Chapter 6. Homotopy, and The Fundamental Group

Homotopy and The Fundamental Group

6.1 Definition: Let X be a topological space and let $\alpha, \beta : [0,1] \to X$ be paths from a to b in X. An (endpoint-fixing) **homotopy** from α to β in X is a continuous map $F : [0,1] \times [0,1] \to X$ such that $F(0,t) = \alpha(t)$ and $F(1,t) = \beta(t)$ for all $t \in [0,1]$, and $F(s,0) = a$ and $F(s,1) = b$ for all $\in [0,1]$. Note that, in this case, for each $s \in [0,1]$ the map $f_s : [0,1] \to X$ given by $f_s(t) = F(s,t)$ is a path from a to b in X. We say that α is **homotopic** to β (or that α is **homotopy-equivalent** to β) in X, and we write $\alpha \sim \beta$ in X, when there exists a homotopy from α to β in X.

6.2 Theorem: *Let X be a topological space, and let $a, b \in X$. Then homotopy-equivalence is an equivalence relation on the set of all paths from a to b in X.*

Proof: Let α, β, γ be paths from a to b in X. Note that $\alpha \sim \alpha$: indeed the map $F : [0, l1] \times [0,1] \to X$ given by $F(s,t) = \alpha(t)$ is a homotopy from α to α in X. Note that if $\alpha \sim \beta$ then $\beta \sim \alpha$: indeed if F is a homotopy from α to β in X then the map $G : [0,1] \times [0,1] \to X$ given by $G(s,t) = F(1-s,t)$ is a homotopy from β to α in X. Finally note that if $\alpha \sim \beta$ in X and $\beta \sim \gamma$ in X then $\alpha \sim \gamma$ in X: indeed, if F is a homotopy from α to β in X and G is a homotopy from β to γ in X, then the map $H : [0,1] \times [0,1] \to X$ given by

$$H(s,t) = \left\{ \begin{array}{cl} F(2s,t) & , \text{if } 0 \leq s \leq \frac{1}{2} \\ G(2s-1,t) & , \text{if } \frac{1}{2} \leq s \leq 1 \end{array} \right\}$$

is a homotopy from α to γ in X.

6.3 Notation: Given a topological space X and a point $a \in X$, we denote the set of homotopy-equivalence classes of loops at a in X by $\pi_1(X,a)$, that is

$$\pi_1(X,a) = \big\{ [\alpha] \,\big|\, \alpha \text{ is a loop at } a \text{ in } X \big\} \text{ , where}$$
$$[\alpha] = \big\{ \beta \,\big|\, \beta \text{ is a loop at } a \text{ in } X \text{ with } \beta \sim \alpha \text{ in } X \big\}.$$

6.4 Notation: Let X be a topological space. When $a \in X$, we write κ_a to denote the **constant loop** at a given by

$$\kappa_a(t) = a$$

for all $t \in [0,1]$. When α is a path from a to b in X, we write α^{-1} to denote the **inverse path** from b to a in X given by

$$\alpha^{-1}(t) = \alpha(1-t)\,.$$

When α is a path from a to b in X and β is a path from b to c in X, we write $\alpha\beta$ to denote the **product path** from a to c in X given by

$$(\alpha\beta)(t) = \left\{ \begin{array}{cl} \alpha(2t) & , \text{if } 0 \leq t \leq \frac{1}{2} \\ \beta(2t-1) & , \text{if } \frac{1}{2} \leq t \leq 1 \end{array} \right\}$$

6.5 Theorem: *Let X be a topological space*

(1) When α and β are paths from a to b in X, if $\alpha \sim \beta$ in X then $\alpha^{-1} \sim \beta^{-1}$ in X.
(2) When α and β are paths from a to b in X, and γ and δ are paths from b to c in X, if $\alpha \sim \gamma$ in X and $\beta \sim \delta$ in X then $\alpha\gamma \sim \beta\delta$ in X.

