Chapter 5. Topological Manifolds

Topological Manifolds

5.1 Definition: An n-dimensional (topological) **manifold**, also called an n-manifold, is a Hausdorff topological space X, with a countable basis, which is **locally homeomorphic to** $\mathbb{R}^n$, meaning that for every $a \in X$ there exists an open set $U_a \subseteq X$, with $a \in U_a$, and a homeomorphism ϕ_a from U_a to an open set $\phi_a(U_a)$ in $\mathbb{R}^n$. Note that $\{U_a | a \in X\}$ is an open cover of X. The homeomorphisms ϕ_a are called (coordinate) **charts** on X, and the set $\{\phi_a | a \in X\}$ is called an **atlas** on X.

5.2 Remark: When $\phi : U \subseteq X \to \phi(U) \subseteq \mathbb{R}^n$ and $\psi : V \subseteq X \to \psi(V) \subseteq \mathbb{R}^n$ are two charts on an n-manifold X with $U \cap V \neq \emptyset$, the map $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ is a homeomorphism between two open sets in $\mathbb{R}^n$, which is called a **transition function**. Various kinds of manifolds are defined by placing restrictions on the transition functions. For example, a **smooth** manifold has an atlas for which the transition functions are smooth.

5.3 Example: The **Euclidean space** $\mathbb{R}^n$ is an n-manifold (using the identity map as a single chart), and more generally every n-dimensional **vector space** is an n-manifold. If X is an n-manifold then so is every open subset $U \subseteq X$, so for example the **general linear group** $GL_n(\mathbb{R})$ is an open subset of the space of $n \times n$ matrices $M_n(\mathbb{R})$, which is an n^2-manifold. The n**-sphere** $\mathbb{S}^n = \left\{x \in \mathbb{R}^{n+1} \big| \|x\| = 1\right\}$ is an n-manifold: indeed, letting $p = e_{n+1}$, $\mathbb{S}^n$ can be covered by the two open sets $U = \mathbb{S}^n \setminus \{p\}$ and $V = \mathbb{S}^n \setminus \{-p\}$, and U and V are both homeomorphic to $\mathbb{R}^n$ using stereographic projection. When X is an n-manifold and Y is an m-manifold, the product $X \times Y$ is an $(n+m)$-manifold: indeed for $a \in X$, $b \in Y$ with charts $\phi_a : U_a \subseteq X \to \phi_a(U_a) \subseteq \mathbb{R}^n$ and $\psi_b : V_b \subseteq Y \to \psi_b(V_b) \subseteq \mathbb{R}^m$, we have the homeomorphism $(\phi_a \times \psi_b) : U_a \times V_b \subseteq X \times Y \to \phi_a(U_a) \times \psi_b(V_b) \subseteq \mathbb{R}^{n+m}$ given by $(\phi_a \times \psi_b)(x, y) = (\phi_a(x), \psi_b(y)) \in \mathbb{R}^{n+m}$. In particular, the n**-torus** $\mathbb{T}^n = \mathbb{S}^1 \times \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ (the product of n copies of $\mathbb{S}^1$) is an n-manifold.

5.4 Definition: Let W be a vector space over a field $\mathbb{F}$. The **projectivization of** W is the set of all 1-dimensional subspaces of W. For $0 \neq x \in W$, write $[x] = \mathrm{Span}_{\mathbb{F}}\{x\}$. Then

$$\mathbb{P}(W) = \left\{ [x] \,\big|\, 0 \neq x \in W \right\}$$

An element of $\mathbb{P}(W)$ (which is a line through the origin in W) is called a **point** in $\mathbb{P}(W)$. If V is a 2-dimensional subspace of W then the set $\mathbb{P}(V) \subseteq \mathbb{P}(W)$ is called a **line** in $\mathbb{P}(W)$. More generally, if V is a $(k+1)$-dimensional subspace of W, then $\mathbb{P}(V)$ is called a k-dimensional projective space in $\mathbb{P}(W)$.

The projectivization of the vector space $\mathbb{R}^{n+1}$ is called the (real) **projective** n**-space**, and it is denoted by $\mathbb{P}^n$ or $\mathbb{P}^n(\mathbb{R})$:

$$\mathbb{P}^n = \mathbb{P}^n(\mathbb{R}) = \mathbb{P}(\mathbb{R}^{n+1}) = \left\{ [x] \,\big|\, 0 \neq x \in \mathbb{R}^{n+1} \right\} \text{, where } [x] = \mathrm{Span}_{\mathbb{R}}\{x\} = \left\{ tx \,\big|\, t \in \mathbb{R} \right\}.$$

The space $\mathbb{P}^1 = \mathbb{P}^1(\mathbb{R})$ is called the (real) **projective line** and the space $\mathbb{P}^2 = \mathbb{P}^2(\mathbb{R})$ is called the (real) **projective plane**.

