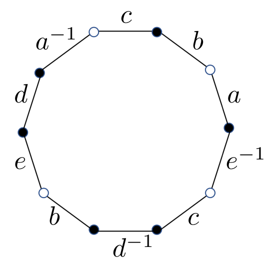
PMATH 367 Topology, Solutions to Assignment 5

1: By counting vertices and determining orientation, determine the topological type (the homeomorphism class) of the polygons with edges identificatied in pairs according to the following words..

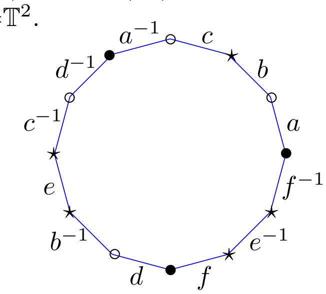
(a) $a\,b\,c\,a^{-1}d\,e\,b\,d^{-1}c\,e^{-1}$

Solution: Since the b-edges are in the same direction, the surface is not orientable, so it is homeomorphic to $(\mathbb{P}^2)^{\#h}$, for some h. Verify that the vertices in the polygon will be identified as indicated in the figure below (with the edges labelled counter-clockwise), so the Euler Characteristic of the surface is $\chi = V - E + F = 2 - 5 + 1 = -2$. Thus the surface is homeomorphic to $(\mathbb{P}^2)^{\#4}$.



(b) $a\,b\,c\,a^{-1}d^{-1}c^{-1}e\,b^{-1}d\,f\,e^{-1}f^{-1}$.

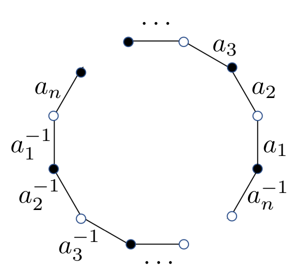
Solution: Draw a dodecagon and label its edges according to the given word. Verify that there are 3 distinct (equivalence classes of) vertices, so the Euler characteristic is $\chi = V - E + F = 3 - 6 + 1 = -2$. Thus the surface is homeomorphic either to $(\mathbb{T}^2)^{\#2}$ or to $(\mathbb{P}^2)^{\#4}$. Since all the edge pairs occur in opposite directions, it is homeomorphic to $(\mathbb{T}^2)^{\#2} = \mathbb{T}^2\#\mathbb{T}^2$.



(c) $a_1a_2\cdots a_na_1{}^{-1}a_2{}^{-1}\cdots a_n{}^{-1}$

Solution: Since all pairs of edges occur in opposite directions, this surface is orientable, so it is homeomorphic to $(\mathbb{T}^2)^{\#g}$ for some g. Check that the start of the a_1 edge is joined to the end of the a_2 edge, which is joined to the start of the a_3 edge, which is joined to the end of the a_4 edge, which is joined to the start of the a_5 edge, and so on.

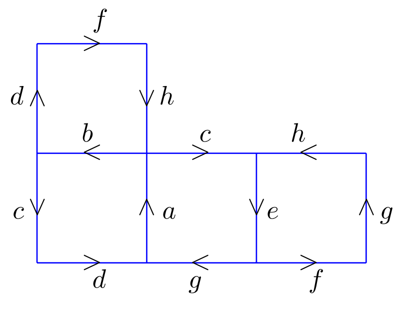
When n is odd, the start of edges $a_1, a_3, a_5, \cdots a_n$ are all joined, then the start of a_n is joined back to the start of a_1, and the vertices are identified as shown below. The Euler characteristic is $\chi = V - E + F = 2 - n + 1 = 3 - n$, and so the surface is homeomorphic to $(\mathbb{T}^2)^{\#g}$ where $g = \frac{n-1}{2}$ so that $\chi = 3 - n = 2 - 2g$.

