Chapter 3. Connected, Path-Connected, and Compact Spaces

Connectedness and Connected Components

3.1 Definition: Let X be a topological space. We say that two subsets $A, B \subseteq X$ **separate** X when $A \neq \emptyset$, $B \neq \emptyset$, $A \cap B = \emptyset$ and $A \cup B = X$. We say that X is **connected** when there do not exist two open sets in X which separate X. Equivalently (as you can verify) X is connected when $\emptyset$ and X are the only two subsets of X which are both open and closed in X. We say that X is **disconnected** when it is not connected.

3.2 Exercise: Prove that the connected subspaces of $\mathbb{R}$ are the intervals (including $\emptyset$, 1-point sets, and $\mathbb{R}$), and the nonempty connected subspaces of $\mathbb{Q}$ are the 1-point sets.

3.3 Theorem: *The image of a connected space under a continuous map is connected. In particular, if two spaces are homeomorphic and one is connected, then so is the other.*

Proof: Let $f : X \to Y$ be a continuous map between topological spaces. Note that, by Theorem 1.35, $f : X \to f(X)$ is also continuous. Suppose that $f(X)$ is disconnected. Choose disjoint nonempty open sets A and B in $f(X)$ with $A \cup B = f(X)$. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint nonempty open sets in X with $f^{-1}(A) \cup f^{-1}(B) = X$, so X is disconnected.

3.4 Lemma: *Let X be a subspace of Y. Suppose that Y is disconnected and that A and B are open sets in Y which separate Y. If X is connected, then either $X \subseteq A$ or $X \subseteq B$.*

Proof: Suppose that X is connected. Note that $A \cap X$ and $B \cap X$ are disjoint open sets in X. If both of the sets $A \cap X$ and $B \cap X$ were nonempty, then they would be open sets in X which separate X. Since X is connected, this is not possible, so either $A \cap X = \emptyset$ or $B \cap X = \emptyset$. If $A \cap X = \emptyset$ then we have $X = X \cap Y = X \cap (A \cup B) = (X \cap A) \cup (X \cap B) = X \cap B$ so that $X \subseteq B$. Similarly, if $B \cap X = \emptyset$ then $X \subseteq A$.

3.5 Theorem: *Let $X = \bigcup_{k \in K} A_k$. If each A_k is a connected subspace of X and $\bigcap_{k \in K} A_k \neq \emptyset$ then X is connected.*

Proof: Suppose that each A_k is connected with $p \in \bigcap_{k \in K} A_k$. Suppose, for a contradiction, that $X = \bigcup_{k \in K} A_k$ is disconnected. Choose open sets U and V in X which separate X. Note that p lies either in U or in V (but not both), say $p \in U$ (and $p \notin V$). Let $k \in K$ be arbitrary. Since A_k is connected, by the above lemma, either $A_k \subseteq U$, or $A_k \subseteq V$. Since $p \in A_k$ and $p \notin V$, we do not have $A_k \subseteq V$ so we must have $A_k \subseteq U$. Since $k \in K$ was arbitrary, we have $A_k \subseteq U$ for every $k \in K$, and hence $X = \bigcup_{k \in K} A_k \subseteq U$. But this contradicts the fact that U and V separate X, giving the desired contradiction.

3.6 Lemma: *Let X be a subspace of Y and let A and B be subsets of X which separate X. Then A and B are open in X (so that X is disconnected) if and only if $\overline{A} \cap B = \emptyset$ and $\overline{B} \cap A = \emptyset$ (where $\overline{A}$ and $\overline{B}$ are the closures in Y).*

Proof: Suppose that A and B are open in X Note that $A = B^c = X \setminus B$ is closed in X so we have $A = \mathrm{Cl}_X(A) = \overline{A} \cap X = \overline{A} \cap (A \cup B) = (\overline{A} \cap A) \cup (\overline{A} \cap B) = A \cup (\overline{A} \cap B)$ and hence $\overline{A} \cap B = \emptyset$. Similarly $\overline{B} \cap A = \emptyset$.

Suppose, conversely, that there exist disjoint nonempty sets $A, B \subseteq X$ with $A \cup B = X$ such that $\overline{A} \cap B = \emptyset$ and $\overline{B} \cap A = \emptyset$. Since $\overline{A} \cap B = \emptyset$, we have $\overline{A} \cap X = A$ so that $\mathrm{Cl}_X(A) = \overline{A} \cap X = A$ and hence A is closed. Similarly B is closed. Since X is the disjoint union of A and B, it follows that A and B are both open in X hence X is not connected.

