Chapter 7. Homotopy Invariance and Retracts

Homotopic Maps and Spaces

7.1 Definition: Let $f, g : (X, a) \to (Y, b)$ be continuous maps of based topological spaces. A **homotopy** from f to g relative to a, is a continuous map $F : [0, 1] \times X \to Y$ such that $F(0, x) = f(x)$ and $F(1, x) = g(x)$ for all $x \in X$, and $F(s, a) = b$ for all $s \in [0, 1]$. When such a homotopy exists, we say that f is **homotopic** to g (relative to a), and we write $f \sim g$ (rel a). Verify, as an exercise, that this is an equivalence relation.

7.2 Theorem: *Let $f, g : (X, a) \to (Y, b)$ be continuous maps of based topological spaces. Suppose that $f \sim g$ (rel a). Then $f_* = g_* : \pi_1(X, a) \to \pi_1(Y, b)$.*

Proof: If $\alpha : [0, 1] \to X$ is a loop at a in X, and $F : [0, 1] \times X \to Y$ is a homotopy from f to g relative to a, then the map $G : [0, 1] \times [0, 1] \to Y$ given by $G(s, t) = F\big(s, \alpha(t)\big)$ is an (endpoint-fixing) homotopy from $f \circ \alpha$ to $g \circ \alpha$, so we have $f_*(\alpha) = [f \circ \alpha] = [g \circ \alpha] = g_*(\alpha)$.

7.3 Corollary: *Let $f : (X, a) \to (Y, b)$ and $g : (Y, b) \to (X, a)$ be continuous maps of based topological spaces. Suppose that $g \circ f \sim \mathrm{id}_X$ (rel a) and $f \circ g \sim \mathrm{id}_Y$ (rel b). Then f_* and g_* are inverses of each other, so we have $\pi_1(X, a) \cong \pi_1(Y, b)$.*

Proof: We have $g_* \circ f_* = (g \circ f)_* = (\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X,a)}$ and similarlt $f_* \circ g_* = \mathrm{id}_{\pi_1(Y,b)}$ so that f_* and g_* are inverses of each other.

7.4 Definition: Let (X, a) and (Y, b) be based topological spaces. When there exist continuous maps $f : (X, a) \to (Y, b)$ and $g : (Y, b) \to (X, a)$ such that $g \circ f \sim \mathrm{id}_X$ (rel a) and $f \circ g \sim \mathrm{id}_Y$ (rel b), we say that the based spaces (X, a) and (Y, b) are **homotopic**, and we write $(X, a) \sim (Y, b)$. You can verify, if you want, that this is an equivalence relation (on the class of based topological spaces). By the above corollary, when $(X, a) \sim (Y, b)$ we have $\pi_1(X, a) \cong \pi_1(Y, b)$.

7.5 Definition: Let $f, g : X \to Y$ be continuous maps of topological spaces. A (free) **homotopy** from f to g is a continuous map $F : [0, 1] \times X \to Y$ such that $F(0, x) = f(x)$ and $F(1, x) = g(x)$ for all $x \in X$. When such a homotopy exists, we say that f is (freely) **homotopic** to g and we write $f \sim g$. You can verify that this is an equivalence relation.

7.6 Note: Let $f, g : X \to Y$ be continuous, and let $a \in X$. Note that if $f \sim g$ then, $f(a)$ and $g(a)$ lie in the same path-component of Y: indeed if $F : [0, 1] \times X \to Y$ is a homotopy from f to g, then $\gamma : [0, 1] \to Y$ given by $\gamma(s) = F(s, a)$ is a path from $f(a)$ to $g(a)$ in Y.

