Chapter 4. Countability and Separation Axioms

The Countability Axioms

4.1 Definition: Let X be a topological space. For a subset $A \subseteq X$, we say that A is **dense** in X when $\overline{A} = X$. Equivalently, A is dense in X when every nonempty open set in X contains a point in A.

4.2 Definition: Let X be a topological space. We say that X is **first-countable** when for every $a \in X$ there exists a (finite or) countable set $\mathcal{B}_a$ of open sets such that for every set U in X with $a \in U$ there is a set $B \in \mathcal{B}_a$ such that $a \in B \subseteq U$. We say that X is **second-countable** when X has a (finite or) countable basis for its topology. We say that X is **Lindelöf** when every open cover of X contains a (finite or) countable sub-cover. We say that X is **separable** when X has a (finite or) countable dense subset.

4.3 Theorem: *Let X be a metric space. Then X is first-countable and the following are equivalent:*
(1) X is second-countable.
(2) X is Lindelöf.
(3) X is separable

Proof: The proof is left as an exercise (this is often proven in a real analysis course).

4.4 Theorem: *Every second-countable topological space is Lindelöf and separable.*

Proof: Let X be a second-countable topological space. Let $\mathcal{B} = \{B_1, B_2, \cdots\}$ be a (finite or) countable basis for X. We claim that X is Lindelöf. Let $\mathcal{S}$ be any open cover of X. Let $K = \left\{k \in \mathbb{Z}^+ \,\middle|\, \exists\, U \in \mathcal{S}\ B_k \subseteq U\right\}$. For each $k \in K$, choose $U_k \in \mathcal{S}$ such that $B_k \subseteq U_k\}$. Then the (finite or) countable set $\left\{U_k \,\middle|\, k \in K\right\}$ is a subcover of $\mathcal{S}$ because for every $a \in X$ we can choose $U \in \mathcal{S}$ with $a \in U$, then we can choose $B_k \in \mathcal{B}$ with $a \in B_k \subseteq U$ and then we have $k \in K$ and $a \in B_k \subseteq U_k$.

We claim that X is separable. Let $K = \left\{k \in \mathbb{Z}^+ \,\middle|\, B_k \neq \emptyset\right\}$. For each $k \in K$, choose $a_k \in B_k$. Then the (finite or) countable set $A = \left\{a_k \,\middle|\, k \in K\right\}$ is dense in X: indeed given any nonempty open set U in X, we can choose $x \in U$, then we can choose $B_k \in \mathcal{B}$ with $x \in B_k$, and then we have $a_k \in B_k \cap A \subseteq U \cap A$.

4.5 Theorem: *Every subspace of a first-countable set is first-countable. Every subspace of a second-countable space is second countable.*

Solution: We prove the first statement and leave the proof of the second statement as an exercise. Let Y be a first-countable space and let $X \subseteq Y$ be a subspace. Let $a \in X$. Since Y is first-counatble and $a \in Y$, we can choose a (finite or) countable set $\mathcal{C}$ of open sets in Y such that for every open set V in Y with $a \in V$, there exists $C \in \mathcal{C}$ such that $a \in C \subseteq V$. Let $\mathcal{B} = \left\{C \cap X \,\middle|\, C \in \mathcal{C}\right\}$. Then $\mathcal{B}$ is (finite or) countable and, given any open set U in X with $a \in U$ we can choose an open set V in Y such that $U = V \cap X$, then we can choose $C \in \mathcal{C}$ such that $a \in C \subseteq V$, and then for $B = C \cap X$ we have $B \in \mathcal{B}$ with $a \in B = C \cap X \subseteq V \cap X = U$.

4.6 Theorem: *The product of two first-coountable spaces is first-countable. The product of two second-countable spaces is second-countable. The product of two separable spaces is separable*

Proof: We prove the first statement. Let X and Y be first-countable spaces. Let $a \in X$ and $b \in Y$ so that $(a,b) \in X \times Y$. Choose a (finite or) countable set $\mathcal{B}$ of open sets in X so that for every open set U in X with $a \in U$ there is a set $B \in \mathcal{B}$ such that $a \in B \subseteq U$, and choose a (finite or) countable set $\mathcal{C}$ of open sets in Y so that for every open set V in Y with $b \in V$ there is a set $C \in \mathcal{C}$ such that $b \in C \subseteq V$. Let $\mathcal{P} = \big\{ B \times C \,\big|\, B \in \mathcal{B}\,,\, C \in \mathcal{C} \big\}$, and note that $\mathcal{P}$ is (finite or) countable. Let W be an open set in $X \times Y$ with $(a,b) \in W$. Since the sets of the form. $U \times V$ with U open in X and V open in Y form a basis for the topology on $X \times Y$, we can choose U open in X and V open in Y such that $(a,b) \in U \times V \subseteq W$. Since $a \in U$ we can choose $B \in \mathcal{B}$ such that $a \in B \subseteq U$, and since $b \in V$ we can choose $C \in \mathcal{C}$ such that $b \in C \subseteq V$. Then we have $B \times C \in \mathcal{P}$ and $(a,b) \in B \times C \subseteq U \times V \subseteq W$.