3.7 Theorem: *Let X be a topological space and let $A \subseteq X$. Suppose that $A \subseteq B \subseteq \overline{A}$. If A is connected, as a subspace of X, then so is B.*

Proof: Suppose that A is connected, and suppose, for a contradiction, that B is disconnected. Let C and D be open sets in B which separate B. By Lemma 3.5, we have $\overline{C} \cap D = \emptyset$ and $\overline{D} \cap C = \emptyset$. Since A is connected, by Lemma 3.3, either $A \subseteq C$ or $A \subseteq D$. Say $A \subseteq C$. Then we have $B \subseteq \overline{A} \subseteq \overline{C}$. Since $D \subseteq B \subseteq \overline{C}$ and $\overline{C} \cap D = \emptyset$, we have $D = \emptyset$, which contradicts the fact that C and D separate B.

3.8 Theorem: *The cartesian product of two connected spaces is connected.*

Proof: Let X and Y be connected topological spaces. If $X = \emptyset$ or $Y = \emptyset$ then $X \times Y = \emptyset$, which is connected. Suppose that $X \neq \emptyset$ and $Y \neq \emptyset$. Choose $a \in X$ and $b \in Y$. Note that $X \times \{b\}$ is connected, since it is homeomorphic to X. Likewise, for each $x \in X$, the subspace $\{x\} \times Y$ is connected since it is homeomorhic to Y. Since $X \times \{b\}$ and $\{x\} \times Y$ are connected with $(x, b) \in (X \times \{b\}) \cap (\{x\} \times Y)$ it follows, from Theorem 3.5, that $(X \times \{b\}) \cup (\{x\} \times Y)$ is connected. Since $(X \times \{b\}) \cup (\{x\} \times Y)$ is connected for every $x \in X$ and $(a, b) \in \bigcap_{x \in X} (X \times \{b\}) \cup (\{x\} \times Y)$ it follows, again from Theorem 3.5, that $X \times Y = \bigcup_{x \in X} (X \times \{b\}) \cup (\{x\} \times Y)$ is connected.

3.9 Theorem: *The cartesian product of an arbitrary set of connected spaces is connected using the product topology.*

Proof: Let X_k be a connected topological space for each $k \in K$. If $X_k = \emptyset$ for some $k \in K$ then $\prod_{k \in K} X_k = \emptyset$. Suppose that $X_k \neq \emptyset$ for all $k \in K$. For each $k \in K$ choose $a_k \in X_k$ and let a be the element in $\prod_{k \in K} X_k$ given by $a(k) = a_k$ for all $k \in K$. Let $\mathcal{F}$ be the set of all finite subsets of K. For each $J \in \mathcal{F}$, let $Y_J = \left\{ y \in \prod_{k \in K} X_k \,\middle|\, y_k = a_k \text{ for all } k \notin J \right\}$, using the subspace topology. We claim that $Y_J \cong \prod_{j \in J} X_j$. Define $f : Y_J \to \prod_{j \in J} X_j$ by $f(y)(j) = y_j$. This map is continuous because given U_j open in X_j for each $j \in J$, so that $\prod_{j \in J} U_j$ is a basic open set in $\prod_{j \in J} X_j$, and letting $U_k = X_k$ for all $k \in K \setminus J$, we have $f^{-1}\big(\prod_{j \in J} U_j\big) = \big\{ y \in Y_J \big| y_j \in U_j \text{ for all } j \in J \big\} = \big\{ y \in Y_J \big| \, y_k \in U_k \text{ for all } k \in K \big\} = \big(\prod_{k \in K} U_k\big) \cap Y_J$, which is a basic open set in Y_J (using the subspace topology). The inverse of f is the map $g = f^{-1} : \prod_{j \in J} X_j \to Y_J$ by $g(x)(k) = \left\{ \begin{matrix} x_k \text{ if } k \in J \\ a_k \text{ if } k \notin J \end{matrix} \right\}$. This map is continuous because given $I \in \mathcal{F}$ and given open sets U_k in X_k with $U_k = X_k$ for all $k \notin I$, so that the set $\big(\prod_{k \in K} U_k\big) \cap Y_J$ is a basic open set in Y_J, we have $g^{-1}\big(\big(\prod_{k \in K} U_k\big) \cap Y_J\big) = \big\{ x \in \prod_{j \in J} X_j \big| x_k \in U_k \text{ for all } k \in J \text{ and } a_k \in U_k \text{ for all } k \notin J \big\} = \big\{ x \in \prod_{j \in J} X_j \big| x_k \in U_k \text{ for all } k \in J \cap I \big\} = \prod_{j \in J} V_j$ where $V_j = U_j$ for $j \in J \cap I$ and $V_j = X_j$ for $j \in J \setminus I$, and this is a basic open set in $\prod_{j \in J} X_j$. This we have $Y_J \cong \prod_{j \in J} X_j$, as claimed.