We make $\mathbb{P}^n$ into a topological space by identifying it with the quotient space when $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ acts by multiplication on $\mathbb{R}^{n+1} \setminus \{0\}$ (the equivalence classes are the lines in $\mathbb{R}^{n+1}$ with the origin removed). Under this identification,

$$\mathbb{P}^n = \left(\mathbb{R}^{n+1} \setminus \{0\}\right) / \mathbb{R}^* = \left\{ [x] \,\big|\, 0 \neq x \in \mathbb{R}^{n+1} \right\} \text{ with } [x] = \left\{ tx \,\big|\, t \in \mathbb{R}^* \right\}.$$

With this topology, $\mathbb{P}^n$ is an n-manifold: indeed, for each $k = 1, 2, \cdots, n+1$, we let $U_k = \big\{[x_1, \cdots, x_{n+1}] \in \mathbb{P}^n \big| x_k \neq 0\big\}$ and define $\phi_k : U_k \subseteq \mathbb{P}^n \to \mathbb{R}^n$ and $\psi_k : \mathbb{R}^n \to U_k \subseteq \mathbb{P}^n$ by

$$\phi_k\big([x_1, \cdots, x_{n+1}]\big) = \big(\tfrac{x_1}{x_k}, \cdots, \tfrac{x_{k-1}}{x_k}, \tfrac{x_{k+1}}{x_k}, \cdots, \tfrac{x_{n+1}}{x_k}\big)$$
$$\psi_k(x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_{n+1}) = [x_1, \cdots, x_{k-1}, 1, x_{k+1}, \cdots, x_{n+1}]$$

The sets U_k are open in $\mathbb{P}^n$, and they cover $\mathbb{P}^n$, and each ϕ_k is a homeomorphisms with inverse ψ_k.

5.5 Exercise: Show that $\mathbb{P}^n \cong \mathbb{S}^n/\{\pm 1\}$.

5.6 Exercise: Find an example of a topoological space which is locally homeomorphic to $\mathbb{R}$ but is not regular, and show that every Hausdorff topological space which is locally homeomorphic to $\mathbb{R}^n$ is regular, and hence that every n-manifold is metrizable.

5.7 Remark: By imitating the proof of Urysohn's Metrization Theorem, one can show that every compact n-manifold is homeomorphic to a subspace of $\mathbb{R}^m$ for some m.

Compact Connected 2-Dimensional Manifolds

5.8 Note: In the remainder of this chapter, we will give an informal, non-rigorous, presentation of the classification of compact connected 2-manifolds, up to homeomorphism.

5.9 Theorem: *Every compact 2-manifold is triangularizable, and hence is homeomorphic to the quotient space of the disjoint union of an even number of disjoint closed solid triangles in $\mathbb{R}^2$ under an equivalence relation which identifies pairs of edges via affine maps (rotations or reflections).*

Proof: We omit the proof, which is long and difficult.

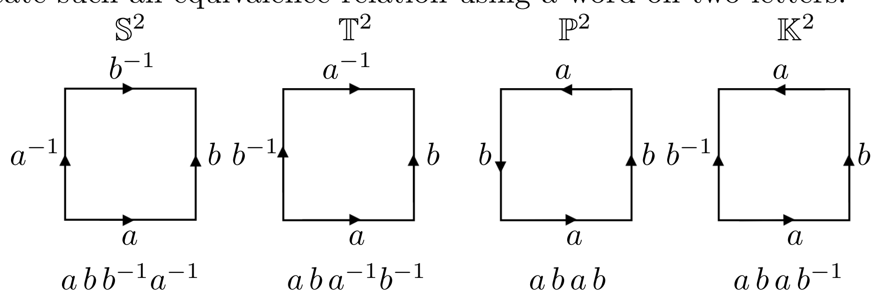
5.10 Corollary: *Every compact connected 2-manifold is homeomorphic to a closed solid regular polygon, with an even number of edges, under an equivalence relation in which the edges are identified in pairs via affine maps. Conversely, every such quotient space is a compact connected 2-dimensional manifold.*

Proof: We shall give an informal explanation. By the previous (unproven) theorem, X is homeomorphic to the quotient space of a disjoint union of an even number of closed equilateral triangles in $\mathbb{R}^2$, under an equivalence relation in which pairs of edges are identified using affine maps. Beginning with one of the triangles as our initial polygon, one by one we join another triangle to our polygon along an identified edge and, at each stage, we deform the resulting polygon to make it regular. Eventually, all triangles will be attached, otherwise X would not be connected.

