Chapter 2. Examples of Topological Spaces

The Standard Topology on Subsets of Euclidean Space

2.1 Definition: Recall that when X is a metric space, the set of open balls in X is a basis of sets in X, and the **metric topology** on X is the topology which is generated by the set of open balls.

The **standard inner product** on $\mathbb{R}^n$ is given by $\langle x, y\rangle = \sum_{k=1}^n x_k y_k$, and it induces the **standard norm** on $\mathbb{R}^n$ which is given by $\|x\| = \sqrt{\sum_{k=1}^n {x_k}^2}$ and this, in turn, induces the **standard metric** on $\mathbb{R}^n$, which is given by $d(x, y) = \sqrt{\sum_{k=1}^n (y_k - x_k)^2}$. The **standard topology** on $\mathbb{R}^n$, is the metric topology using the standard metric.

When $X \subseteq \mathbb{R}^n$, we shall assume, unless otherwise stated, that X is using the **standard topology**, which is metric topology using the standard metric on $\mathbb{R}^n$ (restricted to X), which is equal to the subspace topology on X in $\mathbb{R}^n$ with $\mathbb{R}^n$ using the standard topology.

The Order Topology

2.2 Definition: Let X be a nonempty set. A **linear order** on X is a binary relation $<$ on X such that

(1) For all $x, y \in X$, exactly one of the following three statements holds:

$$x < y \ , \ x = y \ , \ y < x$$

(2) For all $x, y, z \in X$, if $x < y$ and $y < z$ then $x < z$.

An **ordered set** is a set X together with a total order $<$. In an ordered set X, when $x, y \in X$ we write $x > y$ when $y < x$, we write $x \leq y$ when $x = y$ or $x < y$ and we write $x \geq y$ when $x = y$ or $x > y$. When X is an ordered set and $a, b \in X$ with $a \leq b$, the **intervals** between a and b are the sets

$$\begin{aligned}
(a, b) &= \{x \in X \mid a < x \text{ and } x < b\} \\
(a, b] &= \{x \in X \mid a < x \text{ and } x \leq b\} \\
[a, b) &= \{x \in X \mid a \leq x \text{ and } x < b\} \\
[a, b] &= \{x \in X \mid a \leq x \text{ and } x \leq b\}
\end{aligned}$$

2.3 Definition: Let X be an ordered set. For $a \in X$ we say that a is the **minimum** element of X when $a \leq x$ for every $x \in X$, and we say that a is the **maximum** element of X when $a \geq x$ for every $x \in X$. Verify that if X has a minimum element a then it is unique, and we write $a = \min X$. Likewise, verify that if X has a maximum element b then it is unique and we write $b = \max X$.

2.4 Definition: Let X be an ordered set. Let $\mathcal{B}$ consist of the sets of the form (a, b) where $a, b \in X$ with $a < b$, together with the sets of the form $[\min X, b)$ where $b \in X$ with $\min X < b$ (in the case that $\min X$ exists) together with the sets of the form $(a, \max X]$ where $a \in X$ with $a < \max X$ (in the case that $\max X$ exists). Verify that $\mathcal{B}$ is a basis of sets on X. The topology on X generated by $\mathcal{B}$ is called the **order topology** on X.

2.5 Definition: Let X be an ordered set. The topology on X generated by the basis of sets which consists of the sets of the form $[a, b)$ with $a < b$, and sets of the form $[a, m]$ with $a < m = \max X$ (if $\max X$ exists) is called the **lower limit topology** on X.

The Cartesian Product of Two Topological Spaces

2.6 Definition: Let X and Y be topological spaces. The **product space** of X and Y is the set

$$X \times Y = \big\{(x,y)\big| x \in X \text{ and } y \in Y\big\}.$$

The **product topology** on $X \times Y$ is the topology which is generated by the basis which consists of the sets of the form $U \times V$ with U open in X and V open in Y. The maps $p : X \times Y \to X$ and $q : X \times Y \to Y$ given by $p(x,y) = x$ and $q(x,y) = y$ are called the **projections** of $X \times Y$ onto the first and second factors. Note that p and q are both continuous.

2.7 Theorem: *Let $\mathcal{B}$ be a basis for the topology on X and let $\mathcal{C}$ be a basis for the topology on Y. Then $\mathcal{D} = \big\{B \times C \,\big|\, B \in \mathcal{B}, C \in \mathcal{C}\big\}$ is a basis for the product topology on $X \times Y$.*

Proof: The proof is left as an exercise.