Since J is finite, and each X_j is connected, the space $\prod_{j \in J} X_j$ is connected by the previous theorem (and by induction), and hence $Y_J = g\big(\prod_{j \in J} X_j\big)$ is connected by Theorem. 3.3. Since Y_J is connected for every $J \in \mathcal{F}$, and since $a \in Y_J$ for all $J \in \mathcal{F}$, it follows from Theorem 3.5 that $\bigcup_{J \in \mathcal{F}} Y_J$ is connected. Finally, we note that $\overline{\bigcup_{J \in \mathcal{F}} Y_J} = \prod_{k \in K} X_k$: indeed, given $I \in \mathcal{F}$ and given open sets $U_k \subseteq X_k$ with $U_k = X_k$ for all $k \notin I$, so that $\prod_{k \in K} U_k$ is a basic open set in $\prod_{k \in K} X_k$, we have $\emptyset \neq \prod_{k \in K} U_k \cap Y_I \subseteq \prod_{k \in K} X_k \cap \bigcup_{J \in \mathcal{F}} Y_J$. Since $\bigcup_{J \in \mathcal{F}} Y_J$ is connected, and $\prod_{k \in K} X_k = \overline{\bigcup_{J \in \mathcal{F}} Y_J}$, it follows that $\prod_{k \in K} X_k$ is connected by Theorem 3.7.

3.10 Example: The result of the above theorem does not necessarily hold when $\prod_{k\in K} X_k$ uses the box topology. For example you can verify that in the space $\mathbb{R}^\omega$ using the box topology, the sets $U = \left\{x \in \mathbb{R}^\omega \middle| \, \|x\|_\infty < \infty\right\}$ and $V = \left\{x \in \mathbb{R}^\infty \middle| \, \|x\|_\infty = \infty\right\}$ are open sets which separate $\prod_{k\in K} X_k$.

3.11 Definition: Let X be a topological space. Define a relation $\sim$ on X by setting $x \sim y$ if and only if there exists a connected subspace of X which contains both x and y. We note that $\sim$ is an equivalence relation on X: indeed, given $x, y, z \in X$, we have $x \sim x$ because $\{x\}$ is connected, and if $x \sim y$ then clearly $y \sim x$, and if $x \sim y$ and $y \sim z$ then we can choose connected spaces $A, B \subseteq X$ with $x, y \in A$ and $y, z \in B$ and then, by Theorem 3.5, since $y \in A \cap B$ it follows that $A \cup B$ is connected, and we have $x, z \in A \cup B$. The equivalence classes under this equivalence relation are called the **connected components** of X. Note that if X is connected then the only connected component of X is X itself.

3.12 Theorem: *The connected components of a topological space X are connected, and every non-empty connected subspace of X is contained in exactly one of the connected components.*

Proof: We claim that every nonempty connected subspace of X is contained in exactly one connected component. Let A be a nonempty connected subspace of X. Let $a \in A$. Let C be the equivalence class of a, that is $C = [a] = \{x \in X \,|\, x \sim a\}$ and note that $A \cap C \neq \emptyset$ since $a \in A \cap C$. Let D be any equivalence class with $A \cap D \neq \emptyset$. Choose $b \in A \cap D$. Since A is connected with $a \in A$ and $b \in A$ we have $a \sim b$ so that $C = [a] = [b] = D$. Thus A intersects with exactly one connected component, namely C. Since the connected components (being equivalence classes) cover X, it follows that $A \subseteq C$.