7.7 Theorem: *Let $f, g : X \to Y$ be continuous maps of topological spaces. Suppose that $f \sim g$ (freely) and let $F : [0, 1] \times X \to Y$ be a (free) homotopy from f to g. Let $a \in X$ and let γ be the path from $f(a) \to g(a)$ in Y given by $\gamma(s) = F(s, a)$. Then $g_* = \phi_\gamma \circ f_*$ and $f_* = \phi_{\gamma^{-1}} \circ g_*$.*

Proof: Let α be a loop at a in X. Then the map $G : [0, 1] \times [0, 1] \to Y$ given by

$$G(s,t) = \left\{ \begin{array}{cl} \gamma(4t) & \text{if } 0 \le s \le 1\,,\ 0 \le t \le \frac{s}{4} \\ F\big(s\,,\ \alpha\big(\frac{4t-s}{4-3s}\big)\big) & \text{if } 0 \le s \le 1\,,\ \frac{s}{4} \le t \le \frac{2-s}{2} \\ \gamma(2-2t) & \text{if } 0 \le s \le 1\,,\ \frac{2-s}{2} \le t \le 1 \end{array} \right\}$$

is a homotopy from $f \circ \alpha$ to $(\gamma(g \circ \alpha))\gamma^{-1}$ in Y so that $f_*([\alpha]) = [f \circ \alpha] = \big(\gamma(g \circ \alpha)\gamma^{-1}\big] = \phi_\gamma\big([g \circ \alpha]\big) = \phi_\gamma\big(g_*([\alpha])\big) = (\phi_\gamma \circ g_*)([\alpha])$. Thus $f_* = \phi_{\gamma_{-1}} \circ g_*$, hence also $g_* = \phi_\gamma \circ f_*$ (since $\phi_{\gamma^{-1}} = (\phi_\gamma)^{-1}$).

7.8 Theorem: *Let $f : X \to Y$ and $g : Y \to X$ be continuous maps of topological spaces. Suppose that $g \circ f \sim \mathrm{id}_X$ and $f \circ g \sim \mathrm{id}_Y$. Then $\pi_1(X, a) \cong \pi_1(Y, f(a))$.*

Proof: Let F be a (free) homotopy from $g \circ f$ to id_X and let G be a (free) homotopy from $f \circ g$ to id_Y. Let $a \in X$, let $b = f(a) \in Y$, let $c = g(b) \in X$ and let $d = f(c) \in Y$. Let γ be the path in X from c to a given by $\gamma(s) = F(s, a)$, and let δ be the path in Y from d to b given by $\delta(s) = G(s, b)$. Write f_* for the induced homomorphism $f_* : \pi_1(X, a) \to \pi_1(Y, b)$, write g_* for the induced homomorphism $g_* : \pi_1(Y, b) \to \pi_1(X, c)$, and write f'_* for the induced homomorphism $f'_* : \pi_1(X, c) \to \pi_1(Y, d)$. By the above theorem, $g_* \circ f_* = (g \circ f)_* = \phi_{\gamma^{-1}} \circ \mathrm{id}_{\pi_1(X,a)} = \phi_{\gamma^{-1}}$ so that $g_* \circ f_* \circ \phi_\gamma = \mathrm{id}_{\pi_1(X,c)}$. Since the map $g_* : \pi_1(Y, b) \to \pi_1(X, c)$ has a right inverse, it is surjective. Similarly, by the above theorem, we have $f'_* \circ g_* = (f \circ g)_* = \phi_{\delta^{-1}}$ so that $\phi_\delta \circ f'_* \circ g_* = \mathrm{id}_{\pi_1(Y,b)}$. Since g_* has a left inverse, it is injective. Since g_* is surjective and injective, it is bijective (with $(g_*)^{-1} = f_* \circ \phi_\gamma = \phi_\delta \circ f'_*$). Since $g_* \circ f_* = \phi_{\gamma^{-1}}$ and g_* and $\phi_{\gamma^{-1}}$ are both bijective, it follows that f_* is bijective (with $(f_*)^{-1} = \phi_\gamma \circ g_*$).