2.8 Theorem: *Let $A \subseteq X$ and $B \subseteq Y$ be subspaces. Then the product topology on $A \times B$ is equal to the subspace topology on $A \times B \subseteq X \times Y$.*

Proof: The proof is left as an exercise.

2.9 Theorem: *Let X, Y and Z be topological spaces. Let $g : Z \to X$ and $h : Z \to Y$ and define $f : Z \to X \times Y$ by $f(z) = \big(g(z), h(z)\big)$. Then f is continuous if and only if both g and h are continuous.*

Proof: Let $p : X \times Y \to X$ and $q : X \times Y \to Y$ be the projection maps. Note that $g = p \circ f$ and $h = q \circ f$, so we see that if f is continuous, then so are g and h. Suppose that g and h are continuous. Let W be a basic open set in the product topology on $X \times Y$, say $W = U \times V$ where U is open in X and V is open in Y. Then

$$\begin{aligned} f^{-1}(W) = f^{-1}(U \times V) &= \big\{z \in Z \big| f(z) \in U \times V\big\} = \big\{z \in Z \big| \big(g(z), h(z)\big) \in U \times V\big\} \\ &= \big\{z \in Z \big| g(z) \in U \text{ and } h(z) \in V\big\} = g^{-1}(U) \cap h^{-1}(V) \end{aligned}$$

which is open in Z since g and h are continuous.

2.10 Theorem: *Let X, Y and Z be topological spaces, and let $\{p\}$ be any 1-element set. Then, using the product topology,*
(1) $X \times \{p\} \cong X$ and $\{p\} \times X \cong X$,
(2) $X \times Y \cong Y \times X$,
(3) $(X \times Y) \times Z \cong X \times (Y \times Z)$, and
(4) if $X \cong Z$ then $X \times Y \cong Z \times Y$ and $Y \times X \cong Y \times Z$.

Proof: The proof is left as an exercise.

2.11 Example: The cylinder $\mathbb{S}^1 \times \mathbb{R}^1$ is homeomorphic to the punctured plane $\mathbb{R}^2 \setminus \{0\}$. Indeed a homeomorphism $f : \mathbb{S}^1 \times \mathbb{R}^1 \to \mathbb{R}^2 \setminus \{0\}$ is given by $f\big(x,t\big) = e^t x$. The inverse $g : \mathbb{R}^2 \setminus \{0\} \to \mathbb{S}^1 \times \mathbb{R}^1$ is given by $g(u) = \big(\frac{u}{\|u\|}, \ln \|u\|\big)$.

The Cartesian Product of an Indexed Set of Topological Spaces

2.12 Definition: Recall that when $X_1, X_2, \cdots, X_n$ are sets, an n-tuple $(x_1, x_2, \cdots, x_n)$, with each $x_k \in X_k$, is formally defined to be a function $x : \{1, 2, \cdots, n\} \to \bigcup_{k=1}^n X_k$ with $x(k) \in X_k$ for all k, and we write $x_k = x(x)$, and the **cartesian product** of the sets $X_1, X_2, \cdots, X_n$ is the set of all such n-tuples.

More generally, when K is any nonempty set and X_k is a set for each $k \in K$, we define the **cartesian product** of the sets X_k to be the set

$$\prod_{k\in K} X_k = \left\{x : K \to \bigcup_{k\in K} X_k \,\middle|\, \text{each } x(k) \in X_k\right\}$$

and when $x \in \prod_{k\in K} X_k$ we write $x_k = x(k)$. For each $\ell \in K$, the **projection map** associated with ℓ (or the ℓ^{th} **projection map**) is the map $p_\ell : \prod_{k\in K} X_k \to X_\ell$ given by $p_\ell(x) = x_\ell$. In the case that $K = \{1, 2, \cdots, n\}$ we also write $\prod_{k\in K} X_k$ as $\prod_{k=1}^n X_k$, and when $K = \mathbb{Z}^+ = \{1, 2, 3, \cdots\}$ we also write $\prod_{k\in K} X_k$ as $\prod_{k=1}^\infty X_k$.