We claim that each connected component of X is connected. Let C be a connected component and let $a \in C$ so that $C = [a] = \{x \in X \,|\, x \sim a\}$. For each $x \in C$, since $x \sim a$ we can choose a connected set A_x in X with $a, x \in A$. By the previous claim, since A_x is connected with $a \in A_x \cap C$, it follows that $A_x \subseteq C$. Since $x \in A_x \subseteq C$ for all $x \in C$, we have $C = \bigcup_{x\in C} A_x$. This is connected by Theorem 3.5, since each A_x is connected and $a \in \bigcap_{x\in C} A_x$.

3.13 Note: The connected components of a topological space are closed: indeed if C is a connected component of X then, by the above theorem, C is a maximal connected set in X, and by Theorem 3.7, $\overline{C}$ is a connected connected set with $C \subseteq \overline{C}$, and hence $\overline{C} = C$.

3.14 Example: Since $\mathbb{R}$ is connected, it has only one connected component, namely $\mathbb{R}$. The one-point sets are the connected components of $\mathbb{Q}$.

Path-Connectedness and Path-Components

3.15 Definition: Let X be a topological space and let $a, b \in X$. A (continuous) **path** from a to b in X is a continuous map $\alpha : [0, 1] \to X$ with $\alpha(0) = a$ and $\alpha(1) = b$. A **loop** at a in X is a path from a to a in X. We say that X is **path-connected** when for every $a, b \in X$ there exists a path from a to b in X.

3.16 Theorem: *The image of a path-connected space under a continuous map is path-connected. In particular, if $X \cong Y$ and X is path-connected, then so is Y.*

Proof: Let $f : X \to Y$ be continuous and suppose X is path connected. Let $c, d \in f(X)$. Choose $a, b \in X$ such that $f(a) = c$ and $f(b) = d$. Let α be a path in X from a to b. Then $\beta = f \circ \alpha$ is a path in Y from c to d.

3.17 Theorem: *Every path-connected topological space is connected.*

Proof: Let X be a path-connected topological space. Suppose, for a contradiction, that X is not connected. Choose nonempty disjoint open sets U and V in X which separate X (meaning that $X = U \cup V$). Choose $a \in U$ and $b \in V$. Let $\alpha : [0,1] \to X$ be a path from a to b in X. Then $\alpha^{-1}(U)$ and $\alpha^{-1}(V)$ are nonempty disjoint open sets in $[0,1]$ which separate $[0,1]$. This is not possible since $[0,1]$ is connected.

3.18 Example: Every **convex set** in a normed linear space is path-connected, hence connected. Indeed if X is a convex set then, given $a, b \in X$, the map $\alpha : [0,1] \to X$ given by $\alpha(t) = a + t(b-a)$ is a path from a to b in X (α takes values in X because X is convex).

3.19 Theorem: *Let X be a topological space. The relation $\sim$ on X, given by $a \sim b$ when there exists a path from a to b in X, is an equivalence relation on X, which we call* **path-equivalence**.

Proof: We have $a \sim a$ because the constant path $\kappa = \kappa_a : [0,1] \to X$, given by $\kappa(t) = a$ for all t, is a path from a to a in X. Note that if $a \sim b$ then $b \sim a$: indeed if α is a path from a to b in X then the map $\beta = \alpha^{-1} : [0,1] \to X$ given by $\beta(t) = \alpha(1-t)$ is a path from b to a in X. Finally, note that if $a \sim b$ and $b \sim c$ then $a \sim c$: indeed if α is a path from a to b in X and β is a path from b to c in X then the map $\gamma = \alpha\beta : [0,1] \to X$ given by $\gamma(t) = \alpha(2t)$ when $0 \leq t \leq \frac{1}{2}$, and by $\gamma(t) = \beta(2t-1)$ when $\frac{1}{2} \leq t \leq 1$, is a path from a to c in X (γ is continuous by Theorem 1.36).

3.20 Definition: Let X be a topological space. The equivalence classes under the path-equivalence relation on X are called the **path-components** of X.

