Chapter 8. Free Groups and Free Products of Groups

Direct Products and Sums of Groups

8.1 Definition: Let K be a nonempty set, and let G_k be a group for each $k \in K$. The (direct) **product** of the groups G_k is the set

$$\prod_{k\in K} G_k = \left\{ a : K \to \bigcup_{k\in K} G_k \,\middle|\, a(k) \in G_k \right\}.$$

For $a \in \prod_{k\in K} G_\alpha$ we write $a_k = a(k)$. The operation is given by $(ab)(k) = a(k)b(k) \in G_k$. For each $\ell \in K$, we have the **projection map** $p_\ell : \prod_{k\in K} G_k \to G_\ell$ given by $p_\ell(a) = a_\ell$, and the **inclusion map** $i_\ell : G_\ell \to \prod_{k\in K} G_k$ given by $i_\ell(x)(k) = e \in G_k$ when $k \neq \ell$ and $i_\ell(x)(\ell) = x$. The maps p_ℓ and i_ℓ are group homomorphisms.

In the case that $K = \{1, 2, 3, \cdots\}$, we also write $\prod_{k\in K} G_k = \prod_{k=1}^{\infty} G_k$, and in the case that $K = \{1, 2, \cdots, n\}$ we also write $\prod_{k\in K} G_k = \prod_{k=1}^{n} G_k = G_1 \times G_2 \times \cdots \times G_n$.

Recall, or verify, that the product group is characterized, up to isomorphism, by the following, so called, universal mapping property: for every group H, and for all group homomorphisms $f_k : H \to G_k$, there is a unique group homomorphism $f : H \to \prod_{k\in K} G_k$ such that $p_k \circ f = f_k$ for all $k \in K$.

8.2 Remark: The fact that $\prod_{k\in G} G_k$ is characterized by the above mapping property can be summarized by saying that $\prod_{k\in K} G_k$ is a **product** in the category of groups. In the case that each group G_k is abelian, the product $\prod_{k\in K} G_k$ is also abelian, and it is characterized, up to isomorphism, by the same universal mapping property for all abelian groups H, so we can also say that $\prod_{k\in K} G_k$ is a product in the category of abelian groups.

8.3 Definition: Let K be a nonempty set, and let G_k be a group for each $k \in K$. The (direct) **sum** of the groups G_k is the subgroup of $\prod_{k\in K} G_k$ given by

$$\sum_{k\in K} G_k = \left\{ a \in \prod_{k\in K} G_k \,\middle|\, a_k = e \in G_k \text{ for all but finitely many } k \in K \right\}.$$

Recall, or verify, that when the groups G_k are abelian, the sum $\sum_{k\in K} G_k$ is abelian, and it is characterized, up to isomorphism, by the following universal mapping property: for every abelian group H, and for all group homomorphisms $f_k : G_k \to H$, there exists a unique group homomorphism $f : \sum_{k\in K} G_k \to H$ such that $f \circ i_k = f_k$ for all $k \in K$.

In the case that $K = \{1, 2, 3, \cdots\}$, we also write $\sum_{k\in K} G_k = \sum_{k=1}^{\infty} G_k$. In the case that $K = \{1, 2, \cdots, n\}$ we have $\sum_{k\in K} G_k = \prod_{k\in K} G_k = \prod_{k=1}^{n} G_k = G_1 \times G_2 \times \cdots \times G_n$.

In the case that all of the groups G_k are additive abelian groups, we sometimes write $\sum_{k\in K} G_k = \bigoplus_{k\in K} G_k$, and in the case that the groups G_k are additive abelian groups and $K = \{1, 2, \cdots, n\}$, we also write $\bigoplus_{k\in K} G_k = \bigoplus_{k=1}^{n} G_k = G_1 \oplus G_2 \oplus \cdots \oplus G_n$.