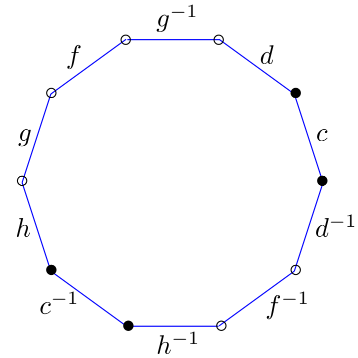


When n is even, the start of the edges $a_1, a_3, a_5, \cdots, a_{n-1}$ are all joined, then the start of a_{n-1} is joined to the end of a_n, which is joined to the end of a_1. Continuing, we find that all of the vertices are identified. The Euler characteristic is $\chi = V - E + F = 1 - n + 1 = 2 - n$, and so the surface is homeomorphic to ${\mathbb{T}^2}_g$, where $g = \frac{n}{2}$ so that $\chi = 2 - n = 2 - 2g$.

2: (a) Determine the topological type of the space obtained from the disjoint union of four squares with edges identified in pairs according to the words $abcd$, $efgh$, $aceg$ and $bdfh$.

Solution: We check to see whether the surface is connected by joining the squares: attach the third square to the first along the a edge, then attach the second square along the e edge, then attach the fourth square along the b edge, as shown below, to obtain the decagon with edges identified according to $c\,d\,g^{-1}f\,g\,h\,c^{-1}h^{-1}f^{-1}d^{-1}$. Verify that there are 2 distinct (equivalence classes of) vertices, so the Euler characteristic is $\chi = V - E + F = 2 - 5 + 1 = -2$. Note that all edge pairs occur in opposite directions, so the space is homeomorphic to $\mathbb{T}^2 \# \mathbb{T}^2$.

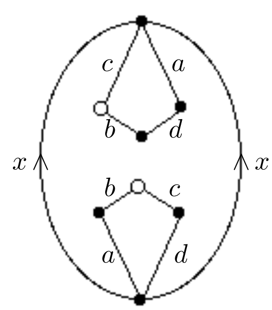


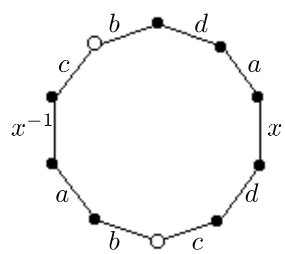


(b) Two disjoint closed squares are chosen on a sphere, their interiors are removed, and then the edges of one are identified with the edges of the other according to $a\,b\,c\,d$ (listed clockwise, from the outside) on the first and $a\,d\,b\,c$ (again listed clockwise) on the second. Determine the topological type of the resulting surface.

Solution: We can place the two squares in $\mathbb{S}^2$ and then open to form a polygon as shown below (you might want to add direction arrows along the edges: for the image on the left, the edges around the quadrilaterals point clockwise and the edges labelled x point upwards, and for the image on the right, the edges are labelled counterclockwise). Since the a edges are in the same direction, the surface is not orientable. The Euler characteristic is $\chi = V - E + F = 2 - 5 + 1 = -2$, and so the surface is homeomorphic to $(\mathbb{P}^2)^{\#4}$.

We remark that if the edges of one of the squares were listed clockwise and the edges of the other were listed counterclockwise, the resulting surface would be orientable. On the other hand, if both were listed counterclockwise, we would obtain the same surface. Indeed, in the picture on the left, if the two x-edges are pushed downwards to be joined below the squares (forming a sphere with the squares on the upper hemisphere), then the edges of the squares are labeled clockwise from the outside, but if the two x-edges are pulled upwards to be joined above the squares (forming a sphere with the squares on the lower hemisphere), then the edges of the squares are labeled clockwise from the inside, and counterclockwise from the outside.