4.7 Exercise: Generalize the above theorem to include countable products, using the product topology.

4.8 Exercise: Let $\mathbb{R}_\ell$ be the set $\mathbb{R}$ using the lower limit topology. Show that $\mathbb{R}_\ell$ is first-countable and Lindelöf and separable, but not second countable.

4.9 Exercise: Let $I = [0,1]$ and let I_o^2 be the set I^2 using the order topology for the dictionary order on I^2. Show that I_o^2 is first countable and compact, hence Lindelöf, but not separable. Also, show that the subspace $A = I \times (0,1)$ is not Lindelöf.

4.10 Exercise: The **Moore plane** Γ is the closed upper half plane $\Gamma = \big\{ (x,y) \in \mathbb{R}^2 \,\big|\, y \geq 0 \big\}$ using the topology with basis consisting of the balls $B\big((a,b), r\big)$ with $0 < r < b$ and the sets $B\big((a,r), r\big) \cup \{(a,0)\}$. Show that Γ is first countable and separable, but not Lindelöff. Also, show that the subspace $\mathbb{R} \times \{0\}$ is not separable.

4.11 Exercise: Let $\mathbb{R}_{cf}$ be the set $\mathbb{R}$ using the co-finite topology (in which the closed proper subsets of $\mathbb{R}$ are the finite sets), let $\mathbb{R}_{cc}$ be the set $\mathbb{R}$ using the co-countable topology (in which the closed proper subsets of $\mathbb{R}$ are the finite or countable sets), and let $\mathbb{R}_d$ be the set $\mathbb{R}$ using the discrete topology (in which all subsets are both open and closed). For each of the spaces $\mathbb{R}_{cf}$, $\mathbb{R}_{cc}$ and $\mathbb{R}_d$, determine whether the space is first countable, whether it is Lindelöff, and whether it is separable.

4.12 Exercise: Let $\mathbb{R}_\ell$ be the set $\mathbb{R}$ using the lower limit topology. Recall, from Exercise 4.7, that $\mathbb{R}_\ell$ is Lindelöf. Show that $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is not Lindelöf.

The Separation Axioms

4.13 Definition: Let X be a topological space. We say that X is $T1$ when for all $a, b \in X$ with $a \neq b$, there exists an open set V in X with $a \notin V$ and $b \in V$. We say that X is T2, or that X is **Hausdorff**, when for all $a, b \in X$ with $a \neq b$, there exist disjoint open sets U and V in X with $a \in U$ and $b \in V$. We say that X is T3, or that X is **regular**, when X is T1 and for every point a in X and every closed set B in X with $a \notin B$, there exist disjoint open sets U and V in X with $a \in U$ and $B \subseteq V$. We say that X is T4, or that X is **normal**, when X is T1 and for every pair of disjoint closed sets A and B in X there exist disjoint open sets U and V in X with $A \subseteq U$ and $B \subseteq V$.

4.14 Theorem: *Let X be a topological space. Then X is T1 if and only if the 1-point subsets of X are closed in X.*

Proof: If X is T1 and we let $a \in X$, then for each $b \in X$ we can choose an open set V_a in X such that $a \notin V$ and $b \in V_b$, and then we have $\{a\}^c = \bigcup_{b \in A^c} U_b$, which is open. Conversely, if the 1-point subsets of X are closed in X, then given $a, b \in X$ with $a \neq b$ we can let V be the open set $V = \{a\}^c$, and then $a \notin V$ and $b \in V$.

4.15 Theorem: *Let X be a T1 topological space. Then X is regular if and only if for every $a \in X$ and for every open set W in X with $a \in W$, there exists an open set U in X with $a \in U \subseteq \overline{U} \subseteq W$.*

Proof: Suppose that X is regular. Let W be open in X with $a \in W$. Then W^c is closed with $a \notin W^c$. Since X is regular, we can choose disjoint open sets U and V in X with $a \in U$ and $W^c \subseteq V$. Since $W^c \subseteq V$ we have $V^c \subseteq W$. Since $U \cap V = \emptyset$ we have $U \subseteq V^c$, which is closed, and hence $a \in U \in \overline{U} \subseteq V^c \subseteq W$.

