SYDE Advanced Math 2, Solutions to Assignment 4

1: Find the 5^{th} Taylor polynomial, centred at 0, for the solution to the IVP $y''+2y'+e^xy=\sin x$ with $y(0)=2$ and $y'(0)=1$.

Solution: We try $y=c_0+c_1x+c_2x^2+c_3x^3+c_4x^4+c_5x^5+\cdots$. Then $y'=c_1+2c_2x+3c_3x^2+4c_4x^3+5c_5x^4+\cdots$, $y''=2c_2+6c_3x+12c_4x^2+20c_5x^3+\cdots$, and

$$\begin{aligned} e^xy &= \big(1+x+\tfrac{1}{2}x^2+\tfrac{1}{6}x^3+\cdots\big)\big(a_0+c_1x+c_2x^2+c_3x^3+\cdots\big) \\ &= \big(a_0+(c_1+c_0)x+(c_2+c_1+\tfrac{1}{2}c_0)x^2+(c_3+c_2+\tfrac{1}{2}c_1+\tfrac{1}{6}c_0)x^3+\cdots\big)\,. \end{aligned}$$

Put these in the DE to get

$$\begin{aligned} &\big(2c_2+6c_3x+12c_4x^2+20c_5x^3+\cdots\big)+2\big(c_1+2c_2x+3c_3x^2+4c_4x^3+\cdots\big) \\ &+\big(c_0+(c_1+c_0)x+(c_2+c_1+\tfrac{1}{2}c_0)x^2+(c_3+c_2+\tfrac{1}{2}c_1+\tfrac{1}{6}c_0)x^3+\cdots\big) \\ &=\big(x-\tfrac{1}{6}x^3+\cdots\big)\,. \end{aligned}$$

Equate coefficients to get $2c_2+2c_1+c_0=0$ (1), $6c_3+4c_2+c_1+c_0=1$ (2), $12c_4+6c_3+c_2+c_1+\frac{1}{2}c_0=0$ (3) and $20c_5+8c_4+c_3+c_2+\frac{1}{2}c_1+\frac{1}{6}c_0=-\frac{1}{6}$ (4). To get $y(0)=2$ we need $c_0=2$ and to get $y'(0)=1$ we need $c_1=1$. Put these in the recursion formulas (1)-(4) to get $c_2=\frac{-2c_1-c_0}{2}=\frac{-2-2}{2}=-2$, $c_3=\frac{1-4c_2-c_1-c_0}{6}=\frac{1+8-1-2}{6}=1$, $c_4=\frac{-6c_3-c_2-c_1-\frac{1}{2}c_0}{12}=\frac{-6+2-1-1}{12}=-\frac{1}{2}$ and $c_5=\frac{-\frac{1}{6}-8c_4-c_3-c_2-\frac{1}{2}c_1-\frac{1}{6}c_0}{20}=\frac{-\frac{1}{6}+4-1+2-\frac{1}{2}-\frac{1}{3}}{20}=\frac{1}{5}$. Thus the 5^{th} Taylor polynomial is $T_5(x)=2+x-2x^2+x^3-\frac{1}{2}x^4+\frac{1}{5}x^5$.

2: Use the Power Series Method to solve the ODE $y''+(x-1)y'+y=0$. Find two linearly independent power series solutions, centred at 0, one satisfying the initial conditions $y(0)=1$, $y'(0)=0$, and the other satisfying $y(0)=0$, $y'(0)=1$. For each solution, state the recurrence relation for the coefficients, and find the 5^{th} Taylor polynomial centred at 0.

Solution: We try $y=\sum\limits_{n\geq 0}c_nx^n$. Then $y'=\sum\limits_{n\geq 1}nc_nx^{n-1}$ and $y''=\sum\limits_{n\geq 2}^{\infty}n(n-1)c_nx^{n-2}$. Put these in the DE to get

$$\begin{aligned} 0 &= y''+(x-1)y'+y \\ &= \sum_{n\geq 2}n(n-1)c_nx^{n-2}+\sum_{n\geq 1}nc_nx^n-\sum_{n\geq 1}nc_nx^{n-1}+\sum_{n\geq 0}c_nx^n \\ &= \sum_{m\geq 0}(m+2)(m+1)c_{m+2}x^m+\sum_{m\geq 1}m\,c_mx^m-\sum_{m\geq 0}(m+1)c_{m+1}x^m+\sum_{m\geq 0}c_mx^m \\ &= (2c_2-c_1+c_0)x^0+\sum_{m\geq 1}\big((m+2)(m+1)c_{m+2}-(m+1)c_{m+1}+(m+1)c_m\big)x^m\,. \end{aligned}$$