3.21 Theorem: *The path components of a topological space X are the maximal path-connected subspaces of X: indeed each path-component of X is path-connected, and every nonempty path-connected subset of X is contained in exactly one of the path-components.*

Solution: Let P be a path-component of X, say $a \in P$ so that $P = [a] = \{x \in X \,|\, x \sim a\}$. Then P is path-connected because if $b, c \in P$ then we have $b \sim a$ and $c \sim a$, and hence $b \sim c$. Now let S be any path-connected subset of X. Suppose that S intersects two path-components, say P and Q, of X. Choose $p \in P \cap S$ and $q \in Q \cap S$. Since $p, q \in S$ and S is path-connected, we have $p \sim q$. Since $p \sim q$ we have $P = [p] = [q] = Q$.

3.22 Note: In a topological space X, since each path-component is path-connected, hence connected, it is contained in one of the connected components of X. It follows that each connected component of X is the (disjoint) union of the path-components which it contains.

3.23 Exercise: Let $A = \{(x, \sin\frac{1}{x}) \,|\, x > 0\}$. The closure $\overline{A}$ of A in $\mathbb{R}^2$ is called the **topologist's sine curve**. Note that $\overline{A} = A \cup B$ where $B = \{(0, y) \,|\, y \in [-1,1]\}$. Show that $\overline{A}$ is connected but not path-connected, and the sets A and B are the path-components.

Compactness

3.24 Definition: Let X be a topological space. For a set $\mathcal{S}$ of subsets of X, we say that $\mathcal{S}$ **covers** X when $\bigcup \mathcal{S} = X$. An **open cover** of X is a set $\mathcal{S}$ of open sets which covers X. When $\mathcal{S}$ is an open cover of X, a **subcover** of $\mathcal{S}$ is a subset $\mathcal{R} \subseteq \mathcal{S}$ with $\bigcup \mathcal{R} = X$. We say that X is **compact** when every open cover of X has a finite subcover. Equivalently, X is compact when it has the property that if K is a set, and U_k is an open set in X for each $k \in K$, and $\bigcup_{k \in K} U_k = X$, then there is a finite subset $L \subseteq K$ such that $\bigcup_{k \in L} U_k = X$.

3.25 Theorem: *(The Heine-Borel Theorem) A subspace of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Proof: We omit the proof. This theorem is proven in a real analysis course.

3.26 Theorem: *The image of a compact space under a continuous map is compact. In particular, if $X \cong Y$ and X is compact, then so is Y.*

Proof: Let $f : X \to Y$ be continuous and suppose X is compact. By restricting the codomain, the map $f : X \to f(X)$ is also continuous. Let $\mathcal{T}$ be an open cover of $f(X)$. Then the set $\mathcal{S} = \left\{ f^{-1}(V) \,\middle|\, V \in \mathcal{T} \right\}$ is an open cover of X. Since X is compact, $\mathcal{S}$ has a finite subcover, so we can choose $V_1, V_2, \cdots, V_n \in \mathcal{T}$ such that $X = \bigcup_{k=1}^n f^{-1}(V_k)$. Then $f(X) = \bigcup_{k=1}^n V_k$ so that $\{V_1, V_2, \cdots, V_n\}$ is a finite subcover of $\mathcal{T}$.

3.27 Theorem: *Let X be a subspace of Y. Then X is compact if and only if for every set $\mathcal{T}$ of open sets in Y with $X \subseteq \bigcup \mathcal{T}$ there exists a finite set $\mathcal{Q} \subseteq \mathcal{T}$ such that $X \subseteq \bigcup \mathcal{Q}$.*

Proof: Suppose that X is compact. Let $\mathcal{T}$ be a set of open sets in Y such that $X \subseteq \bigcup \mathcal{T}$. Let $\mathcal{S} = \{ V \cap X \,|\, V \in \mathcal{T} \}$. Then $\mathcal{S}$ is an open cover of X. Since X is compact, we can choose $V_1, \cdots, V_n \in \mathcal{T}$ such that $X = \bigcup_{k=1}^n (V_k \cap X)$. Since $X = \bigcup_{k=1}^n (V_k \cap X) = \left(\bigcup_{k=1}^n V_k \right) \cap X$ we have $X \subseteq \bigcup_{k=1}^n V_k$.