Suppose, conversely, that for every $a \in X$ and every open set W in X with $a \in W$ there exists an open set U in X with $a \in U \subseteq \overline{U} \subseteq W$. Let $a \in X$ and let B be a closed set in X with $a \notin B$. Then $a \in B^c$, which is open, so we can choose an open set U in X with $a \in U \subseteq \overline{U} \subseteq B^c$. Let $V = \overline{U}^c$, which is open. Since $U \subseteq \overline{U} = V^c$ we have $U \cap V = \emptyset$. Since $V^c = \overline{U} \subseteq B^c$ we have $B \subseteq V$.

4.16 Definition: Let X be a topological space. We say that X is **metrizable** when there exists a metric on X for which the topology on X is the metric topology.

4.17 Theorem: *Every metrizable space is normal, every normal space is regular, every regular space is Hausdorff, and every Hausdorff space is T1.*

Proof: The proof is left as an exercise.

4.18 Theorem: *Every subspace of a T1 space is T1, every subspace of a Hausdorff space is Hausdorff, and every subspace of a regular space is regular.*

Proof: Let X be a subspace of Y. If the 1-point subsets of Y are closed in Y, then given $a \in X$, since $\{a\}$ is closed in Y and $\{a\} = \{a\} \cap X$, it follows that $\{a\}$ is closed in X.

Suppose Y is Hausdorff. Let $a, b \in X$. Choose disjoint open sets U and V in Y with $a \in U$ and $b \in Y$. Then $U \cap X$ and $V \cap X$ are disjoint open sets in X with $a \in U \cap X$ and $b \in V \cap X$.

Now suppose that Y is regular. As shown above, X is T1. Let $a \in X$ and let B be a closed set in X with $a \notin B$. Then we have $B = \mathrm{Cl}_X(B) = \overline{B} \cap X$ where $\overline{B} = \mathrm{Cl}_Y(B)$. Since $a \in X$ and $a \notin B = \overline{B} \cap X$, it follows that $a \notin \overline{B}$. Since Y is regular, we can find open sets U and V in Y with $a \in U$ and $\overline{B} \subseteq V$. Then the sets $U \cap X$ and $V \cap X$ are open in X with $a \in U \cap X$ and $B = \overline{B} \cap X \subseteq B \cap X$.

4.19 Theorem: *When using the product or the box topology, the product of an indexed set of T1 spaces is T1, the product of an indexed set of Hausdorff spaces is Hausdorff, and the product of an indexed set of regular spaces is regular.*

Proof: Let X_k be a topological space for each $k \in K$. Suppose first that each X_k is T1. Let $a \in \prod_{k\in K} X_k$. For each $k \in K$, since X_k is T1, the set $\{a_k\}$ is closed in X_k so that $\overline{\{a_k\}} = \{a_k\}$. Note that $\{a\} = \prod_{k\in K}\{a_k\}$. By Theorem 2.16 we have $\overline{\{a\}} = \overline{\prod_{k\in K}\{a_k\}} = \prod_{k\in K}\overline{\{a_k\}} = \prod_{k\in K}\{a_k\} = \{a\}$ and hence $\{a\}$ is closed in $\prod_{k\in K} X_k$.

Now suppose that each X_k is Hausdorff. Let $a, b \in \prod_{k\in K} X_k$ with $a \neq b$. Since $a \neq b$ we can choose $\ell \in K$ so that $a_\ell \neq b_\ell$. Since X_ℓ is Hausdorff, we can choose disjoint open sets U_ℓ and V_ℓ in X_ℓ with $a_\ell \in U_\ell$ and $b_\ell \in V_\ell$. For each $k \in K$ with $k \neq \ell$, let $U_k = V_k = X_k$. Then the sets $U = \prod_{k\in K} U_k$ and $V = \prod_{k\in K} V_k$ are disjoint basic open sets in $\prod_{k\in K} X_k$ with $a \in U$ and $b \in V$.

