Chapter 9. The Seifert-Van Kampen Thheorem

The Seifert-Van Kampen Theorem

9.1 Note: Let $\alpha_1,\cdots,\alpha_n$ be paths in a topological space X, with the endpoint of α_k equal to the initial point of α_{k+1}. Let $P=(x_0,x_1,\cdots,x_n)$ and $Q=(y_0,y_1,\cdots,y_n)$ be two partitions of the interval $[0,1]$. Let β and γ be the paths in X which follow the paths $\alpha_1,\alpha_2,\cdots,\alpha_n$ with $\beta(t)=\alpha_k\left(\frac{t-x_{k-1}}{x_k-x_{k-1}}\right)$ for $t\in[x_{k-1},x_k]$ and $\gamma(t)=\alpha_k\left(\frac{t-y_{k-1}}{y_k-y_{k-1}}\right)$ for $t\in[y_{k-1},y_k]$. Then we have $\beta\sim\gamma$: indeed a homotopy from β to γ in X is given by

$$F(s,t)=\alpha_k\left(\frac{t-((1-s)x_{k-1}+sy_{k-1})}{((1-s)x_k+sy_k)-((1-s)x_{k-1}+sy_{k-1})}\right)\ \text{ for } t\in\big[(1-s)x_{k-1}+sy_{k-1}\,,\,(1-s)x_k+sy_k\big].$$

9.2 Definition: When $\alpha_1,\alpha_2,\cdots,\alpha_n$ are paths in a topological space, with the endpoint of α_k equal to the endpoint of α_{k+1}, we shall write $\alpha_1\alpha_2\cdots\alpha_n$ to denote the path γ which follows the paths $\alpha_1,\cdots,\alpha_n$ with $\gamma(t)=\alpha_k(nt-(k-1))$ for $t\in\left[\frac{k-1}{n},\frac{k}{n}\right]$, so that α_k is the path obtained by restricting $\gamma=\alpha_1\alpha_2\cdots\alpha_n$ to the interval $\left[\frac{k-1}{n},\frac{k}{n}\right]$.

9.3 Note: Suppose that $F:[a,b]\times[c,d]\to X$ is continuous, and let α_a and α_b, be the paths obtained by restricting F to the intervals $\{a\}\times[c,d]$ and $\{b\}\times[c,d]$, so for example α_a is given by $\alpha_a(t)=F\left(a,\frac{t-c}{d-c}\right)$, and let β_c and β_d be the paths obtained by restricting F to $[a,b]\times\{c\}$ and $[a,b]\times\{d\}$. Then we have $\alpha_a\beta_d\sim\beta_c\alpha_b$: Indeed a homotopy from $\alpha_a\beta_d$ to $\beta_c\alpha_b$ is given by

$$G(s,t)=\left\{\begin{array}{ll} F\big((1-2t)(a,c)+2t((1-s)(b,c)+s(a,d))\big) & \text{if } 0\le t\le\frac12 \\ F\big((2-2t)((1-s)(b,c)+s(a,d))+(2t-1)(b,d)\big) & \text{if } \frac12\le t\le 1\end{array}\right\}.$$

9.4 Theorem: *(The Seifert Van Kampen Theorem) Let X be a topological space with $a\in X$. Suppose that $X=\bigcup_{k\in K}U_k$ where each U_k is open in X with $a\in U_k$. Suppose that U_k, $U_k\cap U_\ell$ and $U_k\cap U_\ell\cap U_m$ are path-connected for all $k,\ell,m\in K$. Then*

$$\pi_1(X,a)\cong\Big(\mathop{\ast}_{k\in K}\pi_1(U_k,a)\Big)/N$$

where N is the normal subgroup generated by elements of the form $[\omega]_k[\omega^{-1}]_\ell$ where ω is a loop at a in $U_k\cap U_\ell$ and $[\omega]_k\in\pi_1(U_k,a)$ and $[\omega]_\ell\in\pi_1(U_\ell,a)$.

Proof: Define $\phi:\mathop{\ast}_{k\in K}\pi_1(U_k,a)\to\pi_1(X,a)$ by

$$\phi\big([\sigma_1]_{\ell_1}[\sigma_2]_{\ell_2}\cdots[\sigma_n]_{\ell_n}\big)=[\sigma_1\sigma_2\cdots\sigma_n]\in\pi_1(X,a)$$

where σ_i is a loop at a in U_{ℓ_i}, and $[\sigma_i]_{\ell_i}\in\pi_1(U_{\ell_i},a)$. Verify, as an exercise, that ϕ is well-defined and ϕ is a group homomorphism.