In the special case that $X_k = X$ for all $k \in K$, we write $\prod_{k\in K} X_k$ as X^K, and we note that X^K is the set of all functions from K to X. When $K = \{1, 2, \cdots, n\}$, we write X^K as X^n, and when $K = \{1, 2, 3, \cdots\}$ we write X^K as X^ω. Note that X^n is the set of n-tuples of elements in X and X^ω is the set of all sequences of elements in X (recalling that the sequence $x = (x_1, x_2, x_3, \cdots)$ in X is formally defined to be equal to the function $x : \mathbb{Z}^+ \to X$ with $x(k) = x_k$).

In the special case that X is a real vector space, we recall that X^n and X^ω are both vector spaces, and we also define X^∞ to be the set of eventually zero sequences, that is the set of all $x = (x_1, x_2, x_3, \cdots) \in X^\omega$ such that $x_k = 0$ for all but finitely many $k \in \mathbb{Z}^+$.

2.13 Definition: Let X_k be a topological space for each $k \in K$. The **box topology** on $\prod_{k\in K} X_k$ is the topology generated by the basis consisting of the sets of the form $\prod_{k\in K} U_k$ where each U_k is open in X_k. The **product topology** on $\prod_{k\in K} X_k$ is the topology generated by the basis consisting of the sets of the form $\prod_{k\in K} U_k$ where each U_k is open in X_k and with $U_k = X_k$ for all but finitely many indices $k \in K$. Note that in the case that the index set K is finite, these two topologies are the same. In the case that K is infinite, the box topology is finer than the product topology.

2.14 Theorem: *For each index $k \in K$, let $\mathcal{B}_k$ be a basis for the topology on X_k, Then the sets of the form $\prod_{k\in K} B_k$ with each $B_k \in \mathcal{B}_k$ form a basis for the box topology on $\prod_{k\in K} X_k$, and the sets of the form $\prod_{k\in K} B_k$, with each $B_k \in \mathcal{B}_k \cup \{X_k\}$ such that $B_k = X_k$ for all but finitely many indices $k \in K$, form a basis for the product topology on $\prod_{k\in K} X_k$.*

Proof: We give the proof in the case of the product topology. Let $\mathcal{C}$ be the basis in the definition of the product topology, that is let $\mathcal{C}$ be set of all sets of the form $\prod_{k\in K} U_k$ where each U_k is open in X_k and $U_k = X_k$ for all but finitely many indices $k \in K$, and let $\mathcal{B}$ be the set of all sets of the form $\prod_{k\in K} B_k$ with each $B_k \in \mathcal{B}_k \cup \{X_k\}$ such that $B_k = X_k$ for all but finitely many indices $k \in K$. Let $a \in \prod_{k\in K} X_k$, that is let $a : K \to \bigcup_{k\in K} X_k$ with $a(k) \in X_k$ for all $k \in K$, and let U be an open set in $\prod_{k\in K} X_k$ (using the product topology) with $a \in U$. By Theorem 1.20, since $\mathcal{C}$ is a basis for the topology on $\prod_{k\in K} X_k$, we can choose $C \in \mathcal{C}$ with $a \in C \subseteq U$. Say $C = \prod_{k\in K} U_k$ where each U_k is open in X_k with $U_k = X_k$ for all but finitely many k. Since $a \in C = \prod_{k\in K} U_k$, we have $a(k) \in U_k$ for all $k \in K$. For each $k \in K$, choose $B_k \in \mathcal{B}_k \cup \{X_k\}$ as follows: if $U_k = X_k$ then choose $B_k = X_k$ and if $U_k \neq X_k$ choose $B_k \in \mathcal{B}_k$ with $a(k) \in B_k \subseteq U_k$. Then let $B = \prod_{k\in K} B_k$ and note that $B \in \mathcal{B}$ with $a \in B \subseteq C \subseteq U$. Thus $\mathcal{B}$ is a basis for $\prod_{k\in K} X_k$, using the product topology (by Theorem 1.20).