Finally, suppose that each X_k is regular. Note that since each X_k is T1 it follows, from the first paragraph, that $\prod_{k\in K} X_k$ is T1. Using Theorem 4.15, in order to show that $\prod_{k\in K} X_k$ is regular, it suffices to show that for every open set W in $\prod_{k\in K} X_k$ and for every $a \in W$, there exists an open set U in $\prod_{k\in K} X_k$ with $a \in U \subseteq \overline{U} \subseteq W$. Let W be an open set in $\prod_{k\in K} X_k$ and let $a \in W$. Choose a basic open set V in $\prod_{k\in K} X_k$ with $a \in V \subseteq W$, say $V = \prod_{k\in K} V_k$ where each V_k is open in X_k with $a_k \in V_k$. For each $k \in K$, since X_k is regular, by Theorem 4.15, we can choose an open set U_k in X_k with $a \in U_k \subseteq \overline{U_k} \subseteq V_k$. Let $U = \prod_{k\in K} U_k$ and note that $a \in U$. By Theorem 2.16, we have $\overline{U} = \overline{\prod_{k\in K} U_k} = \prod_{k\in K} \overline{U_k} \subseteq \prod_{k\in K} V_k = V \subseteq W$, so that $a \in U \subseteq \overline{U} \subseteq W$, as required.

4.20 Exercise: Show that $\mathbb{R}_\ell$ is normal but not metrizable.

4.21 Exercise: Show that $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is regular but not normal.

4.22 Exercise: Let $K = \left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{Z}^+ \right\}$. The K**-topology** on $\mathbb{R}$ is the topology generated by the sets of the form (a, b) with $a < b$ together with the sets of the form $(a, b) \setminus K$ with $a < b$. We write $\mathbb{R}_K$ for the set $\mathbb{R}$ equipped with the K-topology. Show that $\mathbb{R}_K$ is Hausdorff but not regular.

4.23 Exercise: Let X be an infinite set using the co-finite topology. Show that X is T1 but not Hausdorff.

4.24 Remark: A subspace of a normal space is not necessarily normal. As an example, without proof, when K is uncountable, the space $[0, 1]^K$ is compact, using the product topology (by Tichanoff's Theorem), and Hausdorff (by Theorem 4.19), so it is normal (by Theorem 4.24, below), but it can be shown (see Exercise 9 of Chapter 32 on page 206 of Munkres' book) that the subspace $(0, 1)^K$ is not normal. As another example, without proof, the space $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is not normal, by Exercise 4.21, but it can be shown (see Theorem 33.2 on page 211 and Theorem 34.3 on page 218 of Munkres' book) that $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is homeomorphic to a subspace of $[0, 1]^K$ for some (necessarily uncountable) set K.

4.25 Theorem: *Every compact Hausdorff space is normal.*

Proof: Let X be a compact Hausdorff space. First, we claim that X is regular. Let $a \in A$ and let B be a closed set in X with $a \notin B$. For each $b \in B$, since X is Hausdorff we can choose disjoint open sets U_b and V_b in X with $a \in U_b$ and $b \in V_b$. Note that for $\mathcal{S} = \{V_b \big| b \in B\}$ iwe have $B \subseteq \bigcup \mathcal{S}$. Since B is a closed subspace of the Hausdorff space X, it is compact, so we can choose $b_1, b_2, \cdots, b_m \in B$ such that $B \subseteq \bigcup_{k=1}^m V_{b_k}$. Let $U = \bigcap_{k=1}^m U_{b_k}$ and $V = \bigcup_{k=1}^m V_{b_k}$. Then U and V are disjoint open sets in X with $a \in U$ and $B \subseteq V$. This shows that X is regular, as claimed.

Now let us show that X is normal. Let A and B be disjoint closed sets in X. For each $a \in A$, since X is regular we can choose disjoint open sets U_a and V_a in X with $a \in U_a$ and $BH \subseteq V_a$. Note that for $\mathcal{R} = \{U_1 \big| a \in A\}$, we have $A \subseteq \bigcup \mathcal{R}$ Since A is a closed subspace of the Hausdorff space X, it is compact, so we can choose $a_1, a_2, \cdots, a_n \in A$ such that $A \subseteq \bigcup_{k=1}^n U_{a_k}$. Let $U = \bigcup_{k=1}^n U_{a_k}$ and let $V = \bigcap_{k=1}^n V_{a_k}$. Then U and V are disjoint open sets in X with $A \subseteq U$ and $B \subseteq V$. Thus A is normal, as required.

4.26 Theorem: *Every regular space with a (finite or) countable basis is normal.*

Proof: Let X be a regular space with a countable basis $\mathcal{B}$. First note that given $a \in X$ and given an open set W in X with $a \in W$, by Theorem 4.15 we can choose an open set U in X with $a \in U \subseteq \overline{U} \subseteq W$, then we can choose a basic open set $C \in \mathcal{B}$ with $a \in C \subseteq U$, and then we have $\overline{C} \subseteq \overline{U}$ so that $a \in C \subseteq \overline{C} \subseteq W$.