For the converse, consider three kinds of points: a point in the interior of the polygon, an equivalence class consisting of a point from the interior of each of two edges, and an equivalence class of vertices. A point in the interior of the polygon can be surrounded by an open disc which is contained in the interior of the polygon. For an equivalence class consisting of a point from the interior of each of two edges, each point can be surrounded by a half-disc with its diameter along an edge, and under the equivalence relation, the half discs are glued along their diameters to form an open disc. For an equivalence class of vertices, each vertex can be surrounded by a circular sector at the vertex between the two adjacent edges, and under the equivalence relation, the sectors are glued along the edges to form an open disc.

5.11 Corollary: *Every compact connected 2-manifold is homeomorphic to the quotient space of a closed unit disc under an equivalence relation which identifies pairs of arcs between the $2n$ points $p_k = e^{i\pi/k}$ with $k \in \mathbb{Z}_{2n}$ via affine maps.*

5.12 Example: Each of the 2-dimensional manifolds $\mathbb{S}^2$, $\mathbb{T}^2$, $\mathbb{P}^2$ and the **Klein bottle** $\mathbb{K}^2$, can be obtained (up to homeomorphism) by forming the quotient space of a closed square under an equivalence relation which identifies pairs of opposite edges, as shown, and we indicate such an equivalence relation using a word on two letters.



5.13 Note: As in the above example, an equivalence relation which identifies pairs of edges of a closed solid regular $2n$-gon (or an equivalence relation which identifies pairs of arcs between the $2n$ points $p_k = e^{i\pi/k}$ along the boundary circle of the closed unit disc) using a word on n letters in which each letter occurs twice, and each time a letter occurs it occurs with exponent ± 1 (when a letter occurs twice with the same exponent, the corresponding pair of edges is identified via a rotation about the origin, and when the two occurrences have a different exponent, the corresponding pair of edges is identified via a reflection in a line through the origin). Up to homeomorphism, every compact connected 2-manifold can be represented using such a word.

5.14 Definition: Given two 2-dimensional manifolds X and Y, the **connected sum** $X\#Y$ is the 2-manifold formed by removing a small open disc from each of X and Y then glueing the boundary circles together (using an equivalence relation). We shall assume, without proof, that up to homeomorhism, the connected sum does not depend on the position of the small discs, or on the direction in which the boundaries are glued, and that when $X_1 \cong X_2$ and $Y_1 \cong Y_2$ we have $X_1\#Y_1 \cong X_2\#Y_2$ (so that the connected sum is a well-defined operation on 2-manifolds, up to homeomorphism), and also that $X\#Y \cong Y\#X$. We say that the 2-manifold $X\#\mathbb{T}^2$ is obtained by attaching a **handle** to X, and the 2-manifold $X\#\mathbb{P}^2$ is obtained by adding a **crosscap** to X.

5.15 Example: Notice that for any 2-manifold X we have $X\#\mathbb{S}^2 \cong X$.

5.16 Definition: For $g \in \mathbb{N}$ we define $(\mathbb{T}^2)^{\#g}$ and $(\mathbb{P}^2)^{\#h}$ recursively by $(\mathbb{T}^2)^{\#0} = \mathbb{S}^2$ and $(\mathbb{T}^2)^{\#(g+1)} = (\mathbb{T}^2)^{\#g}\#\mathbb{T}^2$ and by $(\mathbb{P}^2)^{\#1} = \mathbb{P}^2$ and $(\mathbb{P}^2)^{\#(h+1)} = (\mathbb{P}^2)^{\#h}\#\mathbb{P}^2$ so that

$$(\mathbb{T}^2)^{\#g} = \mathbb{S}^2\#\mathbb{T}^2\#\cdots\#\mathbb{T}^2$$
$$(\mathbb{P}^2)^{\#h} = \mathbb{P}^2\#\mathbb{P}^2\#\cdots\#\mathbb{P}^2.$$

5.17 Theorem: *When the 2-manifolds X and Y are represented by words σ and τ, the connected sum $X\#Y$ is represented by the concatenated word $\sigma\tau$.*

Proof: Consider two polygons, each with a tear-shaped disc at a vertex.

