1,y) = 1-y^2$. These are similar to the desired boundary conditions for $u = u(x,y)$: indeed if $u = u(x,y)$ satisfies the desired boundary conditions and we let $w(x,y) = u(x,y) - v(x,y)$ then w will satisfy the boundary conditions $w(x,0) = 0$, $w(x,1) = x-x^2$, $w(0,y) = 0$ and $w(1,y) = 0$.

Let us try to find a solution $w = w(x,y)$ to Laplace's equation such that $w(x,0) = 0$, $w(x,1) = x-x^2$, $w(0,y) = 0$ and $w(1,y) = 0$. We try $w = XY$. Laplace's equation becomes $X''Y + XY'' = 0$, that is $\frac{X''}{X} = -\frac{Y''}{Y}$, so that we must have $\frac{X''}{X} = k = -\frac{Y''}{Y}$ for some constant k, and the initial conditions become $X(x)Y(0) = 0$, $X(x)Y(1) = x-x^2$, $X(0)Y(y) = 0$ and $X(1)Y(0) = 0$, so that (for a non-zero solution) we must have $X(0) = 0$, $X(1) = 0$, $Y(0) = 0$ and $w(x,1) = X(x)Y(1) = x-x^2$. First we solve $\frac{X''}{X} = k$, that is $X'' - kX = 0$, with $X(0) = 0$ and $X(1) = 0$. As with the wave equation, when $k \ge 0$ there are no non-zero solutions, and when $k = -\sigma^2$ with $\sigma > 0$, the are non-zeroo solutions only when $k = -\sigma^2 = -(n\pi)^2$ and in this case the solutions are given by $X_n = b_n \sin(n\pi x)$. When $k = -\sigma^2 = -(n\pi)^2$, the second DE $\frac{Y''}{Y} = -k$ becomes $Y'' - (n\pi)^2 Y = 0$ which has solutions $Y = Y_n = a_n e^{n\pi y} + b e^{-n\pi y}$ and the initial condition $Y(0) = 0$ gives $a_n + b_n = 0$ so that $Y_n = a_n e^{n\pi y} - a_n e^{-n\pi y} = 2a_n \sinh(n\pi y)$. Thus for each $n \in \mathbb{Z}^+$ we have found a solution $w = w_n(x,y) = X_n(x)Y_n(y) = c_n \sinh(n\pi y)\sin(n\pi x)$, and this solution satisfies the 3 boundary conditions $w_n(x,0) = w_n(0,y) = w_n(1,y) = 0$. We let

$$w = w(x,y) = \sum_{n=1}^{\infty} c_n \sinh(n\pi y)\sin(n\pi x).$$

To satisfy the last condition $w(x,1) = x-x^2$ we need $\sum_{n=1}^{\infty} c_n \sinh(n\pi)\sin(n\pi x) = x-x^2$ for all $0 \le x \le 1$, so the numbers $c_n \sinh(n\pi)$ must be equal to the Fourier coefficients of the odd periodic function of period 2 which is equal to $f(x) = x-x^2$ for $0 \le x \le 1$. Thus we must have

$$\begin{aligned} c_n \sinh(n\pi) &= 2\int_0^1 (x-x^2)\sin(n\pi x)\,dx = 2\left(\left[-\tfrac{1}{n\pi}(x-x^2)\sin(n\pi x)\right]_0^1 - \int_0^1 \tfrac{2}{n\pi}x\cos(n\pi x)\,dx\right) \\ &= -\tfrac{4}{n\pi}\int_0^1 x\cos(n\pi x)\,dx = -\tfrac{4}{n\pi}\left(\left[\cos(n\pi x)\right]_0^1 - \int_0^1 \tfrac{1}{n\pi}\sin(n\pi x)\,dx\right) \\ &= \tfrac{4}{(n\pi)^2}\int_0^1 \sin(n\pi x)\,dx = -\tfrac{4}{(n\pi)^3}\left[\tfrac{1}{n\pi}x\sin(n\pi x)\right]_0^1 = \tfrac{4}{(n\pi)^3}\big(1-(-1)^n\big), \end{aligned}$$

so we have $c_n \sinh(n\pi) = \frac{8}{(n\pi)^3}$ when n is odd and $c_n = 0$ when n is even. Thus our solution w is given by

$$w(x,y) = \sum_{n\text{ odd}} \tfrac{8}{(n\pi)^3\sinh(n\pi)}\sinh(n\pi y)\sin(n\pi x).$$

The solution $u = u(x,y)$ to the original problem is given by $u(x,y) = v(x,y) + w(x,y)$, that is

$$u(x,y) = x^2 - y^2 + \sum_{n\text{ odd}} \tfrac{8}{(n\pi)^3\sinh(n\pi)}\sinh(n\pi y)\sin(n\pi x).$$