2.15 Theorem: *Let A_k be a subset of X_k for each $k \in K$. If $\prod_{k\in K} X_k$ uses the product topology (or the box topology), then the subspace topology on $\prod_{k\in K} A_k$ is equal to the product topology (or, respectively, the box topology).*

Proof: We give the proof in the case of the box topology. As a basis for the box topology on $\prod_{k\in K} X_k$ we use the set $\mathcal{B}$ of sets of the form $\prod_{k\in K} V_k$ where each V_k is open in X_k. As a basis for the box topology on $\prod_{k\in K} A_k$ we use the set $\mathcal{C}$ of sets of the form $\prod_{k\in K} U_k$ where each U_k is open in A_k. As a basis for the subspace topology on $\prod_{k\in K} A_k$ we use the set $\mathcal{D}$ of sets of the form $B \cap \prod_{k\in K} A_k$ where $B \in \mathcal{B}$. We claim that, in fact, $\mathcal{C} = \mathcal{D}$. Let $C \in \mathcal{C}$, say $C = \prod_{k\in K} U_k$ where each U_k is open in A_k. For each index $k \in K$, since A_k is a subspace of X_k we can choose an open set V_k in X_k such that $U_k = V_k \cap A_k$. Then we have

$$C = \prod_{k\in K} U_k = \prod_{k\in K}(V_k \cap A_k) = \prod_{k\in K} V_k \cap \prod_{k\in K} A_k \in B.$$

On the other hand, given $D = B \cap \prod_{k\in K} A_k \in \mathcal{D}$ with $B = \prod_{k\in K} V_k \in \mathcal{B}$, we have

$$D = \prod_{k\in K} V_k \cap \prod_{k\in K} A_k = \prod_{k\in K}(V_k \cap A_k) \in \mathcal{C}.$$

2.16 Theorem: *Let A_k be a subset of X_k for each $k \in K$. Then, using either the product topology or the box topology, we have $\overline{\prod_{k\in K} A_k} = \prod_{k\in K} \overline{A_k}$.*

Proof: Let $p_\ell : \prod_{k\in K} X_k \to X_\ell$ be the ℓ^{th} projection map. For each $k \in K$, since $\overline{A_k}$ is closed in X_k and p_k is continuous (using either the product or the box topology), it follows that $p_k^{-1}\big(\overline{A_k}\big)$ is closed in $\prod_{k\in K} X_k$. Thus the set $\prod_{k\in K} \overline{A_k} = \bigcap_{k\in K} p_k^{-1}\big(\overline{A_k}\big)$ is closed in $\prod_{k\in K} X_k$. Since $\prod_{k\in K} A_k \subseteq \prod_{k\in K} \overline{A_k}$, which is closed, we have $\overline{\prod_{k\in K} A_k} \subseteq \prod_{k\in K} \overline{A_k}$.

It remains to prove that $\prod_{k\in K} \overline{A_k} \subseteq \overline{\prod_{k\in K} A_k}$. Let $a \in \prod_{k\in K} \overline{A_k}$, and note that $a_k \in \overline{A_k}$ for each $k \in K$. To prove that $a \in \overline{\prod_{k\in K} A_k}$, it suffices (by Theorem 1.11) to show that $\prod_{k\in K} A_k \cap V \neq \emptyset$ for every open set V in $\prod_{k\in K} X_k$ with $a \in V$. Let V be an open set in $\prod_{k\in K} X_k$ with $a \in V$. Choose a basic open set U in $\prod_{k\in K} X_k$ with $a \in U \subseteq V$, and say $U = \prod_{k\in K} U_k$ where each U_k is open in X_k with $a_k \in U_k$. For each $k \in K$, since $a_k \in \overline{A_k}$, we have $A_k \cap U_k \neq \emptyset$. Choose $b_k \in A_k \cap U_k$ for each $k \in K$, and let $b \in \prod_{k\in K} X_k$ be the resulting element, with $b(k) = b_k$ for all k. Then $b \in \prod_{k\in K}(A_k \cap U_k) = \prod_{k\in K} A_k \cap \prod_{k\in K} U_k = \prod_{k\in K} A_k \cap U \subseteq \prod_{k\in K} A_k \cap V$ so that $\prod_{k\in K} A_k \cap V \neq \emptyset$, as required.