5.18 Example: Since $\mathbb{T}^2$ can be represented by the word $aba^{-1}b^{-1}$ it follows that $(\mathbb{T}^2)^{\#g}$ can be represented by $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1}$, and since $\mathbb{P}^2$ can be represented by the word $a\,a$, it follows that $(\mathbb{P}^2)^{\#h}$ can be represented by $a_1a_1a_2a_2\cdots a_ha_h$.

Cut-and-Paste Operations

5.19 Remark: There are a number of operations which can be performed on the words which represent an edge-pairing equivalence relation, without changing the topological type (up to homeomorphism) of the resulting quotient space. For example, rotating the polygon corresponds to cyclically permuting the word (for example the word $a\,b\,c\,a^{-1}c\,b^{-1}$ yields the same 2-manifold, up to homeomorphism, as the word $b\,c\,a^{-1}c\,b^{-1}a$) and reflecting the polygon corresponds to inverting the word (for example the word $a\,b\,c\,a^{-1}c\,b^{-1}$ yields the same 2-manifold, up to homeomorphism, as the word $b\,c^{-1}a\,c^{-1}b^{-1}a^{-1}$). Also, when a word uses, say, the letter a we can replace a by a^{-1} and vice versa, and when a word uses the letter a and does not use the letter x, we can replace a by x and a^{-1} by x^{-1} (for example, the word $a\,b\,c\,a^{-1}c\,b^{-1}$ yields the same 2-manifold as the word $x\,y^{-1}z\,x^{-1}z\,y$).

5.20 Theorem: *When we perform one of the following cut-and-paste operations on a polygon with edges identified in pairs, we obtain a homeomorphic space.*

(1) When two adjacent edges of the polygon are identified in opposite directions, we can remove those two edges.
(2) When a diagonal of the polygon separates a pair of identified edges (one on each side of the diagonal), we can cut the polygon along the diagonal to obtain two disjoint polygons (each includes one of the separated edges and each includes an edge corresponding to the diagonal), then we can reattach the two polygons along the separated pair of edges.

Proof: We omit the proof.

5.21 Example: Consider the regular hexagon with vertices at $p_k = e^{k\pi/3}$ with the edges identified in pairs according to the word $a\,b\,c\,a^{-1}c\,b^{-1}$ (starting at p_0 and moving counterclockwise). Let us perform an operation of Type 2. Note that the diagonal from p_1 to p_3 separates the pair of edges labelled by b and b^{-1}. Cut from vertex p_1 to p_3, labelling the resulting edge using the letter x, to obtain a pentagon and a triangle with edges identified in pairs according to $a\,x\,a^{-1}c\,b^{-1}$ and $b\,c\,x^{-1}$. Reattach the triangle to the pentagon along the edge labelled by b to obtain the hexagon with edges identified according to $a\,x\,a^{-1}c\,c\,x^{-1}$. Thus the 2-manifold represented by the word $a\,b\,c\,a^{-1}c\,b^{-1}$ is homeomorphic to the 2-manifold represented by $a\,x\,a^{-1}c\,c\,x^{-1}$.

5.22 Example: Use the cut-and-paste operations to show that $\mathbb{K}^2 \cong \mathbb{P}^2\#\mathbb{P}^2$ and to show that $\mathbb{P}^2\#\mathbb{T}^2 \cong \mathbb{P}^2\#\mathbb{P}^2\#\mathbb{P}^2$.

Solution: For the first problem, start with $\mathbb{K}^2$ represented by the square with edges identified according to the word $a\,b\,a^{-1}b$ (with the initial letter a from the vertex $p_0 = 1$ to $p_1 = i$). Cut from p_1 to p_3 labelling that edge by c, remove the triangle with vertices p_1, p_2, p_3 (with edges labelled $b\,a^{-1}c^{-1}$) and reattach along the b edge to obtain a quadrilateral with edges labelled $a\,a\,c\,c$, which represents $(\mathbb{P}^2)^{\#2}$.