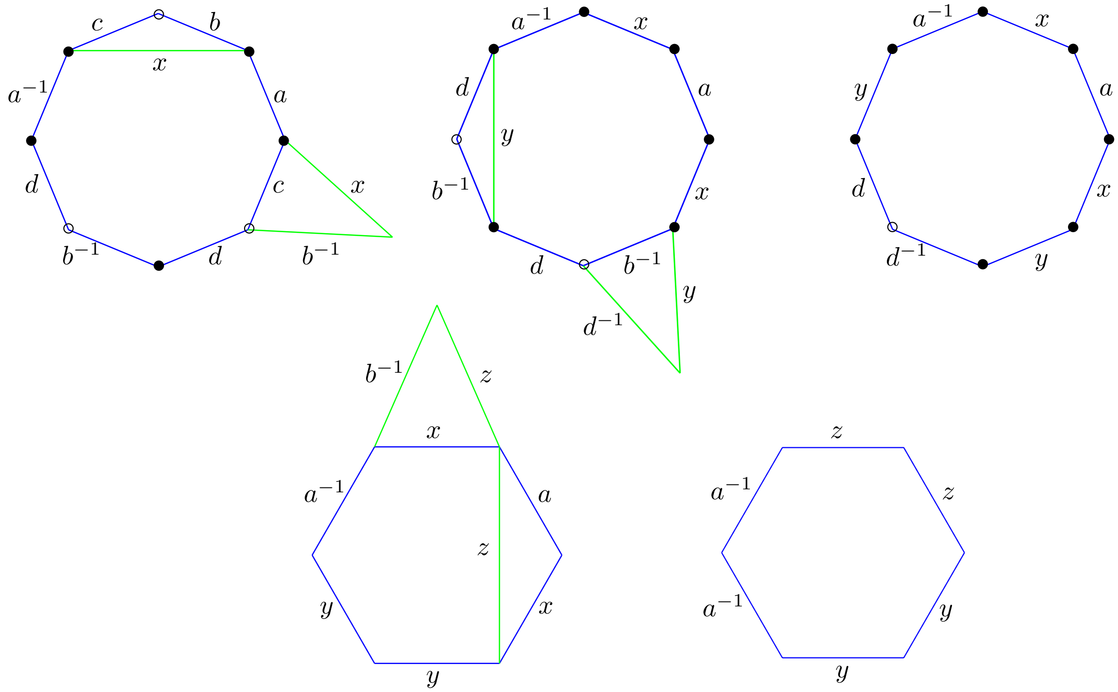




$a\,b\,c\,d\,x\,a\,d\,b\,c\,x^{-1}$

(c) Carry out the cut-and-paste algorithm to determine the topological type of the octagon with its edges identified in pairs according to the word $a\,b\,c\,a^{-1}d\,b^{-1}d\,c$.

Solution: The steps of the algorithm are shown below. The surface is homeomorphic to $\mathbb{P}^2\#\mathbb{P}^2\#\mathbb{P}^2$.



3: (a) Prove that $(\mathbb{R}^3\setminus\{0\})/\mathbb{R}^* \;\cong\; \mathbb{S}^2/\{\pm1\} \;\cong\; D^2/\!\sim$ where D^2 is the closed unit disc $D^2=\big\{z\in\mathbb{C}\,\big|\,\|z\|\leq 1\big\}$ and for $z,w\in D^2$ we have $z\sim w$ if and only if $z=w$ or ($\|z\|=1$ and $w=-z$).