Suppose, conversely, that for every set $\mathcal{T}$ of open sets in Y with $X \subseteq \bigcup \mathcal{T}$ there is a finite subset $\mathcal{Q} \subseteq \mathcal{T}$ such that $X \subseteq \bigcup \mathcal{Q}$. Let $\mathcal{S}$ be a set of open sets in X with $X = \bigcup \mathcal{S}$. For each $U \in \mathcal{S}$, since U is open in X we can choose an open set V_U in Y such that $U = V_U \cap X$. Let $\mathcal{T} = \left\{ V_U \,\middle|\, U \in \mathcal{S} \right\}$. Choose a finite subset $\left\{ V_{U_1}, V_{U_2}, \cdots, V_{U_n} \right\}$ of $\mathcal{T}$ such that $X \subseteq \bigcup_{k=1}^n V_{U_k}$. Then we have $\bigcup_{k=1}^n U_k = \bigcup_{k=1}^n (V_{U_k} \cap X) = \left(\bigcup_{k=1}^n V_{U_k} \right) \cap X = X$ so that $\{U_1, U_2, \cdots, U_n\}$ is a finite subcover of $\mathcal{S}$.

3.28 Theorem: *Every closed subspace of a compact space is compact.*

Proof: Let X be a closed subspace of the compact space Y. Let $\mathcal{T}$ be a set of open sets in Y such that $X \subseteq \bigcup \mathcal{T}$. Then $\mathcal{T} \cup \{X^c\}$ is an open cover of Y, where $X^c = Y \setminus X$. Since Y is compact, we can choose $V_1, V_2, \cdots, V_n \in \mathcal{T}$ such that $V_1 \cup V_2 \cup \cdots \cup V_n \cup X^c = Y$. It follows that $X \subseteq V_1 \cup V_2 \cup \cdots \cup V_n$. Thus X is compact by Theorem 3.27.

3.29 Theorem: *Every compact subspace of a Hausdorff space is closed.*

Proof: Let X be a compact subspace of the Hausdorff space Y. To show that X is closed in Y, we show that for every $b \in X^c = Y \setminus X$ there exists an open set V in Y with $b \in V \subseteq X^c$. Let $b \in X^c$. For each $a \in X$, since Y is Hausdorff we can choose disjoint open sets U_a and V_a in Y with $a \in U_a$ and $b \in V_a$. Since $X \subseteq \bigcup_{a \in X} U_a$ and X is compact, by Theorem 3.27 we can choose $a_1, a_2, \cdots, a_n \in X$ such that $X \subseteq \bigcup_{k=1}^n U_{a_k}$. Let $U = \bigcup_{k=1}^n U_{a_k}$ and $V = \bigcap_{k=1}^n V_{a_k}$. Note that U and V are open in Y with $X \subseteq U$ and $b \in V$. Also note that U and V are disjoint because, for $y \in Y$, if $y \in U = \bigcup_{k=1}^n U_{a_k}$ then we can choose an index ℓ such that $y \in U_{a_\ell}$, and then $y \notin V_{a_\ell}$ and hence $y \notin \bigcap_{k=1}^n V_{a_k} = V$. Since $X \subseteq U$ and $b \in V$ and $U \cap V = \emptyset$, we have found an open set V in Y with $b \in V \subseteq X^c$, as required.

3.30 Theorem: *If X is compact and Y is Hausdorff and $f : X \to Y$ is continuous and bijective, then the inverse of f is also continuous, so that f is a homeomorphism.*

Proof: Let X be compact, let Y be Hausdorff, let $f : X \to Y$ be continuous and bijective, and let $g = f^{-1} : Y \to X$. Let U be an open set in X. Then U^c is closed in X, where $U^c = X \setminus U$. By Theorem 3.26, since U^c is closed in X and X is compact, it follows that U^c is compact. By Theorem 3.25, since U^c is compact and f is continuous, it follows that $f(U^c)$ is compact. Since f is bijective, we have $f(U^c) = f(U)^c = Y \setminus f(U)$. By Theorem 3.27, since $f(U)^c$ is compact and Y is Hausdorff, it follows that $f(U)^c$ is closed in Y, and hence $g^{-1}(U) = f(U)$ is open in Y. Thus g is continuous, as required.

3.31 Example: Show that no two of the spaces $(0,1)$, $(0,1]$ and $[0,1]$ are homeomorphic.

Solution: Since $[0,1]$ is compact while $(0,1)$ and $(0,1]$ are not, we see that $[0,1]$ cannot be homeomorphic either to $(0,1)$ or to $(0,1]$. Also note that $(0,1] \setminus \{1\}$ is connected while $(0,1) \setminus \{p\}$ is not connected for any $p \in (0,1)$, and so it follows that $(0,1]$ cannot be homeomorphic to $(0,1)$. Indeed if $f : (0,1] \to (0,1)$ was a homeomorphism with $p = f(0)$ then the map $f : (0,1] \setminus \{0\} \to (0,1) \setminus \{p\}$ would also be a homeomorphism.