For the second problem, start with $\mathbb{P}^2\#\mathbb{T}^2$ represented by the hexagon with edges identified by $a\,a\,b\,c\,b^{-1}c^{-1}$ (with the initial letter a from $p_0 = 1$ to $p_1 = e^{i\,\pi/3}$). Cut from p_1 to p_3 labelling the edge by y. Remove the triangle with vertices p_1, p_2, p_3 with edges labelled by $a\,b\,y^{-1}$, and reattach along the a edge to obtain a hexagon with edges labelled $y\,b^{-1}y\,c\,b^{-1}c^{-1}$ (with the initial letter y from p_0 to p_1). Cut from p_5 to p_2, labelling the edge by x, remove the quadrilateral with vertices at p_0, p_1, p_2, p_5 labelled by $y\,b^{-1}x^{-1}c^{-1}$ and reattach along the b edge to obtain the hexagon with edges labelled by $x\,x\,y\,c\,y^{-1}c$ (with the initial letter x from p_0 to p_1). Finally, cut from p_4 to p_0, labelling the edge by z, remove the triangle with vertices at p_0, p_4, p_5 labelled by $z^{-1}y^{-1}c$ and reattach along the c edge to obtain a hexagon with edges labelled by $x\,x\,y\,y\,z\,z$, which represents $(\mathbb{P}^2)^{\#3}$.

5.23 Theorem: *Every compact connected 2-manifold is homeomorphic to one of the spaces* $(\mathbb{T}^2)^{\# g} = \mathbb{S}^2 \# \mathbb{T}^2 \# \mathbb{T}^2 \# \cdots \# \mathbb{T}^2$ *with* $g \geq 0$*, or* $(\mathbb{P}^2)^{\# h} = \mathbb{P}^2 \# \mathbb{P}^2 \# \cdots \# \mathbb{P}^2$ *with* $h \geq 1$.

Proof: We describe an algorithm which uses cut-and-paste operations on a polygon with edges identified in pairs, to obtain another such polygon which is homeomorphic to the connected sum of some copies of $\mathbb{P}^2$ with some copies of $\mathbb{T}^2$. The theorem then follows from the fact that $\mathbb{P}^2 \# \mathbb{T}^2 \cong \mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2$ (as shown in Exercise 5.22).

First eliminate any two adjacent edges identified in opposite directions using operations of Type 1. Then determine which vertices are identified under the equivalence relation to find the number of equivalence classes of vertices and the number of vertices in each class. Select the class C (or one of the classes) with the most elements. Perform the following operation (repeatedly as necessary) until all vertices have joined this class (so that all vertices are equivalent and there is only one class): Select an index k such that $p_k \in C$ and $p_{k+1} \notin C$, say $p_{k+1} \in D$. Say the edge from p_k to p_{k+1} is labelled by a, note that the edge from p_{k+1} to p_{k+2} will not be labelled by a or by a^{-1}, and say it is labelled by b. Perform an operation of Type 2, cutting from p_k to p_{k+2} to separate a triangle, and reattach along the edge labelled by b. Verify that this reduces the number of vertices in class D by one and increases the number of vertices in class C by 1. Use operations of Type 1 if necessary, then repeat.

After the first step has been completed, we have a polygon with edges identified in pairs so that all vertices are equivalent. The next step is to isolate copies of $\mathbb{P}^2$ as follows: If there is a non-adjacent pair of edges identified in the same direction, say the edge from p_k to p_{k+1} is labelled by a and the edge from p_ℓ to $p_{\ell+1}$ is labelled by a, then cut from p_k to p_ℓ (labelling this new edge by x) separating the polygon into two polygons, and reattach them along the edges labelled by a. Verify that this results in two adjacent edges both labelled by x (or both labelled by x^{-1}).

After the second step has been completed, any two edges identified by the same letter with the same exponent will be adjacent. If there is no pair of edges with one labelled by a letter, say a and the other labelled by a^{-1}, then we are done and the resulting 2-manifold is a connected sum of some copies of $\mathbb{P}^2$. If there is a pair of edges with one labelled by a and the other labelled by a^{-1}, then verify that there must be a second such pair labelled by b and b^{-1}, and the word (after possibly cyclically permuting and changing letters) must be of the form $a\,\sigma_1 b\,\sigma_2 a^{-1}\sigma_3 b^{-1}\sigma_4$ for some words σ_k (otherwise there would be more than one equivalence class of vertices). Cut from the start of the first edge labelled by a to the start of the second edge labelled by a, label this new cut edge by y^{-1}, separate the polygon with edge identifications $\sigma_3 b^{-1}\sigma_4 y^{-1}$ and reattach along the edge labelled by b to obtain the polygon with edges identified according to $a\sigma_1\sigma_4 y^{-1}\sigma_3\sigma_2 a^{-1} y$. Cut from the end of the first edge labelled by y (the end of σ_4) to the end of the last y edge (the start of the first a edge), label the cut edge by x, remove the polygon with edges labelled by $a\sigma_1\sigma_4 x$ and reattach along the other edge labelled by a to obtain the polygon with edges identified according to $x\,y\,x^{-1}y^{-1}\sigma_3\sigma_2\sigma_1\sigma_4$. This isolates a copy of $\mathbb{T}^2$ labelled by $x\,y\,x^{-1}y^{-1}$. Repeat this step until we are left with copies of $\mathbb{P}^2$ and $\mathbb{T}^2$.

