PMATH 367 Topology, Assignment 5.5 Not to be handed in

1: Let $\alpha(t) = e^{i\,3\pi t} + 2\,e^{i\,12\pi t}$ for $0 \le t \le 1$.

(a) Sketch (the image of) the path α in $\mathbb{C}^*$.

(b) Using the sketch, evaluate each of the path integrals $\int_\alpha \frac{dz}{z}$, $\int_\alpha \frac{dz}{z+2}$ and $\int_\alpha \frac{dz}{z^2+2z}$.

2: Find $\pi_1(X,a)$ for each of the following based spaces (X,a).

(a) $X = \mathbb{P}^2 \setminus \{[0,0,1]\}$, $a = [1,0,0]$

(b) $X = GL_2(\mathbb{R})$, $a = I$

(c) $X = M_2(\mathbb{R}) \setminus GL_2(\mathbb{R})$, $a = O$

(d) $X = \{(x,y,z) \in \mathbb{R}^3 \,|\, z^2 = x^2 + y^2 - 1\}$, $a = (1,0,0)$

3: (a) In the group $\pi_1\big(X \times Y, (a,b)\big)$, loops in $X \times \{b\}$ commute with loops in $\{a\} \times Y$. Let $\sigma(t) = \big(\alpha(t), b\big)$ and $\tau(t) = \big(a, \beta(t)\big)$ be loops in $X \times Y$ at the point (a,b). Find an explicit homotopy from $\sigma\tau$ to $\tau\sigma$ in $X \times Y$.

(b) A **topological group** is a based topological space (G,e) such that G is a group with identity e, and such that the product map $\mu : G \times G \to G$ given by $\mu(a,b) = ab$, and the inversion map $\nu : G \to G$ given by $\nu(a) = a^{-1}$, are both continuous. Show that if (G,e) is a topological group then $\pi_1(G,e)$ is abelian.

4: (a) Show that $\pi_1(X,a)$ is abelian if and only if all change-of-basepoint homomorphisms ϕ_γ depend only on the endpoints of γ (when γ is a path from a to b in X, $\phi_\gamma : \pi_1(X,a) \to \pi_1(X,b)$ is given by $\phi_\gamma(\alpha) = \gamma^{-1}\alpha\gamma$).

(b) For loops α and β in X (possibly at different points), a *free loop-homotopy* from α to β in X is a continuous map $F : [0,1] \times [0,1] \to X$ with $F(0,t) = \alpha(t)$ and $F(1,t) = \beta(t)$ for all t, and $F(s,0) = f(s,1)$ for all s. Show that for loops α and β at a in X, α and β are freely loop-homotopic in X if and only if α and β are conjugate in $\pi_1(X,a)$.

The remaining three problems cover material which will not be on the final exam. For these three problems, if you apply the Seifert-Van Kampen Theorem then you may do so casually, without worrying about a change of basepoint, and you can state the existence of any necessary homeomorphisms or deformation retracts without explicitly providing their formulas.

5: (a) Prove that $\langle a,b \,|\, a^3 = e, b^9 = e, a = bab \rangle \cong \mathbb{Z}_3$.

(b) Let $G = \langle a,b,c \,|\, abcbac = e \rangle$ and let $H = \langle x,y,z \,|\, x^2y^2z^2 = e \rangle$. Show that $G \cong H$ and find an isomorphism $\phi : G \to H$ and its inverse $\psi : H \to G$.

(c) Show that the above group $G = \langle a,b,c \,|\, abcbac = e \rangle$ is not isomorphic to any of the following groups: $\langle x,y \,|\, xy = yx \rangle$, $\langle x,y \,|\, xy^2x = e \rangle$, $\langle x,y,z \,|\, xyz = yzx \rangle$.

6: (a) Let X be the space $\mathbb{P}^2$ with n points identified. Find $\pi_1(X)$ and its abelianization.

(b) Let X be the space $\mathbb{P}^2$ with n points removed. Find $\pi_1(X)$ and its abelianization.

7: For each of the following spaces X, find $\pi_1(X)$. In each case, describe generators for $\pi_1(X)$, and describe $\pi_1(X)$ up to isomorphism using direct products and free products of cyclic groups.

(a) Let X be the union of the x-axis, the y-axis, and the sphere $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$.

(b) Let X be the complement in $\mathbb{R}^3$ of the union of the z-axis and the two circles $x^2 + y^2 = 4$ with $z = \pm 1$.