Let A and B be disjoint closed sets in X. For each $a \in A$, choose a basic open set $C_a \in \mathcal{B}$ with $a \in C_a \subseteq \overline{C_a} \subseteq B^c$ and note that $\overline{C_a} \cap B = \emptyset$. Note that $\mathcal{S} = \{C_a | a \in A\}$ is an open cover of A. Since $\mathcal{S} \subseteq \mathcal{B}$ and $\mathcal{B}$ is (finite or) countable, it follows that $\mathcal{S}$ is (finite or) countable, so we can choose $a_1, a_2, a_3, \cdots$ in A so that $\mathcal{S} = \{C_{a_1}, C_{a_2}, \cdots\}$, and note that for each $a \in A$ we have $C_a = C_{a_k}$ for some k. Similarly, for each $b \in B$, we choose $D_b \in \mathcal{B}$ with $b \in D_b$ such that $\overline{D_b} \cap A = \emptyset$, and say $\mathcal{T} = \{D_b |, b \in B\} = \{D_{b_1}, D_{b_2}, \cdots\}$. Let $U = \bigcup_{n \geq 1} U_n$ where $U_n = C_{a_n} \setminus \bigcup_{k=1}^n \overline{D_{b_n}}$, and let $V = \bigcup_{n \geq 1} V_n$ where $V_n = D_{b_n} \setminus \bigcup_{k=1}^n \overline{C_{a_k}}$. Note that U and V are open sets in X. Note that $A \subseteq \overline{U}$ because for each $a \in A$, a lies in one of the sets C_{a_ℓ} and a lies in none of the sets $\overline{D_{b_k}}$. Similarly, we have $B \subseteq V$. Finally, note that $U \cap V = \emptyset$ because if we had $x \in U \cap V$ then we would have $x \in U_\ell$ for some ℓ and $x \in V_m$ for some m, but if say $\ell \leq m$ then $x \in V_m \implies x \notin \bigcup_{k=1}^m \overline{C}_{a_k} \implies x \notin \overline{C_{a_\ell}} \implies x \notin U_{a_\ell}$.

4.27 Exercise: Show that every ordered set with a minimum element (or with a maximum element) is normal.

Urysohn's Lemma

4.28 Theorem: *(Urysohn's Lemma) Let X be a normal topological space. For any disjoint closed sets $A, B \subseteq X$ there exists a continuous map $f : X \to [0,1]$ with $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$.*

Proof: Let $A, B \subseteq X$ be closed. Say $[0,1] \cap \mathbb{Q} = \{a_0, a_1, a_2, a_3, \cdots\}$ where the terms a_k are distinct with $a_0 = 0$ and $a_1 = 1$. Choose disjoint open sets $U_0, V_0 \subseteq X$ with $A \subseteq U_0$ and $B \subseteq V_0$. Note that

$$U_0 \cap V_0 = \emptyset \Longrightarrow U_0 \subseteq {V_0}^c \Longrightarrow \overline{U}_0 \subseteq {V_0}^c \Longrightarrow \overline{U}_0 \subseteq B^c.$$

Let $U_1 = B^c$ so that $A \subseteq U_0 \subseteq \overline{U}_0 \subseteq U_1 = B^c$. Let $n \geq 2$ and suppose, inductively, that we have defined open sets $U_{a_0}, U_{a_1}, \cdots U_{a_{n-1}}$ such that when $a_k < a_\ell$ we have $\overline{U}_{a_k} \subseteq U_{a_\ell}$. Define U_{a_n} as follows. Rearrange the terms in the set $\{a_0, a_1, \cdots, a_n\}$ in increasing order and say $a_k < a_n < a_\ell$ are consecutive. Since $\overline{U}_{a_k} \subseteq U_{a_\ell}$, we have $\overline{U}_{a_k} \cap {U_{a_\ell}}^c = \emptyset$, so we can choose disjoint open sets $U_{a_n}, V_{a_n} \subseteq X$ with $\overline{U}_{a_k} \subseteq U_{a_n}$ and $U_{a_\ell}^c \subseteq V_{a_n}$, and then

$$U_{a_n} \cap V_{a_n} = \emptyset \Longrightarrow U_{a_n} \subseteq {V_{a_n}}^c \Longrightarrow \overline{U}_{a_n} \subseteq V_{a_n}^c \subseteq U_{a_\ell}.$$

Recursively, we have defined U_{a_n} for all $n \geq 0$, so we have defined U_r for all $r \in [0,1] \cap \mathbb{Q}$. For $r \in \mathbb{Q}$ with $r < 0$ we define $U_r = \emptyset$, and for $r \in \mathbb{Q}$ with $r > 1$ we define $U_r = X$, and then we have defined U_r for all $r \in \mathbb{Q}$ so that whenever $r < s$ we have $\overline{U}_r \subseteq U_s$.