(3) When α is a path from a to b in X, we have $\kappa_a\alpha \sim \alpha$ in X and $\alpha\,\kappa_b \sim \alpha$ in X.
(4) When α is a path from a to b in X, we have $\alpha\,\alpha^{-1} \sim \kappa_a$ in X and $\alpha^{-1}\alpha \sim \kappa_b$ in X.
(5) When α is a path from a to b in X and β is a path from b to c in X, and γ is a path from c to d in X, we have $(\alpha\beta)\gamma \sim \alpha(\beta\gamma)$ in X.

Proof: The proof is left as an exercise.

6.6 Definition: Let X be a topological space and let $a \in X$. By the above theorem, the set $\pi_1(X,a)$, of homotopy-equivalence classes of loops at a in X, is a group under the operation given by $[\alpha][\beta] = [\alpha\beta]$, with identity element $e = [\kappa_a]$ and with the inverse given by $[\alpha]^{-1} = [\alpha^{-1}]$. This group $\pi_1(X,a)$ is called the **fundamental group** of X at a.

6.7 Example: When X is a convex set in a normed linear space and $a \in X$, we have $\pi_1(X,a) = \{e\}$ where $e = [\kappa_a]$. Indeed, for a loop α at a in X, the map $F : [0,1]\times[0,1] \to X$ given by $F(s,t) = \alpha(t) = s(a - \alpha(t))$ is a homotopy from α to κ_a in X.

The Fundamental Group of the Circle

6.8 Theorem: *Let X be a compact metric space and let S be an open cover of X. The there exists a number $\lambda > 0$, called a* **Lebesgue number** *for the open cover S, such that for every $a \in X$ the ball $B(a,\lambda)$ is contained in one of the sets in S.*

Proof: The proof is left as an exercise (this is often proven in an analysis course).

6.9 Theorem: *Let α be a path from a to b in $\mathbb{C}^*$. Choose $\theta_0 \in \mathbb{R}$ such that $a = |a|e^{i\theta_0}$. Then there exist unique continuous fuctions $r, \theta : [0,1] \to \mathbb{R}$, with $r(t) > 0$ for all t and $\theta(0) = \theta_0$, such that $\alpha(t) = r(t)e^{i\,\theta(t)}$ for all t. Moreover, if α is differentiable (or $\mathcal{C}^1$ or smooth) then so are r and θ.*

Proof: Write $\alpha(t) = \big(x(t), y(t)\big) = x(t) + i\,y(t)$ where $x, y : [0,1] \to \mathbb{R}$, and note that x and y are continuous (and indeed differentiable or $\mathcal{C}^1$ or smooth, if α is). It is clear that the map r must be given by $r(t) = \big|\alpha(t)\big| = \sqrt{x(t)^2 + y(t)^2}$ which is continuous (and differentiable, $\mathcal{C}^1$, or smooth, if α is). Let us explain how to construct the map θ. Let $U_1 = \big\{x{+}iy \,\big|\, x{>}0\big\}$, $U_2 = \big\{x{+}iy \,\big|\, y{>}0\big\}$, $U_3 = \big\{x{+}iy \,\big|\, x{<}0\big\}$ and $U_4 = \big\{x{+}iy \,\big|\, y{<}0\big\}$ and, for $k = 1,2,3,4$, define $\theta_k : U_k \to \mathbb{R}$ by $\theta_1(x,y) = \sin^{-1}\frac{y}{\sqrt{x^2+y^2}}$, $\theta_2(x,y) = \cos^{-1}\frac{x}{\sqrt{x^2+y^2}}$, $\theta_3(x,y) = \pi - \sin^{-1}\frac{y}{\sqrt{x^2+y^2}}$ and $\theta_4(x,y) = 2\pi - \cos^{-1}\frac{x}{\sqrt{x^2+y^2}}$. Note that when $\alpha(t) \in U_k$, we must have $\theta(t) = \theta\big(\alpha(t)\big) + 2\pi n_k$ for some $n_k = n_k(t) \in \mathbb{Z}$. In order that θ is continuous, the map $n_k(t)$ must be continuous. Since $n_k(t)$ takes values in the discrete set $\mathbb{Z}$, it must be locally constant (that is constant when t lies in an interval $I \subseteq [0,1]$ and $\alpha(t) \in U_k$ for all $t \in I$). The open sets $\alpha^{-1}(U_k)$ cover the interval $[0,1]$. Choose a Lebesgue number $\lambda > 0$ for this cover. Choose $\ell \in \mathbb{Z}^+$ large enough so that $\frac{1}{\ell} < \lambda$, and note that each subinterval $I_j = \left[\frac{j-1}{\ell}, \frac{j}{\ell}\right]$ is contained in the open ball $B_j = B(x_j, \lambda)$ in $[0,1]$, and B_j is contained in one of the four open sets $\alpha^{-1}(U_k)$, say $B_j \subseteq \alpha^{-1}(U_{k_j})$. It follows that (in order for the required continuous map $\theta(t)$ to exist) there must exist constants $n_j \in \mathbb{Z}$ such that when $t \in B_j$ we have $\theta(t) = \theta_{k_j}\big(\alpha(t)\big) + 2\pi n_j$.