7.9 Definition: Let X and Y be topological spaces. When there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \sim \mathrm{id}_X$ and $f \circ g \sim \mathrm{id}_Y$, we say that X is (freely) **homotopic** to Y, and we write $X \sim Y$. You can verify, as an exercise, that this is an equivalence relation (on the class of topological spaces). By the above theorem, when $X \sim Y$ we have $\pi_1(X, a) \cong \pi_1(Y, b)$ where b is any point in Y which lies in the same path component as $f(a)$.

Retracts and Deformation Retracts

7.10 Definition: Let $f, g : X \to Y$ be continuous maps of topological spaces, let $A \subseteq X$, and suppose that $f(a) = g(a)$ for all $a \in A$. A **homotopy** from f to g, relative to A, is a continuous map $F : [0, 1] \to X \to Y$ such that $F(0, x) = f(x)$ and $F(1, x) = g(x)$ for all $x \in X$, and $F(s, a) = f(a) = g(a)$ for all $s \in [0, 1]$ and all $a \in A$. When such a homotopy exists, we say that f is **homotopic** to g relative to A, and we write $f \sim g$ (rel A).

7.11 Remark: For continuous maps $f, g : X \to Y$, a free homotopy from f to g is the same thing as a homotopy from f to g relative to the set $A = \emptyset$, and if $f(a) = g(a)$ then a homotopy from f to g relative to a is the same thing as a homotopy from f to g relative to the set $A = \{a\}$. For (continuous) paths $\alpha, \beta : [0, 1] \to X$ from a to b in X, an (endpoint-fixing) homotopy of paths from α to β in X is the same thing as a homotopy from α to β relative to the set $A = \{0, 1\}$.

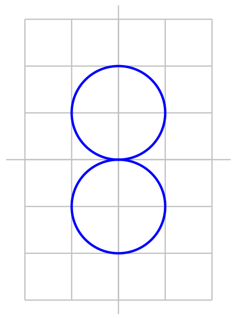
7.12 Definition: Let X be a topological space, let $A \subseteq X$, and let $i : A \to X$ be the inclusion map. A **retraction** from X to A is a continuous map $f : X \to A$ such that $f(a) = a$ for all $a \in A$, or equivalently, such that $f \circ i = \mathrm{id}_A$. When such a retraction exists, we say that A is a **retract** of X. A **deformation retraction** from X to A is continuous map $f : X \to A$ such that $f \circ i = \mathrm{id}_A$ and $i \circ f \sim \mathrm{id}_X$ (freely). When such a deformation retraction exists, we say that A is a **deformation retract** of X. A **strong deformation retraction** from X to A is a continuous map $f : X \to A$ such that $f \circ i = \mathrm{id}_A$ and $i \circ f \sim \mathrm{id}_X$ (rel A). When such a strong deformation retraction exists, we say that A is a **strong deformation retract** of X. Note that every strong deformation retract is also a deformation retract, and every deformation retract is also a retract.

7.13 Note: When $f : X \to A$ is a retraction and $a \in A$, we have $f \circ i = \mathrm{id}_A$ so that $f_* \circ i_* = \mathrm{id}_{\pi_1(X,a)}$, hence $f_* : \pi_1(X,a) \to \pi_1(A,a)$ is surjective and $i_* : \pi_1(A,a) \to \pi_i(X,a)$ is injective. When $f : X \to A$ is a deformation retraction (or a strong deformation retraction), since $f \circ i = \mathrm{id}_A$ and $i \circ f \sim \mathrm{id}_X$, it follows that for all $a \in A$, we have $\pi_1(X,a) \cong \pi_1(A,a)$, indeed the maps $f_* : \pi_1(X,a) \to \pi_1(A,a)$ and $i_* : \pi_1(A,a) \to \pi_1(X,a)$ are inverses of each other.

