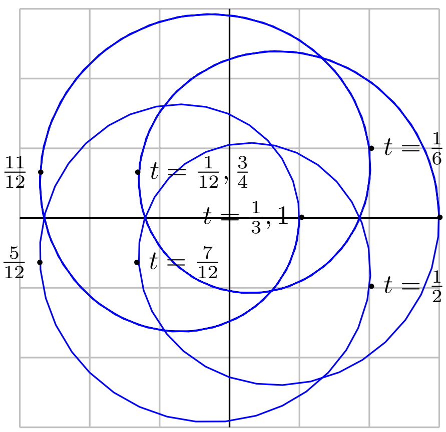
PMATH 367 Topology, Solutions to Assignment 5.5

1: Let $\alpha(t) = e^{i\,3\pi t} + 2\,e^{i\,12\pi t}$ for $0 \le t \le 1$.

(a) Sketch (the image of) the path α in $\mathbb{C}^*$.

Solution: The image can be sketched by plotting points, or by thinking of the curve as the trajectory followed by a point on the rim of a circular disc of radius 2, which is rotating about its centre while its centre revolves around a circular path of radius 1. The path starts at $\alpha(0) = 3$ and ends at $\alpha(1) = 1$.



(b) Using the sketch, evaluate each of the path integrals $\displaystyle\int_\alpha \frac{dz}{z}$, $\displaystyle\int_\alpha \frac{dz}{z+2}$ and $\displaystyle\int_\alpha \frac{dz}{z^2+2z}$.

Solution: With the help of the sketch, we can see that if α is written in polar coordinates as $\alpha(t) = r(t)e^{i\theta(t)}$, with $\theta(0) = 0$, then $\theta\big(\frac{1}{3}\big) = 4\pi$, $\theta\big(\frac{2}{3}\big) = 8\pi$ and $\theta(1) = 12\pi$, and also $r(0) = 3$ and $r(1) = 1$ so

$$\int_\alpha \frac{dz}{z} = \Big[\ln r(t) + i\theta(t)\Big]_0^1 = \ln r(1) - \ln r(0) + i\,\theta(1) - i\,\theta(0) = -\ln 3 + i\,12\pi\,.$$

If α is written in polar coordinates centred at -2 as $\alpha(t) = -2 + s(t)\,e^{i\,\phi(t)}$, with $\phi(0) = 0$, then $\phi\big(\frac{1}{3}\big) = 2\pi$, $\phi\big(\frac{2}{3}\big) = 4\pi$ and $\phi(1) = 6\pi$, and also $s(0) = 5$ and $s(1) = 3$, so

$$\int_\alpha \frac{dz}{z+2} = \Big[\ln s(t) + i\,\phi(t)\Big]_0^1 = \ln s(1) - \ln s(0) + i\,\phi(1) - i\,\phi(0) = \ln 3 - \ln 5 + i\,6\pi\,.$$

Finally,

$$\int_\alpha \frac{dz}{z^2+2z} = \tfrac{1}{2}\int_\alpha \frac{1}{z} - \frac{1}{z+2}\;dz = \tfrac{1}{2}\big(-\ln 3 + i\,12\pi\big) - \tfrac{1}{2}\big(\ln 3 - \ln 5 + i\,6\pi\big) = \tfrac{1}{2}\ln 5 - \ln 3 + i\,3\pi\,.$$

2: Find $\pi_1(X,a)$ for each of the following based spaces (X,a).

(a) $X=\mathbb{P}^2\setminus\{[0,0,1]\}$, $a=[1,0,0]$

Solution: From Problem 4(a) of Assignment 2 we know that $\mathbb{P}^2\setminus\{[0,0,1]\}$ is homeomorphic to the Möbius strip $\mathbb{M}^2$. Also, $\mathbb{M}^2$ is homotopic to its central circle $\mathbb{S}^1$ (indeed $\mathbb{S}^1$ is a strong deformation retract of $\mathbb{M}^2$), and so we have $\pi_1(X,a)\cong\pi_1(\mathbb{S}^1,1)\cong\mathbb{Z}$. In fact, $\pi_1(X,a)$ is generated by $[\sigma]$ where $\sigma(t)=[\cos\pi t,\sin\pi t,0]$.