Define $f : X \to [0,1]$ by

$$f(x) = \inf \left\{ r \in \mathbb{Q} \,\middle|\, x \in U_r \right\}$$

Note that f does take values in $[0,1]$: indeed for all $x \in X$, we have $f(x) \geq 0$ because $r < 0 \Longrightarrow U_r = \emptyset \Longrightarrow x \notin U_r$, and we have $f(x) \leq 1$ because $r > 1 \Longrightarrow U_r = X \Longrightarrow x \in U_r$. Also note that when $x \in A$ we have $x \in U_0$ so that $f(x) = 0$ and when $x \in B$ and $r \leq 1$ we have $U_r \subseteq U_1 = B^c$ so that $x \notin U_r$, and so $f(x) = 1$.

It remains to show that f is continuous. We shall show that the inverse image of every open interval is open. Let $c, d \in \mathbb{R}$ with $c < d$. Let $a \in f^{-1}(c,d)$ so we have $c < f(a) < d$. Choose $r, s \in \mathbb{Q}$ with $c < r < f(a) < s < d$. We claim that $a \in U_s \setminus \overline{U}_r \subseteq f^{-1}(c,d)$. First we make two observations: for $x \in X$ and $p \in \mathbb{Q}$,

(1) if $x \in \overline{U}_p$ then $x \in U_r$ for all $r > p$ and so $f(x) \leq p$, and
(2) if $x \notin U_p$ then $x \notin U_r$ for any $r \leq p$ and so $f(x) \geq p$.

Since $r < f(a)$ it follows from the first observation that $a \notin \overline{U}_r$, and since $f(a) < s$ it follows from the second observation that $a \in U_s$, and this shows that $a \in U_s \setminus \overline{U}_r$. On the other hand, when $x \in U_s \setminus \overline{U}_r$, since $x \in U_s$ it follows from the first observation that $f(x) \leq s$, and since $x \notin \overline{U}_r$ it follows from the second observation that $f(x) \geq r$, and so we have $f(x) \in [r,s] \subseteq (c,d)$. Thus we have $a \in U_s \setminus \overline{U}_r \subseteq f^{-1}(c,d)$, as claimed. Since $U_s \setminus \overline{U}_r$ is open, we can choose a basic open set V with $a \in V \subseteq U_s \setminus \overline{U}_r \subseteq f^{-1}(c,d)$. Since for every $a \in f^{-1}(c,d)$ there is a basic open set V with $a \in V \subseteq f^{-1}(c,d)$, it follows that $f^{-1}(c,d)$ is open, so that f is continuous, as required.

The Tietze Extension Theorem

4.29 Theorem: *(The Tietze Extension Theorem) Let X be a normal topological space, let $A \subseteq X$ be closed, and let $a, b \in \mathbb{R}$ with $a < b$.*

(1) Every continuous map $f : A \to [a, b]$ can be extended to a continuous map $g : X \to [a, b]$.
(2) Every continuous map $f : A \to (a, b)$ can be extended to a continuous map $g : X \to (a, b)$.

Proof: Note that since $[a, b]$ is homeomorphic to the interval $[-1, 1]$, we may replace $[a, b]$ by $[-1, 1]$. Suppose that $f : A \to [-1, 1]$ is continuous.

We begin with an observation. If $h : A \to [-r, r]$ is continuous, then $h^{-1}\big([-r, -\frac{r}{3}]\big)$ and $h^{-1}\big([\frac{r}{3}, r]\big)$ are disjoint closed sets in X, so by scaling and translating the map given by Urysohn's Lemma, we can construct a map $g : X \to \big[-\frac{r}{3}, \frac{r}{3}\big]$ with $g(x) = -\frac{r}{3}$ for all $x \in h^{-1}\big([-r, -\frac{r}{3}]\big)$ and $g(x) = \frac{r}{3}$ for all $x \in h^{-1}\big([\frac{r}{3}, r]\big)$. We then have $|g(x)| \leq \frac{r}{3}$ for all $x \in X$, and we have $|h(x) - g(x)| \leq \frac{2r}{3}$ for all $x \in A$.