Solution: We provide a detailed proof that $\big(\mathbb{R}^3\setminus\{0\}\big)/\mathbb{R}^*\cong\mathbb{S}^2/\{\pm1\}$. The map $f_1:(\mathbb{R}^3\setminus\{0\})\to\mathbb{S}^2$ given by $f_1(x)=\frac{x}{\|x\|}$ is continuous (it is elementary). Composing this with the quotient map from $\mathbb{S}^2$ to $\mathbb{S}^2/\{\pm1\}$ gives the continuous map $f_2:(\mathbb{R}^3\setminus\{0\})\to\mathbb{S}^2/\{\pm1\}$ given by $f_2(x)=\big[\frac{x}{\|x\|}\big]=\big\{\pm\frac{x}{\|x\|}\big\}$. Since $f_2(tx)=f_2(x)$ for all $t\in\mathbb{R}^*$, the map f_2 induces the well-defined and continuous map $f:(\mathbb{R}^3\setminus\{0\})/\mathbb{R}^*\to\mathbb{S}^2/\{\pm1\}$ given by $f\big([x]\big)=\big[\frac{x}{\|x\|}\big]$. The inclusion map $g_1:\mathbb{S}^2\to\mathbb{R}^3\setminus\{0\}$ given by $g_1(u)=u$ is continuous. Composing with the quotient map from $\mathbb{R}^3\setminus\{0\}$ to $(\mathbb{R}^3\setminus\{0\})/\mathbb{R}^*$ gives the continuous map $g_2:\mathbb{S}^2\to(\mathbb{R}^3\setminus\{0\})/\mathbb{R}^*$ given by $g_2(u)=[u]=\big\{tu\,\big|\,0\neq t\in\mathbb{R}\big\}=\text{Span}_{\mathbb{R}}\{u\}\setminus\{0\}$. Since $g_2(-u)=g_2(u)$, the map g_2 induces the well-defined and continuous map $g:\mathbb{S}^2\{\pm1\}\to(\mathbb{R}^3\setminus\{0\})/\mathbb{R}^*$ given by $g\big([u]\big)=[u]$, that is $g\big(\{\pm u\}\big)=\text{Span}_{\mathbb{R}}\{u\}\setminus\{0\}$. Note that the maps f and g are inverses of one another because for all $x\in\mathbb{R}^3\setminus\{0\}$ we have $g\big(f([x])\big)=g\big(\big[\frac{x}{\|x\|}\big]\big)=\big[\frac{x}{\|x\|}\big]=\text{Span}_{\mathbb{R}}\big\{\frac{x}{\|x\|}\big\}\setminus\{0\}=\text{Span}_{\mathbb{R}}\{x\}\setminus\{0\}=[x]$, and for all $u\in\mathbb{S}^2$ we have $\|u\|=1$ so that $f\big(g([u])\big)=f\big([u]\big)=\big[\frac{u}{\|u\|}\big]=[u]$.

We give a slightly less detailed proof that $\mathbb{S}^2/\{\pm1\}\cong D^2/\!\sim$. The map $f_1:\mathbb{S}^2\to D^2/\!\sim$ given by $f_1(u,v,w)=\big[(u,v)\big]$ when $w\geq 0$ and by $f_1(u,v,w)=\big[(-u,-v)\big]$ when $w\leq 0$ is continuous by the glueing lemma, and it satisfies $f_1(u,v,w)=f_1(-u,-v,-w)$, so it induces the well-defined and continuous map $f:\mathbb{S}^2/\{\pm1\}\to D^2/\!\sim$ given by

$$f\big([(u,v,w)]\big)=\left\{\begin{array}{c}[(u,v)]\;\text{ if } w\geq 0\\ [(-u,-v)]\text{ if } w\leq 0\end{array}\right\}.$$

The map $g_1:D^2\to\mathbb{S}^2\setminus\{\pm1\}$ given by $g_1(x,y)=\big[\big(x,y,\sqrt{1-(x^2+y^2)}\big)\big]$ is continuous, and it respects the equivalence relation on D^2 because when $x^2+y^2=1$ we have $g_1(x,y)=g_1(-x,-y)$, so it induces the well-defined and continuous map $g:D^2/\!\sim\to\mathbb{S}^2/\{\pm1\}$ given by $g\big([(x,y)]\big)=\big[\big(x,y,\sqrt{1-(x^2+y^2)}\big)\big]$. It is easy to check that f and g are inverses of one another: for example, when $(u,v,w)\in\mathbb{S}^2$ with $w\leq 0$ we have $g\big(f\big([(u,v,w)]\big)\big)=g\big([(-u,-v)]\big)=\big[\big(-u,-v,\sqrt{1-(u^2+v^2)}\big)\big]=\big[(-u,-v,-w)\big]=\big[(u,v,w)\big]$.