(b) $X=GL_2(\mathbb{R})$, $a=I$

Solution: Note that $GL_2(\mathbb{R})$ is not connected. It is the disjoint union of the open subspaces $GL_2^+(\mathbb{R})$ and $GL_2^-(\mathbb{R})$, which consist of the matrices of positive and negative determinant, respectively. Since the path component of I must be contained in $GL_2^+(\mathbb{R})$, we have $\pi_1\big(GL_2(\mathbb{R}),I\big)=\pi_1\big(GL_2^+(\mathbb{R}),I\big)$. Also, we have $GL_2^+(\mathbb{R})\cong SL_2(\mathbb{R})\times\mathbb{R}^+$: indeed a homeomorphism is given by $f(A)=\Big(\frac{1}{\sqrt{\det(A)}}A,\det(A)\Big)$ with inverse $g(B,d)=\sqrt{d}\,B$. From Problem 4(b) on Assignment 2, we have $SL_2(\mathbb{R})\cong\mathbb{S}^1\times\mathbb{R}^2$, and we have $\mathbb{R}^+\cong\mathbb{R}$, so that $GL_2^+(\mathbb{R})\cong SL_2(\mathbb{R})\times\mathbb{R}^+\cong(\mathbb{S}^1\times\mathbb{R}^2)\times\mathbb{R}\cong\mathbb{S}^1\times\mathbb{R}^3\sim\mathbb{S}^1$. Thus $\pi_1\big(GL_2(\mathbb{R}),I\big)\cong\pi_1(\mathbb{S}^1,1)\cong\mathbb{Z}$. In fact, $\pi_1\big(GL_2(\mathbb{R}),I\big)$ is generated by $[\sigma]$ where $\sigma(t)=\left(\begin{smallmatrix}\cos 2\pi t & -\sin 2\pi t\\ \sin 2\pi t & \cos 2\pi t\end{smallmatrix}\right)$.

(c) $X=M_2(\mathbb{R})\setminus GL_2(\mathbb{R})$, $a=O$

Solution: Let α be a loop at $a=O$ in X. For each $t\in[0,1]$, $\alpha(t)$ is a 2×2 matrix with determinant 0. For all $s\in[0,1]$, the matrix $s\alpha(t)$ also has determinant 0, so the map $F:[0,1]\times[0,1]\to X$ given by $F(s,t)=s\alpha(t)$ is a homotopy from the constant loop κ, given by $\kappa(t)=O$, to the given loop α. Thus $\pi_1(X,a)=0$.

(d) $X=\big\{(x,y,z)\in\mathbb{R}^3\big|z^2=x^2+y^2-1\big\}$, $a=(1,0,0)$

Solution: This is the hyperboloid obtained by revolving the hyperbola $z^2=x^2-1$ (in the xz-plane) about the z-axis. We have $X\cong\mathbb{S}^1\times\mathbb{R}$ with a homeomorphism $f:X\to\mathbb{S}^1\times\mathbb{R}$ given by $f(x,y,z)=\big(\big(\frac{x}{\sqrt{x^2+y^2}},\frac{y}{\sqrt{x^2+y^2}}\big),z\big)$ with inverse $g:\mathbb{S}^1\times\mathbb{R}$ given by $g((u,v),w)=\big(\sqrt{w^2+1}\,u,\sqrt{w^2+1}\,v,w\big)$. It follows that $\pi_1(X,a)\cong\pi_1(\mathbb{S}^1,(1,0))\times\pi_1(\mathbb{R},0)\cong\pi_1(\mathbb{S}^1,(1,0))\cong\mathbb{Z}$, indeed $\pi_1(X,a)$ is generated by $[\sigma]$ where $\sigma(t)=(\cos 2\pi t,\sin 2\pi t,0)$.