3.32 Example: Show that no two of the spaces $\mathbb{R}^1$, $\mathbb{R}^2$, $\mathbb{S}^1$ and $\mathbb{S}^2$ are homeomorphic.

Solution: Since $\mathbb{S}^1$ and $\mathbb{S}^2$ are compact while $\mathbb{R}^1$ and $\mathbb{R}^2$ are not, neither $\mathbb{S}^1$ nor $\mathbb{S}^2$ can be homeomorphic to either $\mathbb{R}^1$ or $\mathbb{R}^2$. Since $\mathbb{R}^2 \setminus \{(0,0)\}$ is connected while $\mathbb{R} \setminus \{x\}$ is not connected for any $x \in \mathbb{R}$, it follows that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^1$. Since $\mathbb{S}^2 \setminus \{(0,0,1)\} \cong \mathbb{R}^2$, and $\mathbb{S}^1 \setminus \{x\} \cong \mathbb{R}^1$ for any $x \in \mathbb{S}^1$ (under the composite of a rotation with the stereographic projection), and since $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^1$, it follows that $\mathbb{S}^2$ is not homeomorphic to $\mathbb{S}^1$.

3.33 Theorem: *Let X and Y be topological spaces.*

(1) If X and Y are connected then so is $X \times Y$.
(2) If X and Y are path-connected then so is $X \times Y$.
(3) If X and Y are compact then so is $X \times Y$.

Proof: The proof is left as an exercise.

3.34 Theorem: *Let $\sim$ be an equivalence relation on a topological space X.*

(1) If X is connected then so is $X/\sim$.
(2) If X is path-connected then so is $X/\sim$.
(3) If X is compact then so is $X/\sim$.

Proof: The proof is left as an exercise.

3.35 Definition: Let X be an ordered set. For $A \subseteq X$ and $b \in X$, we say that b is an **upper bound** for A in X when $b \geq x$ for every $x \in A$, and we say that b is the **supremum** (or the **least upper bound**) of A in X when b is an upper bound for A in X and $b \leq c$ for every upper bound c of A in X. Note that when A has a supremum in X, the supremum is unique, and we denote it by $\sup X$. We say that X has the **supremum property** (or the **least upper bound property) when every nonempty subset of X which has an upper bound in X also has a supremum in X.**

3.36 Theorem: *Let X be an ordered set with the supremum property. Let $a, b \in X$ with $a < b$. Then the interval $[a,b]$ is compact.*

Proof: We leave the proof as an exercise.

3.37 Theorem: *Let X be a topological space. Then X is compact if and only if X has the* **finite intersection property on closed sets***: for every set T of closed sets in X, if every finite subset of T has non-empty intersection, then T has non-empty intersection.*

Proof: Suppose that X is compact. Let T be a set of closed sets in X. Suppose that T has empty intersection, that is suppose $\bigcap_{A\in T} A = \emptyset$. Then $\bigcup_{A\in T} A^c = X$ so the set $S = \{A^c \big| A \in T\}$ is an open cover for X. Since X is compact, we can choose a finite subcover, say $\{{A_1}^c, \cdots, {A_n}^c\}$ of S for X. Then we have $A_1 \cap A_2 \cap \cdots \cap A_n = \emptyset$, showing that some finite subset of T has empty intersection.

Suppose, conversely, that X has the finite intersection property on closed sets. Let S be an open cover of X. Let $T = \{U^c \,\big|\, U \in S\}$. Since $\bigcup S = X$ we have $\bigcap T = \left(\bigcup S\right)^c = \emptyset$. Since X has the finite intersection on closed sets, there exists a finite subset of T with empty intersection. so we can choose $U_1, U_2, \cdots U_n \in S$ such that ${U_1}^c \cap \cdots \cap {U_n}^c = \emptyset$. It follows that $U_1 \cup \cdots \cup U_n = X$, so S has a finite subcover.