(b) Prove that $D^2/\!\sim\;\cong\; I^2/\!\approx\;\cong\; T^2$ where D^2 is the closed unit disc $D^2=\big\{z\in\mathbb{C}\,\big|\,\|z\|\leq1\big\}$ and for $z,w\in D^2$ with $z=x+iy$ and $w=u+iv$, we have $z\sim w$ if and only if $z=w$ or ($\|z\|=1$, $xy\geq0$ and $w=-i\,\overline{z}$) or ($\|z\|=1$, $xy\leq 0$ and $w=i\,\overline{z}$) or ($z\in\{\pm1,\pm i\}$ and $w\in\{\pm1,\pm i\}$), and I^2 is the closed solid unit square $I^2=\big\{(x,y)\in\mathbb{R}^2\,\big|\,0\leq x\leq1\,,\,0\leq y\leq 1\big\}$ and for $(x,y),(u,v)\in I^2$ we have $(x,y)\approx(u,v)$ if and only if $x-u\in\{0,\pm1\}$ and $y-v\in\{0,\pm1\}$, and where $T^2=\big\{(x,y,z)\in\mathbb{R}^3\,\big|\,16\,(x^2+y^2)=(x^2+y^2+z^2+3)^2\big\}\subseteq\mathbb{R}^3$.

Solution: For $z=x+iy\in\mathbb{C}=\mathbb{R}^2$, we write $\|z\|=\|z\|_2=\sqrt{x^2+y^2}$ and $\|z\|_1=|x|+|y|$. Note that $\|z\|_2=\|x+iy\|_2\leq\|x\|_2+\|iy\|_2=|x|+|y|=\|z\|_1$ and that by the Cauchy-Schwartz Inequality in $\mathbb{R}^2$ we have $\|z\|_1=|x|+|y|=\big\langle(1,1),(|x|,|y|)\big\rangle\leq\|(1,1)\|_2\|(|x|,|y|)\|_2=\sqrt{2}\,\|z\|_2$.

Let $Q^2=\big\{z\in\mathbb{C}\,\big|\,\|z\|_1\leq1\big\}$ and let $\simeq$ be the equivalence relation on Q^2 given, for $z,w\in Q^2$ with $z=x+iy$, by $z\simeq w$ if and only if $z=w$ or ($\|z\|_1=1$, $xy\geq0$ and $w=-i\,\overline{z}$) or ($\|z\|_1=1$, $xy\leq1$ and $w=i\,\overline{w}$). Define $f:D^2\to Q^2$ by $f(0)=0$ and $f(z)=\frac{\|z\|_2}{\|z\|_1}\,z$ for $z\neq0$. Note that f is well-defined because when $\|z\|_2\leq1$ we have $\|f(z)\|_1=\|z\|_2\leq 1$, and f is continuous at 0 because when $z\neq0$ we have $\|f(z)\|_2=\frac{\|z\|_2}{\|z\|_1}\,\|z\|_2\leq\|z\|_2\to0$ as $z\to0$. Define $g:Q^2\to D^2$ by $g(w)=\frac{\|w\|_1}{\|w\|_2}\,w$. Note that g is well-defined since when $\|w\|_1\leq1$ we have $\|g(w)\|_2=\|w\|_1\leq1$ and g is continuous at 0 because for $w\neq0$ we have $\|g(w)\|_2=\|w\|_1\leq\sqrt{2}\,\|w\|_2\to0$ as $w\to0$. It is clear that f and g are inverses of one another (so that $D^2\cong Q^2$), and they respect the two equivalence relations so that they induce well-defined continuous maps $\overline{f}:D^2/\!\sim\;\to Q^2/\!\simeq$ and $\overline{g}:Q^2/\!\simeq\;\to D^2/\!\sim$ given by $\overline{f}\big([z]\big)=\big[f(z)\big]$ and $\overline{g}\big([w]\big)=\big[g(w)\big]$ which are inverses of one another. This proves that $D^2/\!\sim\;\cong\;Q^2/\!\sim$.

It is easy to see that $Q^2/\!\simeq\;\cong\;I^2/\!\approx$ by rotating and scaling. To be explicit, we can use the affine maps $f:I^2\to Q^2$ and $g:Q^2\to I^2$ given by $(u,v)=f(x,y)=\big(x-y\,,\,x+y-1\big)$ and $(x,y)=g(u,v)=\frac{1}{2}\big(v+u+1\,,\,v-u+1\big)$, or equivalently by $w=f(z)=(1+i)z-i$ and $z=h(w)=\frac{w+i}{1+i}$. These maps are continuous, and they are inverses of one another, and they respect the equivalence relations so that they induce well-defined continuous maps on the quotient spaces.