Alternatively, note that the circle $x^2+y^2=1$, $z=0$ is a strong deformation retract of X: indeed the map $f(x,y,z)=\big(\frac{x}{\sqrt{x^2+y^2}},\frac{y}{\sqrt{x^2+y^2}}\big)$ is a deformation retraction, with $F(s,(x,y,z))=\big(\frac{x\sqrt{1-s^2z^2}}{\sqrt{x^2+y^2}},\frac{y\sqrt{1-s^2z^2}}{\sqrt{x^2+y^2}},sz\big)$ giving a homotopy from $i\circ f$ to the identity map on X. Thus $\pi_1(X,a)\cong\pi_1(\mathbb{S}^1,1)\cong\mathbb{Z}$, indeed $\pi_1(X,a)$ is generated by $[\sigma]$ where $\sigma(t)=(\cos 2\pi t,\sin 2\pi t,0)$.

3: (a) In the group $\pi_1\big(X\times Y,(a,b)\big)$, loops in $X\times\{b\}$ commute with loops in $\{a\}\times Y$. Let $\sigma(t)=\big(\alpha(t),b\big)$ and $\tau(t)=\big(a,\beta(t)\big)$ be loops in $X\times Y$ at the point (a,b). Find an explicit homotopy from $\sigma\tau$ to $\tau\sigma$ in $X\times Y$.

Solution: The map

$$F(s,t)=\begin{cases}\big(\alpha(2t),b\big) & 0\ \le\ t\ \le\ \frac{1-s}{2}\\ \big(\alpha(1-s),\beta(2t-1+s)\big) & \frac{1-s}{2}\le t\le\frac{2-s}{2}\\ \big(\alpha(2t-1),b\big) & \frac{2-s}{2}\ \le\ t\ \le\ 1\end{cases}$$

is one such homotopy. The map

$$G(s,t)=\begin{cases}\big(a,\beta(2t)\big) & 0\ \le\ t\ \le\ \frac{s}{2}\\ \big(\alpha(2t-s),\beta(s)\big) & \frac{s}{2}\le t\le\frac{1+s}{2}\\ \big(a,\beta(2t-1)\big) & \frac{1+s}{2}\le t\le 1\end{cases}$$

is another.

(b) A **topological group** is a based topological space (G,e) such that G is a group with identity e, and such that the product map $\mu:G\times G\to G$ given by $\mu(a,b)=ab$, and the inversion map $\nu:G\to G$ given by $\nu(a)=a^{-1}$, are both continuous. Show that if (G,e) is a topological group then $\pi_1(G,e)$ is abelian.

Solution: Given two loops α and β at e in G, the map

$$F(s,t)=\begin{cases}\alpha(2t) & 0\ \le\ t\ \le\ \frac{1-s}{2}\\ \alpha(1-s)\beta(2t-1+s) & \frac{1-s}{2}\le t\le\frac{2-s}{2}\\ \alpha(2t-1) & \frac{2-s}{2}\ \le\ t\ \le\ 1\end{cases}$$

is a homotopy from $\alpha\beta$ to $\beta\alpha$ in G. We remark that we used the continuity of μ, but not that of ν.

4: (a) Show that $\pi_1(X,a)$ is abelian if and only if all change-of-basepoint homomorphisms ϕ_γ depend only on the endpoints of γ (when γ is a path from a to b in X, $\phi_\gamma : \pi_1(X,a) \to \pi_1(X,b)$ is given by $\phi_\gamma(\alpha) = \gamma^{-1}\alpha\gamma$).

Solution: Suppose that $\pi_1(X,a)$ is abelian. Let γ and δ be any paths from a to b in X. Let α be any loop at a in X. Then since the loops $\delta\gamma^{-1}$ and α commute as elements in $\pi_1(X,a)$ so that $\delta\gamma^{-1}\alpha \sim \alpha\,\delta\gamma^{-1}$ in X, we have

$$\gamma^{-1}\alpha\,\gamma \sim \delta^{-1}\delta\,\gamma^{-1}\alpha\,\gamma \sim \delta^{-1}\alpha\,\delta\,\gamma^{-1}\gamma \sim \delta^{-1}\alpha\,\delta$$

in X, that is $[\gamma^{-1}\alpha\,\gamma] = [\delta^{-1}\alpha\,\delta]$ in $\pi_1(X,b)$. This shows that $\phi_\gamma = \phi_\delta$.