Finally we note that the constants n_j can be determined, in a unique way, so that the resulting map $\theta(t)$ is continuous with $\theta(0)=\theta_0$: indeed the value of n_1 is uniquely determined so that $\theta_{k_1}(\alpha(0))+2\pi n_1=\theta(0)=\theta_0$, then the value of n_2 is uniquely determined so that $\theta_{k_2}\big(\alpha\big(\frac{1}{\ell}\big)\big)+2\pi n_2=\theta\big(\frac{1}{\ell}\big)=\theta_{k_1}\big(\alpha\big(\frac{1}{\ell}\big)\big)+2\pi n_1$, then the value of n_3 is uniquely determined so that $\theta_{k_3}\big(\alpha\big(\frac{2}{\ell}\big)\big)+2\pi n_3=\theta\big(\frac{2}{\ell}\big)=\theta_{k_2}\big(\alpha\big(\frac{2}{\ell}\big)\big)+2\pi n_2$, and so on. The values n_j are uniquely determined, and the resulting map $\theta:[0,1]\to\mathbb{R}$ given by $\theta(t)=\theta_{k_j}(\alpha(t))+2\pi n_j$ when $t\in I_j$ (or when $t\in B_j$) is continuous (by the Glueing Lemma). Moreover, we note that if α is differentiable (or $\mathcal{C}^1$ or smooth) then so is θ.

6.10 Definition: Let α be a path in $\mathbb{C}\setminus\{a\}$ from b to c. Given $\theta_0\in\mathbb{R}$ such that $b=a+|b-a|e^{i\theta_0}$, since the path given by $\beta(t)=\alpha(t)-a$ is a path in $\mathbb{C}^*$ from $b-a$ to $c-a$, it follows from the above theorem that there exist unique continuous maps $r,\theta:[0,1]\to\mathbb{R}$ with $\theta(0)=\theta_0$ and with $r(t)>0$ for all t, such that $\alpha(t)=a+r(t)e^{i\theta(t)}$. This expression for α is called the **polar representation** of α about a (with initial angle θ_0). We define the **winding number** of α about a to be

$$\operatorname{wind}(\alpha,a)=\tfrac{\theta(1)-\theta(0)}{2\pi}.$$

Note that this does not depend on the choice of θ_0: indeed, if $\alpha(t)=a+r(t)e^{i\phi(t)}$ with $\phi(0)=\phi_0=\theta_0+2\pi k$, then since $a+re^{i\theta(t)}=a+r(t)e^{i(\theta(t)+2\pi k)}$ for all t, we have $\phi(t)=\theta(t)+2\pi k$ for all t (by the uniqueness of the polar representation with initial angle ϕ_0), so that $\phi(1)-\phi(0)=\theta(1)-\theta(0)$. Also note that in the case that α is a loop at b in $\mathbb{C}\setminus\{a\}$, we have $\operatorname{wind}(\alpha,a)\in\mathbb{Z}$.