Finally, let us show that $I^2/\approx\ \cong\ T^2$. Define $f_1 : I^2 \to T^2$ by

$$f_1(s,t) = \big((2+\sin 2\pi t)\cos 2\pi s\,,\ (2+\sin 2\pi t)\sin 2\pi s\,,\ \cos 2\pi t\big).$$

Note that f_1 respects the equivalence relation in that when $(s,t)\approx(u,v)$ we have $f_1(s,t)=f_1(u,v)$, and so it induces the well-defined continuous map $f : I^2/\approx\ \to\ T^2$ given by $f\big([(s,t)]\big)=f_1(s,t)$.

We claim that f is injective. Let $p_1,p_2\in I^2/\approx$. Choose $s_1,t_1,s_2,t_2\in I$ such that $p_1=[(s_1,t_1)]$ and $p_2=([s_2,t_2)]$. Let $(x_1,x_2,x_3)=f(p_1)=f_1(s_1,t_1)$ and $(x_1,x_2,x_3)=f(p_2)=f_1(s_2,t_2)$. Suppose $f(p_1)=f(p_2)$ so that $x_1=x_2$, $y_1=y_2$ and $z_1=z_2$. Since $z_1=z_2$ we have $\cos 2\pi t_1=\cos 2\pi t_2$. Since $\sqrt{x_1^2+y_1^2}=\sqrt{x_2^2+y_2^2}$ we have $2+\sin 2\pi t_1=2+\sin 2\pi t_2$ so that $\sin 2\pi t_1=\sin 2\pi t_2$. Since $\cos 2\pi t_1=\cos 2\pi t_2$ and $\sin 2\pi t_1=\sin 2\pi t_2$ we have $t_2-t_1\in\mathbb{Z}$. Since $t_1,t_2\in[0,1]$ with $t_2-t_1\in\mathbb{Z}$, we have $t_2-t_1\in\{0,\pm1\}$. Since $x_1=x_2$, we have $(2+\sin 2\pi t_1)\cos 2\pi s_1=(2+\sin 2\pi t_2)\cos 2\pi s_2$, and since $2+\sin 2\pi t_1$ and $2+\sin 2\pi t_2$ are equal and positive, it follows that $\cos 2\pi s_1=\cos 2\pi s_2$. Similarly, since $y_1=y_2$ it follows that $\sin 2\pi s_1=\sin 2\pi s_2$. Since $\cos 2\pi s_1=\cos 2\pi s_2$ and $\sin 2\pi s_1=\sin 2\pi s_2$, it follows that $s_2-s_1\in\mathbb{Z}$. Since $s_1,s_2\in[0,1]$ with $s_2-s_1\in\mathbb{Z}$, we have $s_2-s_1\in\{0,\pm1\}$. Since $s_2-s_1\in\{0,\pm1\}$ and $t_2-t_1\in\{0\pm1\}$, it follows that $(s_1,t_1)\approx(s_2,t_2)$ so that $p_1=[(s_1,t1)]=[(s_2,t_2)]=p_2$. Thus f is injective, as claimed.