We claim that ϕ is surjective. Let $\gamma:[0,1]\to X$ be any loop at a in X. The sets $\gamma^{-1}(U_k)$ form an open cover of $[0,1]$, which is compact. Choose a Lebesgue number $\lambda>0$ for this cover. Choose $n\in\mathbb{Z}^+$ large enough so that $\frac1n<\lambda$. Each interval $I_j=\left[\frac{j-1}{n},\frac{j}{n}\right]$ is contained in one of the open sets $\gamma^{-1}(U_k)$, say $I_j\subseteq\gamma^{-1}(U_{\ell_j})$, that is $\gamma(I_j)\subseteq U_{\ell_j}$. For $1\le j\le n$, let α_j be the path obtained by restricting γ to the interval I_j, that is let $\alpha_j(t)=\gamma\left(\frac tn+\frac{j-1}{n}\right)$. For $1\le j\le n-1$ we have $\frac jn\in I_j\cap I_{j+1}$ so that $\gamma\left(\frac jn\right)\in U_{\ell_j}\cap U_{\ell_{j+1}}$, which is path-connected, so we can choose a path ρ_j from a to $\gamma\left(\frac jn\right)$ in $U_{\ell_j}\cap U_{\ell_{j+1}}$. Also, let ρ_0 and ρ_n be the constant loop κ at a. For $1\le j\le n$, let $\sigma_j=\rho_{j-1}\alpha_j\rho_j^{-1}$, which is a loop at a in U_{ℓ_j}. We have $\gamma\sim\alpha_1\alpha_2\cdots\alpha_n\sim\rho_0\alpha_1\rho_1^{-1}\,\rho_1\alpha_2\rho_2^{-1}\cdots\rho_{n-1}\alpha_n\rho_n^{-1}=\sigma_1\sigma_2\cdots\sigma_n$ so that $\phi\big([\sigma_1]_{\ell_1}[\sigma_2]_{\ell_2}\cdots[\sigma_n]_{\ell_n}=[\sigma_1\sigma_2\cdots\sigma_n]=[\gamma]$. Thus ϕ is surjective, as claimed.

Since ϕ is surjective, it follows from the First Isomorphism Theorem that

$$\pi_1(X,a) \cong \big(\mathop{\ast}_{k\in K} \pi_1(U_k,a)\big)/\operatorname{Ker}\phi.$$

We need to prove that $\operatorname{Ker}\phi = N$ where N is the normal subgroup generated by elements of the form $[\omega]_k[\omega^{-1}]_\ell$ where ω is a loop at a in $U_k \cap U_\ell$. Note that when ω is a loop at a in $U_k \cap U_\ell$ we have $\phi\big([\omega]_k[\omega^{-1}]_\ell\big) = [\omega\omega^{-1}] = [\kappa]$, which is the identity element in $\pi_1(X,a)$, so we have $N \subseteq \operatorname{Ker}\phi$.

It remains to show that $\operatorname{Ker}\phi \subseteq N$. For now, suppose that each quadruple intersection $U_k \cap U_\ell \cap U_m \cap U_n$ is path-connected, where $k,\ell,m,n \in K$. Later we shall show how to modify the proof so that it suffices to suppose that each triple intersection $U_k \cap U_\ell \cap U_m$ is path-connected. Let $[\sigma_1]_{\ell_1}[\sigma_2]_{\ell_2}\cdots[\sigma_n]_{\ell_n} \in \operatorname{Ker}\phi$, where each σ_j is a loop at a in U_{ℓ_j} with $\ell_j \in K$. This means that $\sigma_1\sigma_2\cdots\sigma_n \sim \kappa$ in X. Let $F:[0,1]\times[0,1]\to X$ be a homotopy from $\sigma_1\sigma_2\cdots\sigma_n$ to κ in X. The sets $F^{-1}(U_k)$ form an open cover of $[0,1]\times[0,1]$, which is compact. Choose a Lebesgue number $\lambda > 0$ for this open cover. Choose m to be a multiple of n which is large enough so that $\frac{1}{m} < \lambda$. Each square $I_{i,j} = \left[\frac{i-1}{m},\frac{i}{m}\right]\times\left[\frac{j-1}{m},\frac{j}{m}\right]$ is contained in one of the sets $F^{-1}(U_k)$, say $I_{i,j} \subseteq F^{-1}(U_{k_{i,j}})$, that is $F(I_{i,j}) \subseteq U_{k_{i,j}}$.