6.11 Note: Recall that when a continuous (or piecewise continuous) map $f:[0,1]\to\mathbb{C}$ is given by $f(t)=\big(x(t),y(t)\big)=x(t)+i\,y(t)$ with $x,y:[0,1]\to\mathbb{R}$, we define the **integral** $\int_0^1 f$ to be given by $\int_0^1 f=\int_0^1 f(t)\,dt=\int_0^1 x(t)\,dt+i\int_0^1 y(t)\,dt$.

Also recall that when α is a $\mathcal{C}^1$ (or a continuous and piecewise $\mathcal{C}^1$) path in an open set U in $\mathbb{C}$, and f is a continuous map $f:U\subseteq\mathbb{C}\to\mathbb{C}$, we define the **path integral** $\int_\alpha f$ to be $\int_\alpha f=\int_\alpha f(z)\,dz=\int_0^1 f(\alpha(t))\alpha'(t)\,dt$.

6.12 Theorem: *Let α be a $\mathcal{C}^1$ (or a continuous and piecewise $\mathcal{C}^1$) path from b to c in $\mathbb{C}\setminus\{a\}$. Choose $\theta_0\in\mathbb{R}$ such that $b-a=|b-a|e^{i\theta_0}$. Let $\alpha(t)=a+r(t)e^{i\theta(t)}$ be the polar representation of α about a. Then*

$$\int_\alpha\frac{dz}{z-a}=\ln\tfrac{r(1)}{r(0)}+i\,(\theta(1)-\theta(0))=\ln\tfrac{|b-a|}{|c-a|}+2\pi\,i\operatorname{wind}(\alpha,a\,.$$

Proof: The proof is a straightforward calculation: we have

$$\begin{aligned}\int_\alpha\frac{dz}{z-a}&=\int_{t=0}^1\frac{\alpha'(t)\,dt}{\alpha(t)-a}=\int_{t=0}^1\frac{r'(t)e^{i\theta(t)}+r(t)e^{i\theta(t)}\cdot i\theta'(t)}{r(t)e^{i\theta(t)}}\,dt\\&=\int_{t=0}^1\frac{r'(t)}{r(t)}+i\,\theta'(t)\;dt=\Big[\ln r(t)+i\,\theta(t)\Big]_0^1=\ln\tfrac{r(1)}{r(0)}+i\,(\theta(1)-\theta(0)).\end{aligned}$$

6.13 Theorem: *We have* $\pi_1(\mathbb{C}^*,1)=\langle[\sigma]\rangle\cong\mathbb{Z}$ *and* $\pi_1(\mathbb{S}^1,1)=\langle[\sigma]\rangle\cong\mathbb{Z}$, *where* $\sigma:[0,1]\to\mathbb{S}^1\subseteq\mathbb{C}^*$ *is the loop at 1 given by* $\sigma(t)=e^{i\,2\pi t}$.

Proof: We give the proof for $\mathbb{C}^*$ (the proof for $\mathbb{S}^1$ is identical but with $r(t)=1$ for all t). Let α be a loop at 1 in $\mathbb{C}^*$. Let $\alpha(t)=r(t)e^{i\theta(t)}$ be the polar representation of α about 0 with the initial angle $\theta(0)=0$. Let $n=\text{wind}(\alpha,0)$. Let $\tau_n:[0,1]\to\mathbb{S}^1\subseteq\mathbb{C}^*$ be the loop given by $\tau_n(t)=e^{i\,2\pi nt}$, and note that $\text{wind}(\tau_n,0)=n$. Then we have $\alpha\sim\tau_n$ in $\mathbb{C}^*$: indeed the map $F:[0,1]\times[0,1]\to\mathbb{C}^*$ given by

$$F(s,t)=\big(r(t)+s(1-r(t)\big)\,e^{\,i\,(\theta(t)+s(2\pi nt-\theta(t))}$$

is a homotopy from α to τ_n in $\mathbb{C}^*$ (we used a straight-line homotopy in the (r,θ) plane with $(r,\theta)\in\mathbb{R}^+\times\mathbb{R}$, which is convex). This shows that every loop α at 1 in $\mathbb{C}^*$ with $\text{wind}(\alpha,0)=n$ is homotopic to τ_n in $\mathbb{C}^*$.