5.24 Example: Carry out the cut-and-paste algorithm to determine the topological type of the regular polygon with edge identifications given by the word $a\,b\,c\,b\,d\,e\,c^{-1}e\,f\,f^{-1}a^{-1}d$.

Orientability and the Euler Characteristic

5.25 Definition: For a regular polygon with edges identified in pairs, if there exists a pair for which the edges are identified in the same direction (either both clockwise or both counter-clockwise), then we say the resulting 2-manifold is **non-orientable**. Otherwise we say it is **orientable**.

5.26 Theorem: *When X and Y are compact connected 2-manifolds, if $X \cong Y$ then X is orientable if and only if Y is orientable.*

Proof: We do not provide a proof.

5.27 Corollary: *Given $g \geq 0$ and $h \geq 1$, we have $(\mathbb{T}^2)^{\# g} \not\cong (\mathbb{P}^2)^{\# h}$.*

5.28 Definition: Given a polygonalization of a compact connected 2-manifold X (that is given a decomposition of the manifold into polygons, which meet along pairs of edges), if V is the number vertices, E is the number of edges, and F is the number of polygons, then $\chi = V - E + F$ is called the **Euler characteristic** of the polygonalization of X.

5.29 Theorem: *All polygonalizations of a compact connected 2-manifold have the same Euler characteristic.*

Proof: We do not provide a proof. We remark that some students might find it intuitively reasonable to imagine that given two reasonably nice polygonalizations P and Q of a compact 2-manifold X, it is possible to construct another polygonalization R which is finer that both P and Q (so that the vertices of R include all of the vertices in P and Q) and that R can be obtained from P or from Q be performing a series of operations each of which either adds an edge and a vertex, or adds a face and an edge (so that each operation does not change the Euler characteristic).

5.30 Definition: For a compact connected 2-manifold X, the **Euler characteristic** of X, denoted by $\chi(X)$, is the Euler characteristic of any polygonalization of X.

5.31 Theorem: *When X and Y are compact connected 2-manifolds with $X \cong Y$, we have $\chi(X) \cong \chi(Y)$.*

Proof: A homeomorphism sends a polygonalization of X to a polygonalization of Y with the same values of V, E and F, giving the same Euler characteristic.

5.32 Note: Since $(\mathbb{T}^2)^{\# g}$ can be represented by a $4g$-gon with edges identified according the the word $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$, and all vertices are identified under this equivalence relation, we have $\chi\big((\mathbb{T}^2)^{\# g}\big) = V - E + F = 1 - 2g + 1 = 2 - 2g$. Since $(\mathbb{P}^2)^{\# h}$ can be represented by a $2h$-gon with edges identified according to the word $a_1 a_1 a_2 a_2 \cdots a_h a_h$, and all vertices are identified, we have $\chi\big((\mathbb{P}^2)^{\# h}\big) = V - E + F = 1 - g + 1 = 2 - g$.

5.33 Corollary: *If $g_1 \neq g_2$ then $(\mathbb{T}^2)^{\# g_1} \not\cong (\mathbb{T}^2)^{\# g_2}$, and if $h_1 \neq h_2$ then $(\mathbb{P}^2)^{\# h_1} \not\cong (\mathbb{P}^2)^{\# h_2}$.*

5.34 Corollary: *The topological type of a compact connected 2-manifold is determined by its orientability and its Euler class.*

5.35 Exercise: Find the topological type of the regular polygon with edges identified according to the word $a\,b\,c\,a\,d\,b^{-1} d\,c$.

5.36 Exercise: Find a formula for the Euler characteristic of a connected sum.