2.17 Theorem: *Let Z be a topological space, for each index $k \in K$ let X_k be a topological space let $f_k : Z \to X_k$, and let $h : Z \to \prod_{k\in K} X_k$ be the map for which $p_k(h(z)) = f_k(z)$, that is $h(z)(k) = f_k(z)$, for all $k \in K$ and $z \in Z$. When $\prod_{k\in K} X_k$ uses the product topology, h is continuous if and only if each f_k is continuous.*

Proof: If h is continuous then for each $k \in K$, since h and p_k are continuous, so is the composite $f_k = p_k \circ h$. Suppose, conversely, that f_k is continuous for every $k \in K$. Let $\mathcal{C}$ be the basis for the product topology on $\prod_{k\in K} X_k$ which consists of the sets of the form $\prod_{k\in K} V_k$ where each V_k is open in X_k and $V_k = X_k$ for all but finitely many indices $k \in K$. Let $V \in \mathcal{C}$, say $V = \prod_{k\in K} V_k$ where each V_k is open in X_k and $V_k = X_k$ for all but finitely many indices $k \in K$. Let L be the finite set $L = \{k \in K \mid V_k \neq X_k\}$. Then

$$\begin{aligned} h^{-1}(V) &= \big\{z \in Z \,\big|\, h(z) \in \textstyle\prod_{k\in K} V_k\big\} = \big\{z \in Z \,\big|\, f_k(z) \in V_k \text{ for all } k \in K\big\} \\ &= \big\{z \in Z \,\big|\, f_k(z) \in V_k \text{ for all } k \in L\big\} = \textstyle\bigcap_{k\in L} {f_k}^{-1}(V_k) \end{aligned}$$

which is open in Z because it is a finite intersection of open sets in Z (each set ${f_k}^{-1}(V_k)$ is open since each f_k is continuous). Since $h^{-1}(V)$ is open in Z for every basic open set V in $\prod_{k\in K} X_k$, it follows that h is continuous (by Theorem 1.32).

The Quotient Topology

2.18 Definition: Let $\sim$ be an equivalence relation on a topological space X. For $x \in X$, let $[x]$ denote the equivalence class of x, that is

$$[x] = \{y \in X | y \sim x\}.$$

The **quotient space** of X under $\sim$, denoted by $X/\sim$, is the set of equivalence classes, that is

$$X/\sim = \{[x] | x \in X\}.$$

The map $q : X \to X/\sim$ given by $q(x) = [x]$ is called the **quotient map**. In the **quotient topology** on $X/\sim$, for $V \subseteq X/\sim$ we have

$$V \text{ is open in } X/\sim \iff q^{-1}(V) \text{ is open in } X.$$

Note that the quotient map q is continuous in this topology.

2.19 Theorem: *Let X and Y be topological spaces, and let $\sim$ be an equivalence relation on X. Let $g : X \to Y$ be a map which is constant on equivalence classes and define $f : X/\sim \to Y$ by $f([x]) = g(x)$. Then f is continuous if and only if g is continuous.*

Proof: Let $q : X \to X/\sim$ be the quotient map. Note that $g = f \circ q$, so we see that if f is continuous then so is g. Suppose that g is continuous. Let W be open in Y. We have $q^{-1}(f^{-1}(W)) = g^{-1}(W)$, which is open in X since g is continuous, and hence $f^{-1}(W)$ is open in $X/\sim$ by the definition of the quotient topology.

2.20 Definition: An **action** a group G on a topological space X is a map $f : G \times X \to X$ such that for all $a, b \in G$ and all $x \in X$ we have $f(1, x) = x$ and $f(ab, x) = f(a, f(b, x))$. Usually, we write ax for $f(a, x)$, and in this notation we have

$$1x = x \text{ and } (ab)x = a(bx).$$

Given such a group action, we associate the equivalence relation $\sim$ on X given by

$$x \sim y \iff ax = y \text{ for some } a \in G.$$

The quotient space $X/\sim$ is then usually denoted by X/G.

2.21 Example: The additive group $\mathbb{R}$ acts on $\mathbb{R}^2$ by $t(x, y) = (x, y + t)$. The associated equivalence relation is given by $(x_1, y_1) \sim (x_2, y_2) \iff x_1 = x_2$. For $(x, y) \in \mathbb{R}^2$, the equivalence class $[x, y]$ is the vertical line through (x, y). The quotient space $\mathbb{R}^2/\mathbb{R}$ is the set of vertical lines in $\mathbb{R}^2$. This quotient space is homeomorphic to $\mathbb{R}$. A homeomorphism $f : \mathbb{R}^2/\mathbb{R} \to \mathbb{R}$ is given by $f([x, y]) = x$ and its inverse is given by $g(u) = [u, 0]$. Note that the map f is continuous by Theorem 2.19, since $f([x, y]) = h(x, y)$ where $h : \mathbb{R}^2 \to \mathbb{R}$ is the projection $h(x, y) = x$, and note that the map g is continuous since we have $g = q \circ k$ where $k : \mathbb{R} \to \mathbb{R}^2$ is the inclusion $k(u) = (u, 0)$ and $q : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{R}$ is the quotient map.