Also note that, for two loops α,β at 1 in $\mathbb{C}^*$ with $\text{wind}(\alpha,0)=n$ and $\text{wind}(\beta,0)=m$, we have $\text{wind}(\alpha\beta,0)=n+m$: indeed for $\alpha(t)=r(t)e^{i\theta(t)}$ and $\beta(t)=s(t)e^{i\phi(t)}$ with $\theta(0)=\phi(t)=0$ and $\theta(1)=n$ and $\phi(t)=m$, then the polar representation of $\alpha\beta$ about 0 with initial angle 0 is given by $(\alpha\beta)(t)=R(t)e^{i\Psi(t)}$ where

$$R(t)=\left\{\begin{array}{c}r(2t)\quad,\text{ if }0\le t\le\frac12\\ s(2t-1)\ ,\text{ if }\frac12\le t\le1\end{array}\right\}\ \text{and}\ \Phi(t)=\left\{\begin{array}{r}\theta(2t)\quad,\text{ if }0\le t\le\frac12\\ 2\pi n+\phi(2t-1)\ ,\text{ if }\frac12\le t\le1\end{array}\right\}$$

and we have $\Psi(1)=2\pi n+2\pi m$.

In particular, if we write $\sigma^1=\sigma$ and $\sigma^n=\sigma^{n-1}\sigma$, so that for example $\sigma^4=\sigma^3\sigma=(\sigma^2\sigma)\sigma=((\sigma\sigma)\sigma)\sigma$, then $\text{wind}(\sigma^n,0)=n$. Thus, when α is any loop at 1 in $\mathbb{C}^*$ with $\text{wind}(\alpha,0)=n$, we have $\alpha\sim\tau_n\sim\sigma^n$, and hence $[\alpha]=[\sigma]^n$ in $\pi_1(\mathbb{C}^*,1)$. This proves that $\pi_1(\mathbb{C}^*,1)=\langle[\sigma]\rangle$ (the cyclic group generated by $[\sigma]$).

It remains to show that when $n\neq m$ we have $\sigma^n\not\sim\sigma^m$ (so that the cyclic group $\langle[\sigma]\rangle$ is infinite). This follows directly from Cauchy's Theorem for Paths, from complex analysis, which states that when α and β are two $\mathcal{C}^1$ (or continuous and piecewise $\mathcal{C}^1$) paths in an open set U in $\mathbb{C}$, and $f:U\subseteq\mathbb{C}\to\mathbb{C}$ is holomorphic in U, if $\alpha\sim\beta$ in U then $\int_\alpha f=\int_\beta f$. It follows that if $\sigma^n\sim\sigma^m$, or equivalently if $\tau_n\sim\tau_m$, then $n=\frac{1}{2\pi i}\int_{\tau_n}\frac{dz}{z}=\frac{1}{2\pi i}\int_{\tau_m}\frac{dz}{z}=m$.

For those students who have not seen Cauchy's Theorem for Paths, we briefly describe an alternate method for proving that if $\alpha\sim\beta$ in $\mathbb{C}^*$ then $\text{wind}(\alpha,0)\neq\text{wind}(\beta,0)$. Let α and β be paths from a to b in $\mathbb{C}^*$. Suppose that $\alpha\sim\beta$ in $\mathbb{C}^*$ and let $F:[0,1]\times[0,1]\to\mathbb{C}^*$ be a homotopy from α to β in $\mathbb{C}^*$. By imitating the proof of Theorem 6.9, one can show that, given a choice of $\theta_0\in\mathbb{R}$ such that $a=|a|e^{i\theta_0}$, there exist unique continuous functions $r,\theta:[0,1]\to\mathbb{R}$, with $r(s,t)>0$ for all s,t and $\theta(0,0)=\theta_0$, such that $F(s,t)=r(s,t)e^{i\,\theta(s,t)}$ for all s,t (we will provide the details of this proof later, in a more general context, when we discuss homotopy lifting for covering spaces). Since $\theta(s,0)$ is continuous and takes values in $\mathbb{Z}$, it is constant, so we have $\theta(1,0)=\theta(0,0)=\theta_0$, and since $\theta(s,1)$ is continuous and takes values in $\mathbb{Z}$, it is constant, so we have $\theta(1,1)=\theta(0,1)$. It follows that