For $0 \le i,j \le m$, let $x_{i,k} = F\big(\frac{i}{m},\frac{j}{m}\big)$. Note that $x_{i,0} = x_{i,m} = x_{m,j} = a$. For $0 \le i \le m$ and $1 \le j \le m$, let $\alpha_{i,j}$ be the path from $x_{i,j-1}$ to $x_{i,j}$ obtained by restricting F to the interval $\left\{\frac{i}{m}\right\}\times\left[\frac{j-1}{m},\frac{j}{m}\right]$. For $1 \le i \le m$ and $0 \le j \le m$, let $\beta_{i,j}$ be the path from $x_{i-1,j}$ to $x_{i,j}$ obtained by restricting F to the interval $\left[\frac{i-1}{m},\frac{i}{m}\right]\times\left\{\frac{j}{m}\right\}$. Recall that m is a multiple of n, say $m = pn$. Then we have $\sigma_1 = \alpha_{0,1}\alpha_{0,2}\cdots\alpha_{0,p}$ and $\sigma_2 = \alpha_{0,p+1}\alpha_{0,p+2}\cdots\alpha_{0,2p}$, and so on. Let $k_{0,1} = k_{0,2} = \cdots = k_{0,p} = \ell_1$ and $k_{0,p+1} = k_{0,p+2} = \cdots = k_{0,2p} = \ell_2$ and so on.

Note that (if $j > 0$) $x_{i,j}$ lies in $U_{k_{i,j}}$ and (if $i < m$ and $j > 0$) in $U_{k_{i+1,j}}$ and (if $j < m$) in $U_{k_{i,j+1}}$ and (if $i < m$ and $j < m$) in $U_{k_{i+1,j+1}}$. For $0 \le i < m$ and $1 \le j < m$, choose a path $\rho_{i,j}$ from a to $x_{i,j}$ which lies in all the relevant sets $U_{k_{i,j}}$, $U_{k_{i+1,j}}$, $U_{k_{i,j+1}}$ and $U_{k_{i+1,j+1}}$ (we can do this since quadruple intersections are path-connected). Also, noting that $x_{i,0} = x_{i,m} = x_{m,j} = a$, for all i,j, we choose $\rho_{i,0} = \rho_{i,m} = \rho_{m,j} = \kappa$, the constant loop at a. For $0 \le i \le m$ and $1 \le j \le m$, let $\sigma_{i,j} = \rho_{i,j-1}\alpha_{i,j}\rho_{i,j}^{-1}$. Note that $\sigma_{i,j}$ is a loop at a which lies in $U_{k_{i,j}}$ and (if $i < m$) in $U_{k_{i+1,j}}$, and that $\sigma_{m,j} = \kappa$. For $1 \le i \le m$ and $0 \le j \le m$, let $\tau_{i,j} = \rho_{i-1,j}\beta_{i,j}\rho_{i,j}^{-1}$. Note that $\tau_{i,j}$ is a loop at a which (if $j > 0$) lies in $U_{k_{i,j}}$ and (if $j < m$) in $U_{k_{i,j+1}}$, and that $\tau_{i,0} = \tau_{m,0} = \kappa$.

For $0 \le i \le m$, let $u_i = [\sigma_{i,1}]_{k_{i,1}}[\sigma_{i,2}]_{k_{i,2}}\cdots[\sigma_{i,m}]_{k_{i,m}}$. Note that

$$[\sigma_1]_{\ell_1}[\sigma_2]_{\ell_2}\cdots[\sigma_n]_{\ell_n} = [\sigma_{0,1}]_{k_{0,1}}[\sigma_{0,2}]_{k_{0,2}}\cdots[\sigma_{0,m}]_{k_{0,m}} = u_0.$$