2.22 Example: The multiplicative group $\mathbb{S}^1 \subseteq \mathbb{C}^1$ acts on the plane $\mathbb{R}^2 = \mathbb{C}^1$ by complex multiplication. The associated equivalence relation is given by $z \sim w \iff |z| = |w|$. For $0 \neq z \in \mathbb{R}^2 = \mathbb{C}^1$, the equivalence class $[z]$ is the circle of radius $|z|$ centered at 0, and the equivalence class of 0 is $[0] = \{0\}$. The quotient space $\mathbb{C}^1/\mathbb{S}^1$ consists of the origin and all circles centered at the origin. This quotient space is homeomorphic to the interval $[0,\infty) \subseteq \mathbb{R}$. A homeomorphism $f : \mathbb{C}^1/\mathbb{S}^1 \to [0,\infty)$ is given by $f\big([z]\big) = |z|$. Its inverse $g : [0,\infty) \to \mathbb{C}^1/\mathbb{S}^1$ is given by $g(x) = [x]$. Note that f is continuous by Theorem 2.19, since $f\big([z]\big) = h(z)$ where $h : \mathbb{C}^1 \to [0,\infty)$ is given by $h(z) = |z|$ (which is continuous), and note that g is continuous since $g = q \circ k$ where $k : [0,\infty) \to \mathbb{C}^1$ is the inclusion and q is the quotient map.

2.23 Example: The multiplicative group $(0,\infty)$ acts on the plane $\mathbb{R}^2$ by $t(x,y) = (tx,ty)$. We obtain the relation $(x_1,y_1) \sim (x_2,y_2)$ if and only if $(x_2,y_2) = t(x_1,y_1)$ for some $t > 0$. For $(0,0) \neq (x,y) \in \mathbb{R}^2$, the equivalence class $[x,y]$ is the ray through (x,y) from the origin, and the equivalence class of $(0,0)$ is $[0,0] = \{(0,0)\}$. The quotient space $\mathbb{R}^2/(0,\infty)$ consists of the origin and all of the rays from the origin. There is a fairly natural bijection $f : \mathbb{R}^2/(0,\infty) \to \mathbb{S}^1 \cup \{(0,0)\}$ but this bijection cannot be a homeomorphism (at least not if $\mathbb{S}^1 \cup \{(0,0)\}$ is considered as a subspace of $\mathbb{R}^2$), because the quotient space $\mathbb{R}^2/(0,\infty)$ is not Hausdorff. Indeed, the only open set in $\mathbb{R}^2/(0,\infty)$ which contains $[0,0]$ is the entire space $\mathbb{R}^2/(0,\infty)$.

2.24 Example: The additive group $\mathbb{Z}$ acts on $\mathbb{R}$ by addition. The associated equivalence relation is given by $x \sim y \iff y - x \in \mathbb{Z}$. Show that $\mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1$.

Solution: Define $f : \mathbb{R}/\mathbb{Z} \to \mathbb{S}^1$ by $f\big([t]\big) = \big(\cos 2\pi t, \sin 2\pi t\big)$. Note that f is continuous by Theorem 2.19. The inverse of f is the map $g : \mathbb{S}^1 \to \mathbb{R}/\mathbb{Z}$ given by

$$g(x,y) = \begin{cases} \left[\frac{1}{2\pi}\sin^{-1}\left(\frac{y}{\sqrt{x^2+y^2}}\right)\right] & \text{, if } x \geq 0 \\ \left[\frac{1}{2} - \frac{1}{2\pi}\sin^{-1}\left(\frac{y}{\sqrt{x^2+y^2}}\right)\right] & \text{, if } x \leq 0\,. \end{cases}$$

Note that $\mathbb{S}^1 = A \cup B$ where $A = \big\{(x,y) \in \mathbb{S}^1 \big| x \geq 0\big\}$ and $B = \big\{(x,y) \in \mathbb{S}^1 \big| x \leq 0\big\}$, which are both closed, and the restriction of g to each of the sets A and B is continuous (since each restriction is the composite of an elementary function with the quotient map), and so g is continuous.