$$\text{wind}(\beta,0)=\tfrac{\theta(1,1)-\theta(1,0)}{2\pi}=\tfrac{\theta(0,1)-\theta(0,0)}{2\pi}=\text{wind}(\alpha,0).$$

Basic Properties

6.14 Note: When X is path-connected and P is the path-component of X which contains the point $a \in X$, we have $\pi_1(X,a) = \pi_1(P,a)$. Indeed, every loop α at a in X also lies in P, and when α and β are homotopic loops at a in X, every homotopy F from α to β in X takes values in P so that it is also a homotopy from α to β in P.

6.15 Note: When γ is a path from a to b in X, the map $\phi_\gamma : \pi_1(X,a) \to \pi_1(X,b)$ given by $\phi_\gamma([\alpha]) = [\gamma^{-1}\alpha\gamma]$ is a well-defined group isomorphism: It is well-defined because for loops α, β at a in X, if $\alpha \sim \beta$ then $\gamma^{-1}\alpha\,\gamma \sim \gamma^{-1}\beta\gamma$ in X. It is a group homomorphism because for loops α, β at a in X, we have $\gamma^{-1}\alpha\beta\,\gamma \sim \gamma^{-1}\alpha\gamma\,\gamma^{-1}\beta\gamma$ in X. It is bijective because it has an inverse ${\phi_\gamma}^{-1} = \phi_{\gamma^{-1}} : \pi_1(X,b) \to \pi_1(X,a)$ which is given by $\phi_{\gamma^{-1}}([\beta]) = \gamma\beta\gamma^{-1}$.

6.16 Notation: When X is path-connected and $a \in X$, it is fairly common to write $\pi_1(X,a)$ simply as $\pi_1(X)$.

6.17 Definition: A **based topological space** (or a **pointed topological space**) is a pair (X,a) where X is a topological space and $a \in X$ (the point a is called the **base point**). A (continuous) **map of based spaces** $f : (X,a) \to (Y,b)$ is a continuous map $f : X \to Y$ with $f(a) = b$. A **homeomorphism** from $(X,a) \to (Y,b)$ is a continuous map $f : (X,a) \to (Y,b)$ with a continuous inverse map $f^{-1} : (Y,b) \to (X,a)$. We say that (X,a) is **homeomorphic** to (Y,b), and write $(X,a) \cong (Y,b)$, when there exists a homeomorphism $f : (X,a) \to (Y,b)$.

6.18 Definition: Given a map $f : (X,a) \to (Y,b)$ of based spaces, we define the **induced group homomorphism** $f_* : \pi_1(X,a) \to \pi_1(Y,b)$ by $f_*([\alpha]) = [f \circ \alpha]$. Note that f_* is well-defined because, for loops α and β at a in X, if F is a homotopy from α to β in X then $G = f \circ F$ is a homotopy from $f \circ \alpha$ to $f \circ \beta$ in Y. Also note that f_* is a group homomorphism because, for loops α and β at a in X, we have $f \circ (\alpha\beta) = (f \circ \alpha)(f \circ \beta)$.

6.19 Note: Note that $\mathrm{id}_* = \mathrm{id}$, meaning that when $\mathrm{id} : (X,a) \to (X,a)$ is the identity map (given by $\mathrm{id}(x) = x$ for all $x \in X$, the induced map $\mathrm{id}_* : \pi_1(X,a) \to \pi_1(X,a)$ is also the identity map. Also note that when $f : (X,a) \to (Y,b)$ and $g : (Y,b) \to (Z,c)$ we have $(g \circ f)_* = g_* \circ f_*$ because $(g \circ f) \circ \alpha = g \circ (f \circ \alpha)$ for all loops α at a in X.