For $u,v \in \mathop{\ast}_{k\in K}\pi_1(U_k,a)$, write $u \equiv v$ to indicate that $uN = vN$. We shall complete the proof by showing that $u_0 \equiv u_1 \equiv \cdots \equiv u_m$ and noting that $u_m = 0$ (the empty string in $\mathop{\ast}_{k\in K}\pi_1(U_k,a)$), so that $u_0 \in N$. We do this using a sequence of steps, at each step using one of the following two observations. First, note that when ω is a loop at a in $U_k \cap U_\ell$, since $[\omega]_k[\omega^{-1}]_\ell \in N$, we have $[\omega]_\ell \equiv [\omega]_k[\omega^{-1}]_\ell[\omega]_\ell = [\omega]_k$. Second, note that by Note 6.3, in the set $U_{k_{i,j}}$ we have $\alpha_{i-1,j}\beta_{i,j} \sim \beta_{i,j-1}\alpha_{i,j}$ so that $\sigma_{i-1,j}\tau_{i,j} \sim \tau_{i,j-1}\sigma_{i,j}$, hence $\tau_{i,j-1}^{-1}\sigma_{i-1,j} \sim \sigma_{i,j}\tau_{i,j}^{-1}$, and so we have $[\tau_{i,j-1}^{-1}]_{k_{i,j}}[\sigma_{i-1,j}]_{k_{i,j}} = [\sigma_{i,j}]_{k_{i,j}}[\tau_{i,j}^{-1}]_{k_{i,j}}$.

Using the above two observations, repeatedly, gives

$$\begin{aligned}
u_{i-1} &= [\sigma_{i-1,1}]_{k_{i-1,1}}[\sigma_{i-1,2}]_{k_{i-1,2}}[\sigma_{i-1,3}]_{k_{i-1,3}}\cdots[\sigma_{i-1,m}]_{k_{i-1,m}}\\
&= [\tau_{i,0}^{-1}]_{k_{i,0}}[\sigma_{i-1,1}]_{k_{i-1,1}}[\sigma_{i-1,2}]_{k_{i-1,2}}[\sigma_{i-1,3}]_{k_{i-1,3}}\cdots[\sigma_{i-1,m}]_{k_{i-1,m}}\\
&\equiv [\tau_{i,0}^{-1}]_{k_{i,1}}[\sigma_{i-1,1}]_{k_{i,1}}[\sigma_{i-1,2}]_{k_{i-1,2}}[\sigma_{i-1,3}]_{k_{i-1,3}}\cdots[\sigma_{i-1,m}]_{k_{i-1,m}}\\
&= [\sigma_{i,1}]_{k_{i,1}}[\tau_{i,1}^{-1}]_{k_{i,1}}[\sigma_{i-1,2}]_{k_{i-1,2}}[\sigma_{i-1,3}]_{k_{i-1,3}}\cdots[\sigma_{i-1,m}]_{k_{i-1,m}}\\
&\equiv [\sigma_{i,1}]_{k_{i,1}}[\tau_{i,1}^{-1}]_{k_{i,2}}[\sigma_{i-1,2}]_{k_{i,2}}[\sigma_{i-1,3}]_{k_{i-1,3}}\cdots[\sigma_{i-1,m}]_{k_{i-1,m}}\\
&= [\sigma_{i,1}]_{k_{i,1}}[\sigma_{i,2}]_{k_{i,2}}[\tau_{i,2}^{-1}]_{k_{i,2}}[\sigma_{i-1,3}]_{k_{i-1,3}}\cdots[\sigma_{i-1,m}]_{k_{i-1,m}}\\
&\quad\vdots\\
&= [\sigma_{i,1}]_{k_{i,1}}[\sigma_{i,2}]_{k_{i,2}}\cdots[\sigma_{i,m-1}]_{k_{i,m-1}}[\tau_{i,m-1}^{-1}]_{k_{i,m-1}}[\sigma_{i-1,m}]_{k_{i-1,m}}\\
&\equiv [\sigma_{i,1}]_{k_{i,1}}[\sigma_{i,2}]_{k_{i,2}}\cdots[\sigma_{i,m-1}]_{k_{i,m-1}}[\tau_{i,m-1}^{-1}]_{k_{i,m}}[\sigma_{i-1,m}]_{k_{i,m}}\\
&= [\sigma_{i,1}]_{k_{i,1}}[\sigma_{i,2}]_{k_{i,2}}\cdots[\sigma_{i,m-1}]_{k_{i,m-1}}[\sigma_{i,m}]_{k_{i,m}}[\tau_{i,m}^{-1}]_{k_{i,m}}\\
&= [\sigma_{i,1}]_{k_{i,1}}[\sigma_{i,2}]_{k_{i,2}}\cdots[\sigma_{i,m-1}]_{k_{i,m-1}}[\sigma_{i,m}]_{k_{i,m}} = u_{i+1}.
\end{aligned}$$

Thus $[\sigma_1]_{\ell_1}\cdots[\sigma_n]_{\ell_n} \equiv u_1 \equiv u_m = [\sigma_{m,1}]_{k_{m,1}}\cdots[\sigma_{m,m}]_{k_{m,m}} = 0$ since each $\sigma_{m,j} = \kappa$. This proves that $[\sigma_1]_{\ell_1}[\sigma_2]_{\ell_2}\cdots[\sigma_n]_{\ell_n} \in N$ and hence $\operatorname{Ker}\phi \subseteq N$, as required.