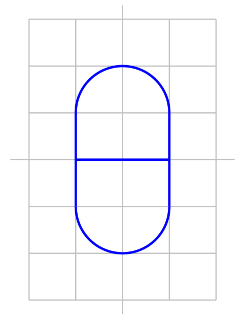
7.14 Example: The origin $\{0\}$ is a strong deformation retract of $\mathbb{R}^n$. The sphere $\mathbb{S}^{n-1}$ is a strong deformation retract of $\mathbb{R}^n \setminus \{0\}$. When p is the north pole $p = (0, \cdots, 0, 1) \in \mathbb{S}^n$, the one-point set $\{-p\}$ is a strong deformation retract of the punctured sphere $\mathbb{S}^n \setminus \{p\}$ (which is homeomorphic, via stereographic projection, to $\mathbb{R}^n$). Also, $\mathbb{S}^{n-1}$ is a strong deformation retract of $\mathbb{S}^n \setminus \{\pm p\}$ (which is homeomorphic, via stereographic projection, to $\mathbb{R}^n \setminus \{0\}$).

7.15 Example: The **figure eight space** 8 and the **theta space** Θ, as shown, are both strong deformation retracts of $\mathbb{C}\setminus\{\pm i\}$, hence $8 \sim \Theta$. But note that $8 \not\cong \Theta$ because $8 \setminus \{0\}$ is disconnected but $\Theta \setminus \{p\}$ is path-connected, hence connected, for all $p \in \theta$.

The Figure Eight Space 8



The Theta Space Θ



7.16 Exercise: Let X be the **comb space** $X = \bigcup_{n=1}^{\infty} \big([0,1]\times\{\frac{1}{n}\}\big) \cup \big(\{0\}\times[0,1]\big) \cup \big([0,1]\times\{0\}\big)$. Show that $\{(0,0)\}$ is a strong deformation retract of X and that $\{(1,0)\}$ is a deformation retract, but not a strong deformation retract, of X.

7.17 Example: The set $\{1\}$ is a retract of $\mathbb{S}^1$, but not a deformation retract $\big($since $\pi_1(\{1\},1)$ is not isomorphic to $\pi_1(\mathbb{S}^1,1)\big)$. The boundary circle $\mathbb{S}^1$ is not a retract of the closed unit disc $D^2 = \overline{B}(0,1)$ since $\pi_1(\mathbb{S}^1,1) \cong \mathbb{Z}$ but $\pi_1(D^2,1) = 0$, so the map $i_* : \pi_1(\mathbb{S}^1,1) \to \pi_1(D^2,1)$ is not injective.

7.18 Definition: Let X be a topological space. We say that X is **simply connected** when X is path connected and $\pi_1(X,a) = 0$ for some (hence for any) $a \in X$. We say that X is **contractible** when X is homotopic to a 1-point set or, equivalently, when for some (hence for every) $a \in X$, the set $\{a\}$ is a deformation retract of X. Note that every contractible space is simply connected.

Applications

7.19 Theorem: *(The Fundamental Theorem of Algebra) Every non-constant polynomial in* $\mathbb{C}[x]$ *has a root in* $\mathbb{C}$.

Proof: I may include a proof later.

7.20 Theorem: *(The Brouwer Fixed Point Theorem) Let* D *be the closed unit disc in* $\mathbb{R}^2$. *Every continuous map* $f : D \to D$ *has a fixed point.*

Proof: I may include a proof later.

7.21 Theorem: *(The Borsuk-Ulam Theorem) For every continuous map* $f : \mathbb{S}^2 \to \mathbb{R}^2$, *there is a point* $x \in \mathbb{S}^2$ *such that* $f(-x) = f(x)$.

Proof: I may include a proof later.

7.22 Theorem: *(The Lusternik-Schirelmann Theorem) Suppose that* $\mathbb{S}^2 = A_1 \cup A_2 \cup A_3$ *where each set* A_k *is closed in* $\mathbb{S}^2$. *Then one of the sets* A_k *contains a pair of antipodal points* $\pm x$.

Proof: I may include a proof later.