6.20 Remark: The above note can be summarized by saying the we have a (covariant) **functor** F, from the category of based topological spaces to the category of groups, given by $F(X,a) = \pi_1(X,a)$ with $F(f) = f_*$ when $f : (X,a) \to (Y,b)$.

6.21 Theorem: *When* $(X,a) \cong (Y,b)$ *we have* $\pi_1(X,a) \cong \pi_1(Y,b)$.

Proof: This is immediate from the above note: indeed when $f : (X,a) \to (Y,b)$ is a homeomorphism with inverse $g = f^{-1} : (Y,b) \to (X,a)$, we have $g \circ f = \mathrm{id}$ and $f \circ g = \mathrm{id}$, and hence $g_* \circ f_* = (g \circ f)_* = \mathrm{id}_* = \mathrm{id}$ and $f_* \circ g_* = (f \circ g)_* = \mathrm{id}_* = \mathrm{id}$, so that f_* is invertible with inverse $g_* = (f_*)^{-1}$.

6.22 Theorem: $\pi_1\big(X \times Y, (a,b)\big) \cong \pi_1(X,a) \times \pi_1(Y,b)$.

Proof: Let $p : X \times Y \to X$ and $q : X \times Y \to Y$ be the projection maps. Since every loop γ at (a,b) in $X \times Y$ is of the form $\gamma(t) = \big(\alpha(t), \beta(t)\big)$ where $\alpha = p \circ \gamma$ (which is a loop at a in X) and $\beta = q \circ \gamma$ (which is a loop at b in Y), the map $\phi : \pi_1\big(X \times Y, (a,b)\big) \to \pi_1(X,a) \times \pi_1(Y,b)$ given by $\phi([\gamma]) = \big([p \circ \gamma], [q \circ \gamma]\big) = \big(p_*([\gamma]), q_*([\gamma])\big)$ is a surjective group homorphism, and ϕ is injective because if F is a homotopy from α to κ_a in X and G is a homotopy from β to κ_b in Y, then (F,G) is a homotopy from (α,β) to $\kappa_{(a,b)}$ in $X \times Y$.

6.23 Corollary: $\pi_1(\mathbb{T}^n) \cong \mathbb{Z}^n$.

6.24 Definition: Let $f, g : (X, a) \to (Y, b)$ be continuous maps of based topological spaces. A **homotopy** from f to g relative to a, is a continuous map $F : [0,1] \times X \to Y$ such that $F(0,x) = f(x)$ and $F(1,x) = g(x)$ for all $x \in X$, and $F(s,a) = b$ for all $s \in [0,1]$. When such a homotopy exists, we say that f is **homotopic** to g (relative to a), and we write $f \sim g$ (rel a). Verify, as an exercise, that this is an equivalence relation.

6.25 Theorem: *Let $f, g : (X, a) \to (Y, b)$ be continuous maps of based topological spaces. Suppose that $f \sim g$ (rel a). Then $f_* = g_* : \pi_1(X, a) \to \pi_1(Y, b)$.*

Proof: If $\alpha : [0,1] \to X$ is a loop at a in X, and $F : [0,1] \times X \to Y$ is a homotopy from f to g relative to a, then the map $G : [0,1] \times [0,1] \to Y$ given by $G(s,t) = F\big(s, \alpha(t)\big)$ is an (endpoint-fixing) homotopy from $f \circ \alpha$ to $g \circ \alpha$, so we have $f_*(\alpha) = [f \circ \alpha] = [g \circ \alpha] = g_*(\alpha)$.