8.4 Remark: The fact that $\sum_{k\in K} G_k$ is characterized by the above mapping property, when the groups G_k and H are abelian, can be summarized by saying that $\sum_{k\in K} G_k$ is a **coproduct** in the category of abelian groups. In the case that the groups G_k are not abelian, the sum $\sum_{k\in K} G_k$ does not satisfy the above mapping property for all groups H, and so $\sum_{k\in K} G_k$ is *not* a coproduct in the category of all groups.

Free Products of Groups

8.5 Definition: Let K be a nonempty set and let G_k be a group for each $k \in K$. A **word** on the groups G_k is a string, on the disjoint union $\bigsqcup_{k\in K} G_k$, of the form $w = a_1 a_2 \cdots a_n$ where $n \geq 0$ (when $n = 0$ this gives the empty string $\emptyset$) and each $a_i \in G_{k_i}$ for some $k_i \in K$. This word is said to be **reduced** when $a_i \neq e \in G_{k_i}$ for all i, and $k_i \neq k_{i+1}$ for all i. Any word can be reduced using the following **reduction** operations: if $a_i = e \in G_{k_i}$ then we can remove the term a_i from the word, and if $k_i = k_{i+1}$ and say $a_i a_{i+1} = b \in G_{k_i}$ then we can replace the pair of terms $a_i a_{i+1}$ by the single term b, with $b \in G_k$. The **free product** of the groups G_k, denoted by $\ast_{k\in K} G_k$, is the set of reduced words on $\bigsqcup_{k\in K} G_k$ (or, alternatively, the quotient of the set of all words under the relation given by equivalence under the reduction operations). The operation on $\ast_{k\in K} G_k$ is given by concatenation, followed by reduction. Thus

$$\mathop{\ast}_{k\in K} G_k = \left\{ a_1 a_2 \cdots a_n \,\middle|\, n \geq 0\,,\, a_i \in G_{k_i} \right\}$$

and, if we wish, we can require that $a_i \neq e \in G_{k_i}$ and that $k_i \neq k_{i+1}$. For each $\ell \in K$ we have the natural **inclusion** map $i_\ell : G_\ell \to \ast_{k\in K} G_k$ given by $i_\ell(a) = a$, where $a \in G_\ell$.

In the case that $K = \{1, 2, 3, \cdots\}$ we also write $\ast_{k\in K} G_k = \ast_{k=1}^{\infty} G_k$. In the case that $K = \{1, 2, \cdots, n\}$ we also write $\ast_{k\in K} G_k = \ast_{k=1}^{n} G_k = G_1 * G_2 * \cdots * G_n$.

Recall, or verify, that the free product is characterized by the following universal mapping property: for every group H and for all group homomorphisms $f_k : G_k \to H$, there exists a unique group homomorphism $f : \ast_{k\in K} \to H$ such that $f \circ i_k = f_k$ for all $k \in K$. This group homomorphism is given by $f(a_1 a_2 \cdots a_n) = f_{k_1}(a_1) f_{k_2}(a_2) \cdots f_{k_n}(a_n) \in H$.

8.6 Remark: The fact that $\ast_{k\in K} G_k$ is characterized by the above mapping property can be summarized by saying that $\ast_{k\in K} G_k$ is a **coproduct** in the category of groups.

8.7 Example: We have $G * H = \left\{ \emptyset, a_1, b_1, a_1 b_2, b_1 a_2, a_1 b_2 a_3, b_1 a_2 b_3, \cdots \,\middle|\, a_i \in G, b_j \in H \right\}$.