3.38 Theorem: *(Tychanoff's Theorem) The product of any indexed set of compact spaces is compact, using the product topology.*

Proof: Let X_k be compact for each $k \in K$. We shall prove that $\prod X_k$ has the finite intersection property on closed sets. Let T be a set of closed sets in $\prod X_k$ such that every finite subset of T has non-empty intersection. We need to show that $\bigcap T \neq \emptyset$. By Zorn's Lemma, we can choose a maximal set S of subsets of $\prod X_k$ with $T \subseteq S$ such that every finite subset of S has non-empty intersection (let $\mathcal{R}$ be the set of all such sets S and note that for every chain $\mathcal{C}$ in $\mathcal{R}$ we have $\bigcup \mathcal{C} \in \mathcal{R}$). Note that the maximality of S implies that S is closed under finite intersection (since if $A_1, \cdots, A_n \in S$ then every intersection of a finite subset of $S \cup \{A_1 \cap \cdots \cap A_n\}$ is also an intersection of a finite subset of S).

We shall show that $\bigcap \{\overline{A} \,\big|\, A \in S\} \neq \emptyset$, hence $\bigcap T \neq \emptyset$ since if $A \in T$ then $A = \overline{A} \in S$. Let $k \in K$. Note that finite subsets of $\{p_k(A) \,\big|\, A \in S\}$ have non-empty intersection (because if $A_1, \cdots, A_n \in S$ then $p_k(A_1) \cap \cdots \cap p_k(A_n) = p_k(A_1 \cap \cdots \cap A_n) \neq \emptyset$), and hence finite subsets of $\{\overline{p_k(A)} \,\big|\, A \in S\}$ also have nonempty intersection. Since X_k is compact, so X_k has the finite intersection property on closed sets, it follows that $\bigcap \{\overline{p_k(A)} \big| A \in S\} \neq \emptyset$, so we can choose $a_k \in X_k$ such that $a_k \in \overline{p_k(A)}$ for every $A \in S$. We do this for each $k \in K$, that is for each $k \in K$ we choose $a_k \in X_k$ with $a_k \in \overline{p_k(A)}$ for every $A \in S$, then we let $a = (a_k)_{k\in K} \in \prod_{k\in K} X_k$.

We claim that $a \in \overline{A}$ for every $A \in S$. Let $k \in K$. Let U_k be an open set in X_k with $a_k \in U_k$. Then for every $A \in S$, we have $a_k \in \overline{p_k(A)} \cap U_k$ so that $\overline{p_k(A)} \cap U_k \neq \emptyset$ hence $p_k(A) \cap U_k \neq \emptyset$ (if we had $p_k(A) \cap U_k = \emptyset$ then $p_k(A) \subseteq {U_k}^c$ hence $\overline{p_k(A)} \subseteq {U_k}^c$ so that $\overline{p_k(A)} \cap U_k = \emptyset$). For each $A \in S$, since $p_k(A) \cap U_k \neq \emptyset$, we can choose $b \in A$ such that $p_k(b) \in U_k$, that is $b \in {p_k}^{-1}(U_k)$, and hence ${p_k}^{-1}(U_k) \cap A \neq \emptyset$. Since S is closed under finite intersection and ${p_k}^{-1}(U_k) \cap A \neq \emptyset$ for every $A \in S$, the maximality of S implies that ${p_k}^{-1}(U_k) \in S$. Let V be any basic open set in $\prod X_k$ with $a \in V$, say $V = \prod U_k$ where each $U_k \subseteq X_k$ is open with $a_k \in U_k$, and with $U_k = X_k$ for all $k \in F$ where F is a finite subset of K. Since ${p_k}^{-1}(U_k) \in S$ for every $k \in K$ and S is closed under finite intersection, we have

$$V = \{(x_k)_{k\in K} \,\big|\, x_k \in U_k \text{ for all } k \in F\} = \textstyle\bigcap_{k\in F} {p_k}^{-1}(U_k) \in S.$$

Since $V \in S$ and every finite subset of S has non-empty intersection, we have $A \cap V \neq \emptyset$ for all $A \in S$. Given $A \in S$, since $A \cap V \neq \emptyset$ for every basic open set V in $\prod X_k$ with $a \in V$, it follows that $a \in \overline{A}$. Thus $a \in \overline{A}$ for all $A \in S$, so $\bigcap \{\overline{A} \,\big|\, A \in S\} \neq \emptyset$, as required.