6.26 Corollary: *Let $f : (X, a) \to (Y, b)$ and $g : (Y, b) \to (X, a)$ be continuous maps of based topological spaces. Suppose that $g \circ f \sim \mathrm{id}_X$ (rel a) and $f \circ g \sim \mathrm{id}_Y$ (rel b). Then f_* and g_* are inverses of each other, so we have $\pi_1(X, a) \cong \pi_1(Y, b)$.*

6.27 Definition: Let (X, a) and (Y, b) be based topological spaces. When there exist continuous maps $f : (X, a) \to (Y, b)$ and $g : (Y, b) \to (X, a)$ such that $g \circ f \sim \mathrm{id}_X$ (rel a) and $f \circ g \sim \mathrm{id}_Y$ (rel b), we say that the based spaces (X, a) and (Y, b) are **homotopic**, and we write $(X, a) \sim (Y, b)$. You can verify, if you want, that this is an equivalence relation (on the class of based topological spaces). By the above corollary, when $(X, a) \sim (Y, b)$ we have $\pi_1(X, a) \cong \pi_1(Y, b)$.

6.28 Definition: Let $f, g : X \to Y$ be continuous maps of topological spaces, let $A \subseteq X$, and suppose that $f(a) = g(a)$ for all $a \in A$. A **homotopy** from f to g, relative to A, is a continuous map $F : [0,1] \times X \to Y$ such that $F(0,x) = f(x)$ and $F(1,x) = g(x)$ for all $x \in X$, and $F(s,a) = f(a) = g(a)$ for all $s \in [0,1]$ and all $a \in A$. When such a homotopy exists, we say that f is **homotopic** to g relative to A, and we write $f \sim g$ (rel A). Note that when $a \in A$, if $f \sim g$ (rel A) then $f \sim g$ (rel a).

6.29 Definition: Let X be a topological space, let $A \subseteq X$, and let $i : A \to X$ be the inclusion map. A **retraction** from X to A is a continuous map $f : X \to A$ such that $f(a) = a$ for all $a \in A$, or equivalently, such that $f \circ i = \mathrm{id}_A$. When such a retraction exists, we say that A is a **retract** of X. A **strong deformation retraction** from X to A is a continuous map $f : X \to A$ such that $f \circ i = \mathrm{id}_A$ and $i \circ f \sim \mathrm{id}_X$ (rel A). When such a strong deformation retraction exists, we say that A is a **strong deformation retract** of X. Note that every strong deformation retract is also a retract.

6.30 Example: The origin $\{0\}$ is a strong deformation retract of $\mathbb{R}^n$. The sphere $\mathbb{S}^{n-1}$ is a strong deformation retract of $\mathbb{R}^n \setminus \{0\}$. When p is the north pole $p = (0, \cdots, 0, 1) \in \mathbb{S}^n$, the one-point set $\{-p\}$ is a strong deformation retract of the punctured sphere $\mathbb{S}^n \setminus \{p\}$ (which is homeomorphic, via stereographic projection, to $\mathbb{R}^n$). Also, $\mathbb{S}^{n-1}$ is a strong deformation retract of $\mathbb{S}^n \setminus \{\pm p\}$ (which is homeomorphic, via stereographic projection, to $\mathbb{R}^n \setminus \{0\}$). The **figure eight space** $E = C \cup (-C)$, where C is the circle $(x-1)^2 + y^2 = 1$, is a strong deformation retract of $\mathbb{R}^2 \setminus \{\pm(1,0)\}$.

6.31 Note: When $f : X \to A$ is a retraction and $a \in A$, we have $f \circ i = \mathrm{id}_A$ so that $f_* \circ i_* = \mathrm{id}_{\pi_1(X,a)}$, hence $f_* : \pi_1(X, a) \to \pi_1(A, a)$ is surjective and $i_* : \pi_1(A, a) \to \pi_1(X, a)$ is injective. When $f : X \to A$ is a strong deformation retraction, since $f \circ i = \mathrm{id}_A$ and $i \circ f \sim \mathrm{id}_X$ (rel a), we have $(X, a) \sim (A, a)$ and the maps $f_* : \pi_1(X, a) \to \pi_1(A, a)$ and $i_* : \pi_1(A, a) \to \pi_1(X, a)$ are inverses of one another.