Since $f : A \to [-1, 1]$ is continuous, by the above observation we can construct a continuous map $g_1 : X \to \big[-\frac{1}{3}, \frac{1}{3}\big]$ such that $|f(x) - g_1(x)| \leq \frac{2}{3}$ for all $x \in A$. Since $(f - g_1) : A \to \big[-\frac{2}{3}, \frac{2}{3}\big]$ is continuous, we can apply the obove observation again to construct a continuous map $g_2 : X \to \big[-\frac{2}{9}, \frac{2}{9}\big]$ such that $|f(x) - g_1(x) - g_2(x)| \leq \frac{4}{9}$ for all $x \in A$. Repeating this procedure, we construct maps $g_k : X \to \big[-\frac{2^{k-1}}{3^k}, \frac{2^{k-1}}{3^k}\big]$ such that $\big|f(x) - \sum_{k=1}^{n} g_k(x)\big| \leq \frac{2^n}{3^n}$ for all $x \in A$. Since $|g_k(x)| \leq \frac{2^{k-1}}{3^k}$ for all $x \in X$, the series $\sum_{k=1}^{\infty} g_k$ converges uniformly on X by the Weierstrass M-Test. Define $g(x) = \sum_{k=1}^{\infty} g_k(x)$ for all $x \in X$. Note that g is continuous by uniform convergence, note that for all $x \in X$ we have $|g(x)| \leq \sum_{k=1}^{\infty} |g_k(x)| \leq \sum_{k=1}^{\infty} \frac{2^{n-1}}{3^n} = 1$ so that $g : X \to [-1, 1]$, and note that for all $x \in A$, since $\big|f(x) - \sum_{k=1}^{n} g_k(x)\big| \leq \frac{2^n}{3^n}$ we have $f(x) = \sum_{k=1}^{\infty} g_k(x) = g(x)$, and so g extends f. This completes the proof of Part 1.

To prove Part 2, suppose that $f : A \to (a, b)$ is continuous. Note that f is also continuous as a map $f : A \to [a, b]$ so, by Part 1, we can extend f to a continuous map $h : X \to [a, b]$. Let $B = h^{-1}(a) \cup h^{-1}(b)$ and note that B is closed in X and B is disjoint from A. By Urysohn's Lemma, we can construct a continuous map $k : X \to [0, 1]$ with $k(x) = 0$ for all $x \in B$ and $k(x) = 1$ for all $x \in A$. Then $g = kh : X \to (a, b)$ is continuous on X with $g(x) = h(x) = f(x)$ for all $x \in A$.

Urysohn's Metrization Theorem

4.30 Exercise: The following theorem shows that $\mathbb{R}^\omega$ is metrizable, using the product topology. Show that $\mathbb{R}^\omega$ is not metrizable in the box topology, and show that when K is uncountable, $\mathbb{R}^K$ is not metrizable in the product topology.

4.31 Theorem: *Let X_k be metrizable for each $k \in \mathbb{Z}^+$. Then $\prod_{k=1}^{\infty} X_k$ is metrizable, in the product topology*

Proof: We outline a proof, and leave the details as an exercise. Verify that when d is a metric on a set X, the map $\overline{d} : X \times X \to \mathbb{R}$ given by $\overline{d}(x,y) = \min\big\{d(x,y), 1\big\}$, is another metric on X which induces the same topology as d. This metric $\overline{d}$ is called the **bounded metric** on X corresponding to d.

For each $k \in \mathbb{Z}^+$, let d_k be a metric which induces the topology on X_k, and let $\overline{d}_k$ be the corresponding bounded metric. For $x, y \in \prod_{k=1}^{\infty} X_k$, let

$$d(x,y) = \sup\left\{ \tfrac{\overline{d}_k(x_k,y_k)}{k} \,\middle|\, k \in \mathbb{Z}^+ \right\}.$$

Verify that d is a metric on $\prod_{k=1}^{\infty} X_k$ which induces the product topology.