Free Abelian Groups and Free Groups

8.8 Definition: Let A be a nonempty set. A (formal) **linear combination** on A is an expression of the form $\sum_{k=0}^{n} k_i a_i$ where $n \geq 0$ (when $n = 0$ we obtain the empty sum which we write as 0) and each $a_i \in A$ and each $k_i \in \mathbb{Z}$. The above linear combination is **reduced** when each $k_i \neq 0$ and the elements a_i are distinct. Any linear combination can be reduced using the following **reduction operations**: the terms can be reordered, when $a_i = a_j = a$ the two terms $k_i a_i$ and $k_j a_j$ (that is the terms $k_i a$ and $k_j a$) can be replaced by the single term $(k_i + k_j)a$, and when $k_i = 0$ the term $k_i a_i$ (that is the term $0a_i$) can be omitted. The **free abelian group** on A (or **generated** by A), denoted by $FAb(A)$, is the set of linear combinations on A (or, to be more precise, the quotient of the set of linear combinations under the relation given by the reduction operations). The operation is addition, which is assumed to be abelian. Thus we have

$$FAb(A) = \text{Span}_{\mathbb{Z}} A = \left\{ \sum_{k=1}^{n} k_i a_i \,\middle|\, n \in \mathbb{N}\,,\ a_i \in A\,,\ k_i \in \mathbb{Z} \right\}$$

and, if we want, we can require that each $k_i \neq 0$ and that the elements a_i are distinct. The natural **inclusion** map $i : A \to FAb(A)$ is given by $i(a) = a = 1\,a$.

Recall, or verify, that the free abelian group on A is characterized, up to isomorphism, by the following universal mapping property: for every additive abelian group G and for every map of sets $f : A \to G$, there is a unique group homomorphism $g : FAb(A) \to G$ such that $g \circ i = f$. This map g is given by $g\big(\sum_{i=1}^{n} k_i a_i\big) = \sum_{i=1}^{n} k_i f(a_i)$.

8.9 Remark: The fact that $FAb(A)$ is characterized by the above mapping property can be summarized by saying that $FAb(A)$ is a **free object** in the category of abelian groups.

8.10 Definition: Let A be a nonempty set. A **word** on the set A is a string of the form $w = \prod_{i=1}^{n} {a_i}^{k_i} = {a_1}^{k_1} {a_2}^{k_2} \cdots {a_n}^{k_n}$ where $n \geq 0$ ($n = 0$ gives the empty word $\emptyset$) and each $a_i \in A$ and each $k_i \in \mathbb{Z}$. This word is **reduced** when $k_i \neq 0$ for all i and $a_i \neq a_{i+1}$ for all i. Any word can be reduced using the following **reduction** operations: if $k_i = 0$ then we can remove the term ${a_i}^{k_i}$ (that is the term ${a_i}^0$) from the string, and if $a_i = a_{i+1} = a$ then we can replace the pair of terms ${a_i}^{k_i} a_{i+1}^{k_{i+1}}$ by the single term $a^{k_i + k_{i+1}}$. The **free** group on A (or the free group **generated** by A), denoted by $F(A)$, is the set of all reduced words on the set A (or, equivalently, the quotient of the set of all words on A under the relation given by the reduction operations). The operation on $F(A)$ is given by concatenation followed by reduction. Thus we have

$$F(A) = \left\{ \prod_{i=1}^{n} {a_i}^{k_i} \,\middle|\, n \geq 0\,,\ a_i \in A\,,\ k_i \in \mathbb{Z} \right\}$$

and, if we want, we can require that $k_i \neq 0$ and that $a_i \neq a_{i+1}$. The natural **inclusion** map $i : A \to F(A)$ is given by $i(a) = aa^1$.

Recall, or verify, that the free group is characterized, up to isomorphism, by the following property: for every group G and for every map of sets $f : A \to G$, there is a unique group homomorphism $g : F(A) \to G$ such that $g \circ i = f$. This map g is given by $g\big(\prod_{i=1}^{n} {a_i}^{k_i}\big) = \prod_{i=1}^{n} f(a_i)^{k_i}$.

8.11 Remark: The fact that the free group is characterized by the above universal mapping property may be summarized by saying the $F(A)$ is a **free object** in the category of groups.