All coefficients vanish, so $c_2=\frac{c_1-c_0}{2}$ and $c_{m+2}=\frac{(m+1)c_{m+1}-(m+1)c_m}{(m+2)(m+1)}=\frac{c_{m+1}-c_m}{m+2}$ for $m\geq 1$. If $c_0=1$ and $c_1=0$ then the recursion formulas give $c_2=\frac{0-1}{2}=-\frac{1}{2}$, $c_3=\frac{-\frac{1}{2}-0}{3}=-\frac{1}{6}$, $c_4=\frac{-\frac{1}{6}+\frac{1}{2}}{4}=\frac{1}{12}$ and $c_5=\frac{\frac{1}{12}+\frac{1}{6}}{5}=\frac{1}{20}$, so the 5^{th} Taylor polynomial is

$$T_5(y_1)=1-\tfrac{1}{2}x^2-\tfrac{1}{6}x^3+\tfrac{1}{12}x^4+\tfrac{1}{20}x^5\,.$$

If $c_0=0$ and $c_1=1$ then the recursion formulas give $c_2=\frac{1-0}{2}=\frac{1}{2}$, $c_3=\frac{\frac{1}{2}-1}{3}=-\frac{1}{6}$, $c_4=\frac{-\frac{1}{6}-\frac{1}{2}}{4}=-\frac{1}{6}$ and $c_5=\frac{-\frac{1}{6}+\frac{1}{6}}{5}=0$ and so the 5^{th} Taylor polynomial for the solution y_2 is

$$T_5(y_2)=x+\tfrac{1}{2}x^2-\tfrac{1}{6}x^3-\tfrac{1}{6}x^4\,.$$

3: Use Frobenius' Method to solve the ODE $4xy'' + 2y' = y$. Find two linearly independent series solutions, centred at 0. For each solution, solve the recurrence relation to obtain an explicit formula for the n^{th} coefficient, then find a closed form formula for the solution.

Solution: We try $y = \sum\limits_{n\geq 0} c_n x^{n+r}$ so $y' = \sum\limits_{n\geq 0}(n+r)c_n x^{n+r-1}$ and $y'' = \sum\limits_{n\geq 0}(n+r)(n+r-1)c_n x^{n+r-2}$. Put these in the DE to get

$$\begin{aligned}
0 &= 4xy'' + 2y' - y \\
&= \sum_{n\geq 0} 4(n+r)(n+r-1)c_n x^{n+r-1} + \sum_{n\geq 0} 2(n+r)c_n x^{n+r-1} - \sum_{n\geq 0}^{\infty} c_n x^{n+r} \\
&= x^r\Big(\sum_{m\geq -1} 4(m+r+1)(m+r)c_{m+1}x^m + \sum_{m\geq -1} 2(m+r+1)c_{m+1}x^m - \sum_{m\geq 0} c_m x^m\Big) \\
&= x^r\Big(\sum_{m\geq -1} 2(m+r+1)(2m+2r+1)c_{m+1}x^m - \sum_{m\geq 0} c_m x^m\Big) \\
&= x^r\Big(2r(2r-1)c_0 x^{-1} + \sum_{m\geq 0}\big(2(m+r+1)(2m+2r+1)c_{m+1} - c_m\big)x^m\Big).
\end{aligned}$$

All coefficients must vanish, so we have $r(2r-1) = 0$ and $c_{m+1} = \frac{c_m}{2(m+r+1)(2m+2r+1)}$ for $m \geq 0$. When $r = 0$, the recursion formula becomes $c_{m+1} = \frac{c_m}{2(m+1)(2m+1)} = \frac{c_m}{(2m+1)(2m+2)}$, so if $c_0 = 1$ then we get $c_1 = \frac{1}{1\cdot 2}$, $c_2 = \frac{1}{1\cdot 2\cdot 3\cdot 4}$, $c_3 = \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}$, and in general $c_n = \frac{1}{(2n)!}$. In this case the solution is

$$y_1 = x^0\left(\sum_{n=0}^{\infty}\frac{x^n}{(2n)!}\right) = \cosh\sqrt{x}\,.$$

When $r = \frac{1}{2}$ the recursion formula becomes $c_{m+1} = \frac{c_m}{2\left(m+\frac{3}{2}\right)(2m+2)} = \frac{c_m}{(2m+2)(2m+3)}$, so if $c_0 = 1$ then we get $c_1 = \frac{1}{2\cdot 3}$, $c_2 = \frac{1}{2\cdot 3\cdot 4\cdot 5}$ and in general $c_n = \frac{1}{(2n+1)!}$. In this case the solution is

$$y_2 = x^{1/2}\left(\sum_{n=0}^{\infty}\frac{x^n}{(2n+1)!}\right) = \sqrt{x}\cdot\frac{\sinh\sqrt{x}}{\sqrt{x}} = \sinh\sqrt{x}\,.$$

The general solution is $y = a\cosh\sqrt{x} + b\sinh\sqrt{x}$, for $x \geq 0$.