Conversely, suppose that all change-of-basepoint homomorphisms ϕ_γ depend only on the endpoints of γ. Let α and β be any loops at a in X. Then since $\phi_\alpha = \phi_\beta$ we have $\phi_\alpha(\beta) = \phi_\beta(\beta)$, that is $[\alpha^{-1}\beta\,\alpha] = [\beta^{-1}\beta\,\beta]$, so $[\alpha]^{-1}[\beta][\alpha] = [\beta]$, and hence $[\beta][\alpha] = [\alpha][\beta]$.

(b) For loops α and β in X (possibly at different points), a *free loop-homotopy* from α to β in X is a continuous map $F : [0,1]\times[0,1] \to X$ with $F(0,t) = \alpha(t)$ and $F(1,t) = \beta(t)$ for all t, and $F(s,0) = f(s,1)$ for all s. Show that for loops α and β at a in X, α and β are freely loop-homotopic in X if and only if α and β are conjugate in $\pi_1(X,a)$.

Solution: Let α and β be loops at a in X.

Suppose that α and β are freely loop-homotopic. Let F be a free loop-homotopy from α to β. Let $\gamma(s) = F(s,0) = F(s,1)$. Note that $\gamma(0) = F(0,0) = \alpha(0) = a$ and $\gamma(1) = F(1,0) = \beta(0) = a$, so γ is a loop at a. The map

$$H(s,t) = \begin{cases} \gamma(2t) = F(2t,0) & 0 \le t \le \frac{s}{2} \\ F\big(s, \frac{4t-2s}{4-3s}\big) & \frac{s}{2} \le t \le \frac{4-s}{4} \\ \gamma(4-4t) = F(4-4t,1) & \frac{4-s}{4} \le t \le 1 \end{cases}$$

is an (endpoint-fixing) homotopy from α to $\gamma(\beta\gamma^{-1})$ in X. Thus α and β are conjugate in $\pi_1(X,a)$.

Conversely, suppose α and β are conjugate in $\pi_1(X,a)$, say $\alpha \sim \gamma\beta\gamma^{-1}$ where γ is a loop at a in X. Notice that for any two loops σ and τ at a in X, $\sigma\tau$ is freely loop-homotopic to $\tau\sigma$; indeed the map

$$F(s,t) = \begin{cases} \tau(2t-s+1) & 0 \le t \le \frac{s}{2} \\ \sigma(2t-s) & \frac{s}{2} \le t \le \frac{1+s}{2} \\ \tau(2t-s-1) & \frac{1+s}{2} \le t \le 1 \end{cases}$$

is a free loop-homotopy from $\sigma\tau$ to $\tau\sigma$. In particular, $\gamma\beta\gamma^{-1}$ is freely loop-homotopic to $\beta\gamma^{-1}\gamma$. We also note that free loop-homotopy is an equivalence relation (the proof is identical to the proof that homotopy of paths is an equivalence relation). Since α is homotopic (and hence freely loop-homotopic) to $\gamma\beta\gamma^{-1}$, which is freely loop-homotopic to $\beta\gamma^{-1}\gamma$, which, in turn, is homotopic (hence freely loop-homotopic) to β, the loops α and β are freely loop-homotopic.

5: (a) Prove that $\big\langle a,b\,\big|\,a^3\!=\!e, b^9\!=\!e, a\!=\!bab\big\rangle \cong \mathbb{Z}_3$.