Generators and Relations

8.12 Note: Note that every group G is a homomorphic image of a free group. Indeed if $A \subseteq G$ is any subset of G which generates G, then we have a natural surjective group homomorphism $\phi : F(A) \to G$ given by $\phi\big(\prod_{i=1}^n {a_i}^{k_i}\big) = \prod_{i=1}^n {a_i}^{k_i}$. It follows, from the First Isomorphism Theorem, that $G \cong F(A)/K$ where $K = \text{Ker}\phi$. Thus every group G is of the form $F(A)/N$ for some nonempty set A and some normal subgroup $N \trianglelefteq G$.

8.13 Definition: When A is a nonempty set and $W \subseteq F(A)$, we define

$$\langle A|W\rangle = F(A)/N$$

where N is the normal subgroup of $F(A)$ generated by W. In the case that $A = \{a_1, \cdots, a_n\}$ and $W = \{w_1, \cdots, w_m\}$, we often omit set brackets, and we often write $w_i = e$ to indicate that $w_i \in N$, so

$$\langle A|W\rangle = \langle a_1, \cdots, a_n \,|\, w_1, \cdots, w_m\rangle = \langle a_1, \cdots, a_n \,|\, w_1 = e, w_2 = e, \cdots, w_m = e\rangle.$$

In addition, when $w = uv$ with $u, v \in F(A)$, we might write $w = e$ as $u = v^{-1}$.

8.14 Example: Here are a few examples

$$\begin{aligned}
F(A) &\cong \langle A \,|\, \emptyset\rangle \\
\mathbb{Z} &\cong \langle a \,|\, \emptyset\rangle \\
\mathbb{Z}_n &\cong \langle a \,|\, a^n\rangle = \langle a \,|\, a^n = e\rangle \\
\mathbb{Z}^2 &\cong \langle a, b \,|\, aba^{-1}b^{-1}\rangle = \langle a, b \,|\, ab = ba\rangle \\
\mathbb{Z}_n \times \mathbb{Z}_m &\cong \langle a, b \,|\, a^n, b^m, aba^{-1}b^{-1}\rangle = \langle a, b \,|\, a^n = e, b^m = e, ab = ba\rangle \\
\mathbb{Z}_n * \mathbb{Z}_m &\cong \langle a, b \,|\, a^n, b^m\rangle
\end{aligned}$$

8.15 Example: Show that $D_n \cong \langle \sigma, \tau \,|\, \sigma^n, \tau^2, \sigma\tau\sigma\tau\rangle$.

8.16 Example: When A and B are disjoint we have $\langle A \,|\, W\rangle * \langle B \,|\, V\rangle \cong \langle A \cup B \,|\, V \cup W\rangle$.

8.17 Example: Show that $\langle A \,|\, W \cup V\rangle \cong \big(F(A)/N\big)/M$ where N is the normal subgroup of $F(A)$ generated by the elements in W, and M is the normal subgroup of $F(A)/N$ generated by elements in V (or, to be precise, by elements $vN \in F(A)/N$ where $v \in V$).

8.18 Definition: For a group G, the **abelianization** of G is the group $Ab(G) = G/N$ where N is the normal subgroup of G generated by the elements of the form $aba^{-1}b^{-1}$ where $a, b \in G$. Recall, or verify, that $Ab(G)$ is an abelian group.

8.19 Example: Show that $Ab\big(F(A)\big) \cong FAb(A) = \text{Span}_{\mathbb{Z}}(A)$.

8.20 Example: For $w = \prod_{i=1}^n {a_i}^{k_i} \in F(A)$, let $\varphi(w) = \sum_{i=1}^n k_i a_i \in \text{Span}_{\mathbb{Z}}(A)$. Show that

$$Ab\big(\langle A \,|\, W\rangle\big) \cong \text{Span}_{\mathbb{Z}}(A)/N$$

where N is the subgroup of $\text{Span}_{\mathbb{Z}}(A)$ generated by $\big\{\varphi(w) \,\big|\, w \in W\big\}$.

8.21 Example: Show that

$$Ab(D_4) \cong \left\{ \begin{array}{cl} \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{if } n \text{ is even} \\ \mathbb{Z}_2 & \text{if } n \text{ is odd} \end{array} \right\}.$$