This completes the proof, under the assumption that quadruple intersections are path-connected. We can modify the proof so that only triple intersections need to be path-connected as follows. Rather than partitioning the domain of F into the squares $I_{i,j} = \left[\frac{i-1}{m}, \frac{i}{m}\right] \times \left[\frac{j-1}{m}, \frac{j}{m}\right]$, which sometimes meet four squares at a vertex, we can partition the domain of F into squares and rectangles $R_{i,j}$ with at most three meeting at each vertex: when i is even, let $R_{i,j} = I_{i,j}$, when i is odd, move the horizontal edges up by $\frac{1}{3m}$ letting $R_{i,1} = \left[\frac{i-1}{m}, \frac{i}{m}\right] \times \left[0, \frac{4}{3m}\right]$ and $R_{i,j} = I_{i,j} + \left(0, \frac{1}{3m}\right)$ for $1 < j < m$, and $R_{i,m} = \left[\frac{i-1}{m}, \frac{i}{m}\right] \times \left[\frac{m-1}{m} + \frac{1}{3m}, 1\right]$. Note that the largest rectangles have sides of length $\frac{1}{m}$ and $\frac{4}{3m}$, hence their diameter is $\frac{5}{3m} < \frac{2}{m} < 2\lambda$, so they lie in an open ball of radius λ, and hence they lie in one of the open sets $F^{-1}(U_k)$, $k \in K$. Thus we can repeat the same argument used above to show that $\operatorname{Ker}\phi \subseteq N$, and we only need to assume that triple intersections are path-connected.

9.5 Corollary: *Let X be a topological space with $a \in X$. Suppose that $X = U \cup V$ where U and V are open in X with $a \in U \cap V$. Suppose that U, V and $U \cap V$ are path-connected. Then*

$$\pi_1(X,a) \cong \big(\pi_1(U,a) * \pi_1(V,a)\big)/N$$

where N is the normal subgroup generated by elements of the form $[\omega]_U[\omega^{-1}]_V$ and $[\omega]_V[\omega^{-1}]_U$, where ω is a loop at a in $U \cap V$. Also, we have the following two particular cases:

(1) If $\pi_1(U \cap V) = 0$ then $\pi_1(X,a) \cong \pi_1(U,a) * \pi_1(V,a)$.
(2) If $\pi_1(V,a) = 0$ then $\pi_1(X,a) \cong \pi_1(U,a)/N$ where N is the normal subgroup generated by elements of the form $[\omega]_U$ where ω is a loop at a in $U \cap V$.

9.6 Example: Note that when $n \geq 2$ we have $\pi_1(\mathbb{S}^n) = 0$: Indeed, let $1 = (1, 0, \cdots, 0)$ and $p = (0, \cdots, 0, 1)$, and take $U = \mathbb{S}^n \setminus \{p\}$ and $V = \mathbb{S}^n \setminus \{-p\}$. Then, using stereographic projection, we have $U \cong \mathbb{R}^n$ so that $\pi_1(U, 1) = 0$ and $V \cong \mathbb{R}^n$ so that $\pi_1(V, 1) = 0$, and $U \cap V \cong \mathbb{R}^2 \setminus \{0\}$ which is path-connected, and hence $\pi_1(X, 1) = 0$ by the Seifert Van Kampen Theorem.

9.7 Definition: For based topological spaces (X_k, a_k), where K is a nonempty set, the **wedge product** $\bigwedge_{k\in K}(X_k, a_k)$ is the quotient space of the disjoint union $\bigsqcup_{k\in K} X_k$ under the equivalence relation which identifies all the basepoints. The equivance class containing the basepoints is the basepoint of the wedge product.

9.8 Example: The finite wedge product of circles $\bigwedge_{k=1}^{n}(\mathbb{S}^1, 1)$ is homeomorphic to the n**-loop space**, which is the union of the images of the loops $\alpha_k(t) = (\sin \pi t)e^{i\,2\pi(k+t)/n}$ for $1 \le k \le n$, and also to the **shrinking wedge of** n **circles**, which is the union of the images of the loops $\alpha_k(t) = \frac{1}{k}(\sin \pi t)e^{i\,\pi t}$ for $1 \le k \le n$.