Solution: Since $a = bab$ we have $b = ab^{-1}a^{-1}$ and $b^{-1} = aba^{-1}$, and so since $a^3 = 1$ we have

$$b^2 = b(ab^{-1}a^{-1}) = bab^{-1}a^{-1} = ba(aba^{-1})a^{-1} = ba^2ba^{-2} = ba^2(ab^{-1}a^{-1})a^{-2} = ba^3b^{-1}a^{-3} = bb^{-1} = 1\,.$$

Since $b^2 = 1$ and $b^9 = 1$ we have $b = b^9(b^2)^{-4} = 1$. Thus $G = \langle a \big| a^3 = 1\big\rangle \cong \mathbb{Z}_3$.

(b) Let $G = \big\langle a,b,c\,\big|\,abcbac\!=\!e\big\rangle$ and let $H = \big\langle x,y,z\,\big|\,x^2y^2z^2\!=\!e\big\rangle$. Show that $G \cong H$ and find an isomorphism $\phi : G \to H$ and its inverse $\psi : H \to G$.

Solution: The hexagon with edges identified in pairs according to the word $abcbac$ has fundamental group G, and the hexagon with edges identified according to $xxyyzz$ has fundamental group H. By finding the Euler characteristic, we see that both spaces are homeomorphic to $\mathbb{P}^2{}_3$, so they have isomorphic fundamental groups. Thus $G \cong H$.

We can also use the cut-and-paste algorithm to find an isomorphism explicitly, as follows: Draw the hexagon with edges identified according to $abcbac$. Cut from the initial point of the second a edge to the initial point of the first a edge. Label this new edge by x. Note that, associating directed edges with paths, we have $x \sim ac$. Remove the triangle with edges labled acx^{-1} and reglue it along the a edge to obtain the hexagon with edges identified according to $xxc^{-1}bcb$. Cut from the final point of the first b edge to the final point of the second b edge. Label this new edge by z. Note that $z \sim cb$. Remove the triangle with edges $bz^{-1}c$ and reglue along the b edge to obtain the hexagon with edges identified according to $xxc^{-1}c^{-1}zz$. Redirect the c edges and relabel them y, so $y = c^{-1}$. We obtain the hexagon with edges labeled $xxyyzz$. Since $x \sim ac$, $y \sim c^{-1}$ and $z \sim cb$, we have an isomorphism $\psi : H \to G$ given by $\psi(x) = ac$, $\psi(y) = c^{-1}$ and $\psi(z) = cb$. Also, since $x \sim ac$, $z \sim cb$ and $y = c^{-1}$, we have $c = y^{-1}$, $b \sim c^{-1}z = yz$ and $a \sim xc^{-1} = xy$, so the inverse $\phi : G \to H$ is given by $\phi(a) = xy$, $\phi(b) = yz$ and $\phi(c) = y^{-1}$.

(c) Show that the above group $G = \big\langle a,b,c\,\big|\,abcbac\!=\!e\big\rangle$ is not isomorphic to any of the following groups:

$$\big\langle x,y\,\big|\,xy\!=\!yx\big\rangle\ ,\ \big\langle x,y\,\big|\,xy^2x\!=\!e\big\rangle\ ,\ \big\langle x,y,z\,\big|\,xyz\!=\!yzx\big\rangle.$$

Solution: The abelianization of G is $\text{Ab}(G) \cong \mathbb{Z}^3/\langle(2,2,2)\rangle \cong \mathbb{Z}^2 \times \mathbb{Z}_2$, and the abelianizations of the above three groups are

$$\text{Ab}\big(\langle x,y|xy\!=\!yx\rangle\big) \cong \mathbb{Z}^2\ ,\ \text{Ab}\big(\langle x,y|xy^2x\!=\!e\rangle\big) \cong \mathbb{Z}^2/\langle(2,2)\rangle \cong \mathbb{Z}\times\mathbb{Z}_2\ ,\ \text{and}\ \text{Ab}\big(\langle x,y,z|xyz\!=\!yzx\rangle\big) \cong \mathbb{Z}^3\,,$$

and so G is not isomorphic to any of them.

6: (a) Let X be the space $\mathbb{P}^2$ with n points identified. Find $\pi_1(X)$ and its abelianization.