4.32 Theorem: *(Urysohn's Metrization Theorem) Every regular space with a countable basis for its topology is metrizable.*

Proof: Let X be a regular topological space with a countable basis $\mathcal{B} = \{B_1, B_2, B_3, \cdots\}$ for its topology. Note that X is normal, by Theorem 4.25. We shall construct a sequence $(f_n)_{n\geq 1}$ of continuous functions $f_n : X \to [0,1]$ with the property that for every $a \in X$ and for every open set U in X with $a \in U$, there exists $n \in \mathbb{Z}^+$ such that $f_n(a) = 1$ and $f_n(x) = 0$ for all $x \in U^c = X \setminus U$. Let $\mathcal{S} = \big\{(k,\ell) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \,\big|\, \overline{B}_k \subseteq B_\ell\big\}$, and note that $\mathcal{S}$ is countable. For each pair $(k,\ell) \in \mathcal{S}$, since $\overline{B}_k \subseteq B_\ell$ we have $\overline{B}_k \cap B_\ell^c = \emptyset$ so, that by Urysohn's Lemma, we can choose a continuous map $g_{k,\ell} : X \to [0,1]$ with $g_{k,\ell}(x) = 1$ for all $x \in B_k$ and $g_{k,\ell}(x) = 0$ for all $x \in B_\ell^c$. Since the set $\big\{g_{k,\ell} \big| (k,\ell) \in \mathcal{S}\big\}$ is countable, we can list the elements as $\big\{g_{k,\ell} \big| (k,\ell) \in \mathcal{S}\big\} = \big\{f_1, f_2, f_3, \cdots\big\}$ where $(f_n)_{n\geq 1}$ is a sequence of functions $f_n : X \to [0,1]$. Given an open set U in X and given $a \in U$, we can choose a basic open set B_ℓ with $a \in B_\ell \subseteq U$ then, since X is regular, we can choose an open set V with $a \in V \subseteq \overline{V} \subseteq B_\ell$, then we can choose another basic open set B_k with $a \in B_k \subseteq V$, and then we have $a \in B_k \subseteq \overline{B_k} \subseteq \overline{V} \subseteq B_\ell \subseteq U$, and finally we can choose $n \in \mathbb{Z}^+$ so that $f_n = g_{k,\ell}$ to get $f(x) = 1$ for all $x \in B_k$ (so that $f(a) = 1$) and $f(x) = 0$ for all $x \in B_\ell^c$ (so that $f(x) = 0$ for all $x \in U^c$).

Let $f : X \to [0,1]^\omega \subseteq \mathbb{R}^\omega$ be the function given by $f(x) = \big(f_1(x), f_2(x), \cdots\big)$. Note that f is continuous because $\mathbb{R}^\omega$ is using the product topology and each f_k is continuous. Note that f is injective because given $a, b \in X$ with $a \neq b$, we can choose disjoint open sets U and V in X with $a \in U$ and $b \in V$, then we can choose $n \in \mathbb{Z}^+$ such that $f_n(a) = 1$ and $f_n(x) = 0$ for all $x \in U^c$ so that $f_n(b) = 0$ and hence, since $f_n(a) \neq f_n(b)$ we have $f(a) \neq f(b)$. By restricting the codomain, we obtain a bijective continuous map $f : X \to f(X) \subseteq [0,1]^\omega \subseteq \mathbb{R}^\omega$.

To complete the proof it suffices to show that $g = f^{-1} : f(X) \to X$ is continuous, when $f(X) \subseteq \mathbb{R}^\omega$ uses the subspace topology. It suffices to show that $g^{-1}(U)$ is open in $f(X)$ for every open set U in X. Let U be open in X and note that $g^{-1}(U) = f(U)$. We must show that $f(U)$ is open in $f(X)$. Let $b \in f(U)$. Let $a \in U$ with $f(a) = b$. Choose $n \in \mathbb{Z}^+$ such that $f_n(a) = 1$ and $f_n(x) = 0$ for all $x \notin U$. Let $p_n : \mathbb{R}^\omega \to \mathbb{R}$ be the n^{th} projection map. Let $V = p_n^{-1}\big((0,\infty)\big) = \big\{y \in \mathbb{R}^\omega \big| y_n > 0\big\}$ and note that V is open in $\mathbb{R}^\omega$. Let $W = V \cap f(X)$, which is open in $f(X)$. Note that $b \in W$ because $b_n = p_n(b) = p_n(f(a)) = f_n(a) = 1 > 0$. We claim that $W \subseteq f(U)$. Let $y \in W = V \cap f(X)$. Since $y \in f(X)$ we can choose $x \in X$ such that $y = f(x)$. Since $f(x) = y \in V$ we have $f_n(x) = y_n > 0$. Since $f_n(x) > 0$ we have $x \in U$ (since $f_n(x) = 0$ for all $x \notin U$). Thus $y = f(x) \in f(U)$ so that $W \subseteq f(U)$, as claimed. For each $b \in f(U)$ we have found an open set W in $f(X)$ with $b \in W \subseteq f(X)$, which shows that $f(X)$ is open, as required.