4: Use Frobenius' Method to solve the ODE $3x^2y'' + x(x-1)y' + y = 0$. Find two linearly independent series solutions, centred at 0. For each solution, solve the recurrence relation to obtain an explicit formula for the n^{th} coefficient. Find a closed form formula for one of the two solutions.

Solution: We try $y = \sum\limits_{n\geq 0} c_n x^{n+r}$ so $y' = \sum\limits_{n\geq 0}(n+r)c_n x^{n+r-1}$ and $y'' = \sum\limits_{n\geq 0}(n+r)(n+r-1)c_n x^{n+r-2}$. Put these in the DE to get

$$
\begin{aligned}
0 &= 3x^2y'' + x(x-1)y' + y \\
&= \sum_{n\geq 0} 3(n+r)(n+r-1)c_n x^{n+r} + \sum_{n\geq 0}(n+r)c_n x^{n+r+1} - \sum_{n\geq 0}(n+r)c_n x^{n+r} + \sum_{n\geq 0} c_n x^{n+r} \\
&= x^r\Big(\sum_{m\geq 0} 3(m+r)(m+r-1)c_m x^m + \sum_{m\geq 1}(m+r-1)c_{m-1}x^m - \sum_{m\geq 0}(m+r)c_m x^m + \sum_{m\geq 0} a_m x^m\Big) \\
&= x^r\Big(\sum_{m\geq 0}\big(3(m+r)(m+r-1)-(m+r)+1\big)c_m x^m + \sum_{m\geq 1}^{\infty}(m+r-1)c_{m-1}x^m\Big) \\
&= x^r\Big(\big(3r(r-1)-r+1\big)c_0x^0 + \sum_{m\geq 1}\big(\big(3(m+r)(m+r-1)-(m+r)+1\big)c_m + (m+r-1)c_{m-1}\big)x^m\Big).
\end{aligned}
$$

All coefficients must vanish, so we have $3r(r-1)-r+1=0$, that is $3r^2-4r+1=0$ or equivalently $(3r-1)(r-1)=0$ so $r=1$ or $r=\frac{1}{3}$, and we have $c_m = \frac{-(m+r-1)c_{m-1}}{3(m+r)(m+r-1)-(m+r)+1} = \frac{-c_{m-1}}{3(m+r)-1}$.

When $r=1$ the recursion formula becomes $c_m = \frac{-c_{m-1}}{3m+2}$. If we take $c_0=1$ then we have $c_1 = -\frac{1}{5}$, $c_2 = \frac{1}{5\cdot 8}$, $c_3 = -\frac{1}{5\cdot 8\cdot 11}$, and in general $c_n = \frac{(-1)^n}{5\cdot 8\cdot 11\cdot\cdots\cdot(3n+2)}$. In this case we obtain the solution

$$y_1 = x^1\left(1 + \sum_{n=1}^{\infty}\frac{(-1)^n x^n}{5\cdot 8\cdot 11\cdot\cdots\cdot(3n+2)}\right) = x - \tfrac{1}{5}x^2 + \tfrac{1}{5\cdot 8}x^3 - \tfrac{1}{5\cdot 8\cdot 11}x^4 + \cdots.$$

When $r=\frac{1}{3}$ the recursion formula becomes $c_m = -\frac{c_{m-1}}{3m}$. If we set $c_0=1$ then we obtain $c_2 = -\frac{1}{3}$, $c_2 = \frac{1}{3\cdot 6}$, $c_3 = -\frac{1}{3\cdot 6\cdot 9}$, and in general $c_n = \frac{(-1)^n}{3\cdot 6\cdot 9\cdot\cdots\cdot(3n)} = \frac{(-1)^n}{3^n n!}$. In this case we obtain the solution

$$y_2 = x^{1/3}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n n!}\right) = x^{1/3}e^{-x/3}.$$