Solution: $\mathbb{P}^2$ is homeomorphic to $D/\sim$ where D is the closed unit disc centered at 0 and $\sim$ is the equivalence relation which identifies points on the boundary S of D according to the word α^2. Choosing the n points to lie in $S/\sim$ we find that X is homeomorphic to $D/\sim$ where now $\sim$ is the equivalence relation which identifies points on S according to the word $\alpha_1\alpha_2\cdots\alpha_n\alpha_1\alpha_2\cdots\alpha_n$. As with similar examples in class, we can use the Van Kampen Theorem to obtain $\pi_1(X,1) = \langle \alpha_1, \alpha_2, \cdots, \alpha_n \big| (\alpha_1\alpha_2\cdots,\alpha_n)^2\rangle$. The abelianization of this group is isomorphic to $\mathbb{Z}^n/\langle(2,2,\cdots,2)\rangle \cong \mathbb{Z}^{n-1}\oplus\mathbb{Z}_2$.

To apply Van Kampen's Theorem, we let $U = X\setminus\{0\}$ and $V = B$ where B is the interior of D. Then U deformation-retracts to $S/\sim$, which is homeomorphic to the wedge sum of n circles, and we have $\pi_1(U,1) = \langle\alpha_1,\cdots,\alpha_n\rangle$. We let λ be the line segment from $\frac{1}{2}$ to 1 in X, and then a change of basepoint gives $\pi_1\big(U,\frac{1}{2}\big) = \langle\sigma_1,\cdots,\sigma_n\rangle$ where $\sigma_i = \lambda\alpha_i\lambda^{-1}$. Since $V = B$ is contractible, we have $\pi_1\big(V,\frac{1}{2}\big) = 0$. Also, $U\cap V = B\setminus\{0\}$ deformation-retracts to the circle of radius $\frac{1}{2}$ about 0, and we have $\pi_1\big(U\cap V,\frac{1}{2}\big) = \langle\gamma\rangle$ where γ is the loop at $\frac{1}{2}$ about this circle. In U, $\gamma \sim \lambda(\alpha_1\alpha_2\cdots\alpha_n)^2\lambda^{-1} \sim (\sigma_1\sigma_2\cdots\sigma_n)^2$. By Van Kampen's Theorem $\pi_1\big(X,\frac{1}{2}\big) = \langle\sigma_1,\sigma_2,\cdots,\sigma_n\big|(\sigma_1\sigma_2\cdots\sigma_n)^2\rangle$. Changing the basepoint back to 1 gives $\pi_1(X,1) = \langle\alpha_1,\alpha_2,\cdots,\alpha_n\big|(\alpha_1\alpha_2\cdots,\alpha_n)^2\rangle$.

(b) Let X be the space $\mathbb{P}^2$ with n points removed. Find $\pi_1(X)$ and its abelianization.

Solution: Let D be the closed unit disc, let p_i be the n points $p_i = \big(0,-1+\frac{2i-1}{n}\big)$ for $i = 1,2,\cdots,n$. Then X is homeomorphic to $\big(D/\sim\big)\setminus\{p_1,\cdots,p_n\}$ where $\sim$ is the equivalence relation which identifies points on the boundary of D according to the word α^2. For $i = 1,2,\cdots,n$ let $q_i = \big(0,-1+\frac{2i}{n}\big)$, so that each q_i lies between p_i and p_{i+1}, and let α_i be the loop at $1=-1\in X$ which follows the arc of the circle from 1 through the point q_i to the point -1 in D. Then X deformation-retracts to the union of the images of the loops α_i, so we have $X \sim \bigvee_{i=1}^{n}\mathbb{S}^1$ and $\pi_1(X,1) = \langle\alpha_1,\alpha_2,\cdots,\alpha_n\rangle$. The abelianization of this is isomorphic to $\mathbb{Z}^n$.

7: For each of the following spaces X, find $\pi_1(X)$. In each case, describe generators for $\pi_1(X)$, and describe $\pi_1(X)$ up to isomorphism using direct products and free products of cyclic groups.

(a) Let X be the union of the x-axis, the y-axis, and the sphere $x^2+y^2+z^2=1$ in $\mathbb{R}^3$.