The countable wedge of circles $\bigwedge_{k=1}^{\infty}(\mathbb{S}^1, 1)$, by contrast, is not homeomorphic to the countable **shrinking wedge of circles**, which is the union of the images of the loops $\alpha_k(t) = \frac{1}{k}(\sin \pi t)e^{i\,\pi t}$ for $k \in \mathbb{Z}^+$. One way to see this is to note that the countable wedge of circles is locally simply connected, but the countable shrinking wedge of circles is not.

9.9 Example: Show that $\pi_1\big(\bigwedge_{k=1}^{n}(\mathbb{S}^1, 1)\big) = \big\langle \alpha_1, \cdots, \alpha_n \big\rangle \cong \ast_{k=1}^{n}\,\mathbb{Z}$ where α_k is the loop which goes once around the k^{th} circle $S_k = \mathbb{S}^1$.

9.10 Example: Let G be a finite connected graph (consisting of a finite set of vertices and a finite set of edges), and let a be a vertex of G. Let T be a maximal tree in G (that is a maximal subgraph which contains no cycles). Let $E_1, \cdots, E_n$ be the edges in G which do not lie in T (so for each k, the graph $T \cup E_k$ contains a cycle). For each k, let α_k be a loop in $G \cup E_k$ which follows a path γ along T from a to an endpoint of E_k, then follows a cycle in $T \cup E_k$, then follows γ^{-1} back to a. Show that $\pi_1(G, a) \cong \langle \alpha_1, \cdots, \alpha_n \rangle \cong \ast_{k=1}^{n}\,\mathbb{Z}$.

9.11 Example: Recall that $(\mathbb{T}^2)^{\#g}$ is homeomorphic to the quotient space $T_g^2 = D/\sim$ where D is the closed unit disc and $\sim$ is the equivalence relation which identifies points on the boundary $S = \partial D$ according to the word $\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\alpha_2\beta_2\alpha_2^{-1}\beta_2^{-1}\cdots\alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1}$. Show that $\pi_1(T_g^2, 1) = \big\langle \alpha_1, \beta_1, \alpha_2, \beta_2, \cdots, \alpha_g\beta_g \,\big|\, \alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\alpha_2\beta_2\alpha_2^{-1}\beta_2^{-1}\cdots\alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1}\big\rangle$. Also recall that $(\mathbb{P}^2)^{\#h} \cong P_h^2 = D/\sim$ where $\sim$ identifies points on $S = \partial D$ according to $\alpha_1^2\alpha_2^2\cdots\alpha_h^2$. Show that $\pi_1(P_h^2, 1) = \big\langle \alpha_1, \alpha_2, \cdots, \alpha_h \,\big|\, \alpha_1^2\alpha_2^2\cdots\alpha_h^2\big\rangle$. Deduce that $Ab\big(\pi_1(T_g^2)\big) \cong \mathbb{Z}^{2g}$ and $Ab(\pi_1(P_h^2)) \cong \mathbb{Z}^{h-1} \times \mathbb{Z}_2$.

9.12 Example: Show that given any group of the form $G \cong \langle \alpha_1, \cdots, \alpha_n \,\big|\, w_1, \cdots, w_\ell \rangle$, we can construct a based topological space (X, a) with $\pi_1(X, a) \cong G$ as follows. Let (W, a) be the wedge product of n circles, and let α_k be the loop at a which goes once around the k^{th} circle $S_k = \mathbb{S}^1$. Let X be the quotient space of the disjoint union of W with ℓ closed discs $D_1, D_2, \cdots, D_\ell$ under the equivalence relation which identifies points on the boundary of the circle $T_j = \partial D_j$ with points on W according to word w_j.

9.13 Definition: A (finite) **CW complex** is a topological space X which is obtained as follows: We begin with a finite discrete set of points X^0. Having constructed X^{k-1}, we let X^k be the quotient space of the disjoint union of X^{k-1} with finitely many closed k-balls $D_1, D_2, \cdots, D_\ell$, under the equivalence relation which identifies points on the boundary $S_j = \partial D_j$ with points on X^{k-1} in accordance with a continuous map $f_j : S_j \to X^{k-1}$. Eventually the construction ends with $X = X^n$. The space X^k is called the k **skeleton** of X.

9.14 Remark: The fundamental group of a CW complex is equal to the fundamental group of its 2-skeleton.