Solution: Let Y be the union of the unit sphere with the line segment from $(-1,0,0)$ to $(1,0,0)$ and the line segment from $(0,1,0)$ to $(0,1,0)$. Then Y is a strong deformation retract of X so $\pi_1(X)=\pi_1(Y)$. Let Y^1 be the intersection of Y with the xy-plane, so Y^1 is the union of the circle $x^2+y^2=1$ with the line segment from $(-1,0)$ to $(1,0)$ and the line segment from $(0,-1)$ to $(0,1)$. Label the line segment from $(0,0)$ to $(1,0)$ by ρ_1, the line segment from $(0,0)$ to $(0,1)$ by ρ_2, the line segment from $(0,0)$ to $(-1,0)$ by ρ_3 and the line segment from $(0,0)$ to $(0,-1)$ by ρ_4. Label the arc along the circle from $(1,0)$ to $(0,1)$ by α_1, the arc from $(0,1)$ to $(-1,0)$ by α_2, the arc from $(-1,0)$ to $(0,-1)$ by α_3 and the arc from $(0,-1)$ to $(1,0)$ by α_4. Also, let $\sigma_1=\rho_1\alpha_1{\rho_2}^{-1}$, $\sigma_2=\rho_2\alpha_2{\rho_3}^{-1}$, $\sigma_3=\rho_3\alpha_3{\rho_4}^{-1}$ and $\sigma_4=\rho_4\alpha_4{\rho_1}^{-1}$. Note that Y is obtained from Y^1 by attaching two discs along their boundary circles each according to the word $\alpha_1\alpha_2\alpha_3\alpha_4$, and that $\alpha_1\alpha_2\alpha_3\alpha_4\sim\sigma_1\sigma_2\sigma_3\sigma_4$. By the Seifert-VanKampen Theorem we have $\pi_1(Y^1)=\langle\sigma_1,\sigma_2,\sigma_3,\sigma_4\,|\,\emptyset\rangle$ and

$$\pi_1(X)=\pi_1(Y)=\langle\sigma_1,\sigma_2,\sigma_3,\sigma_4\,|\,\sigma_1\sigma_2\sigma_3\sigma_4\rangle=\langle\sigma_1,\sigma_2,\sigma_3\,|\,\emptyset\rangle\cong\mathbb{Z}*\mathbb{Z}*\mathbb{Z}\,.$$

(b) Let X be the complement in $\mathbb{R}^3$ of the union of the z-axis and the two circles $x^2+y^2=4$ with $z=\pm 1$.

Solution: Let Y be the union of the two tori obtained by revolving the circles $(x-2)^2+(z\pm 1)^2=1$, $y=0$ about the z-axis. These two tori meet along the circle $x^2+y^2=4$, $z=0$. Note that Y is a strong deformation retract of X so we have $\pi_1(X)=\pi_1(Y)$. Let Y^1 be the union of the two circles $(x-2)^2+(z\pm 1)^2=1$, $z=0$ with the circle $x^2+y^2=4$, $z=0$. Label the circle $(x-2)^2+(z-1)^2=1$, $y=0$ by α, label the circle $(x-2)^2+(z+1)^2=1$, $y=0$ by β and label the circle $x^2+y^2=4$, $z=0$ by γ, so that α, β and γ are loops at $(2,0,0)$. Note that Y is obtained from Y^1 by attaching two squares along their boundaries according to the words $\alpha\gamma\alpha^{-1}\gamma^{-1}$ and $\beta\gamma\beta^{-1}\gamma^{-1}$. By the Seifert-VanKampen Theorem $\pi_1(Y^1)=\langle\alpha,\beta,\gamma\,|\,\emptyset\rangle$ and

$$\pi_1(X)=\pi_1(Y)=\langle\alpha,\beta,\gamma\,|\,\alpha\gamma\alpha^{-1}\gamma^{-1},\beta\gamma\beta^{-1}\gamma^{-1}\rangle\cong\mathbb{Z}\times(\mathbb{Z}*\mathbb{Z})\,.$$