We claim that f is surjective. Let $(x,y,z)\in\mathbb{T}^2$. Since $16(x^2+y^2)=(x^2+y^2+z^2-3)^2$, we have $4\sqrt{x^2+y^2}=x^2+y^2+z^2-3$ so that $(\sqrt{x^2+y^2}\ -2)^2+z^2=1$. Note that $(\sqrt{x^2+y^2}-2)^2=1-z^2\le 1$ so that $-1\le\sqrt{x^2+y^2}-2\le 1$ and hence $1\le\sqrt{x^2+y^2}\le 3$. Since $\sqrt{x^2+y^2}\ge 1$ so that $(x,y)\ne(0,0)$, we can choose $\theta\in[0,2\pi]$ so that $(\cos\theta,\sin\theta)=\big(\frac{x}{\sqrt{x^2+y^2}},\frac{y}{\sqrt{x^2+y^2}}\big)$. Since $(\sqrt{x^2+y^2}-2)^2+z^2=1$, we can choose $\phi\in[0,2\pi]$ so that $(\sin\phi,\cos\phi)=\big(\sqrt{x^2+y^2}-2\,,\,z\big)$. For $s=\frac{\theta}{2\pi}\in[0,1]$ and $t=\frac{\phi}{2\pi}\in[0,1]$, we have $f\big([s,t]\big)=f_1(s,t)\big((2+\sin\phi)\cos\theta\,,\,(2+\sin\phi)\sin\theta\,,\,\cos\phi\big)=(x,y,z)$. Thus f is surjective, as claimed.

Finally, note that since $\mathbb{T}^2$ is compact and $I/\approx$ is Hausdorff and $f : I^2/\approx\to\mathbb{T}^2$ is continuous and bijective, f is a homeomorphism (by Theorem 3.30).

Alternatively, we can find an explicit formula for $g=f^{-1}$ as follows: Let A_k be the four closed subsets of T^2 given by

$$\begin{aligned}A_1&=\big\{(x,y,z)\in T^2\,\big|\,y\ge 0\,,\ x^2+y^2\ge 4\big\}\\ A_2&=\big\{(x,y,z)\in T^2\,\big|\,y\ge 0\,,\ x^2+y^2\le 4\big\}\\ A_3&=\big\{(x,y,z)\in T^2\,\big|\,y\le 0\,,\ x^2+y^2\ge 4\big\}\\ A_4&=\big\{(x,y,z)\in T^2\,\big|\,y\le 0\,,\ x^2+y^2\le 4\big\}.\end{aligned}$$

Let $g_k : A_k\to I^2$ be the continuous maps given by

$$\begin{aligned}g_1(x,y,z)&=\big(\tfrac{1}{2\pi}\cos^{-1}\tfrac{x}{\sqrt{x^2+y^2}}\,,\ \tfrac{1}{2\pi}\cos^{-1}z\big),\\ g_2(x,y,z)&=\big(\tfrac{1}{2\pi}\cos^{-1}\tfrac{x}{\sqrt{x^2+y^2}}\,,\ 1-\tfrac{1}{2\pi}\cos^{-1}z\big),\\ g_3(x,y,z)&=\big(1-\tfrac{1}{2\pi}\cos^{-1}\tfrac{x}{\sqrt{x^2+y^2}}\,,\ \tfrac{1}{2\pi}\cos^{-1}z\big),\\ g_4(x,y,z)&=\big(1-\tfrac{1}{2\pi}\cos^{-1}\tfrac{x}{\sqrt{x^2+y^2}}\,,\ 1-\tfrac{1}{2\pi}\cos^{-1}z\big).\end{aligned}$$

By composing with the quotient map from I^2 to $I^2/\approx$, we obtain the continuous maps $\overline{g}_k : A_k\to I^2/\approx$ given by $\overline{g}_k(x,y,z)=\big[g_k(x,y,z)\big]$. Note that $\overline{g}_k$ and $\overline{g}_\ell$ agree on the intersection $A_k\cap A_\ell$, so they can be combined using the glueing lemma to obtain the well-defined continuous map $g : T^2\to I^2/\sim$ given by $g(x,y,z)=\overline{g}_k(x,y,z)=\big[g_k(x,y,z)\big]$ for all $(x,y,z)\in A_k$. A slightly long calculation shows that f and g are inverses of one another. For example, when $(x,y,z)\in A_3$, we have $y\le 0$ and $x^2+y^2\ge 4$, and since $(x,y,z)\in T^2$, we have $4\sqrt{x^2+y^2}=x^2+y^2+z^2+3$ so that $1-z^2=(\sqrt{x^2+y^2}-2)^2$, and hence

$$\begin{aligned}f\big(g(x,y,z)\big)&=f_1\big(g_3(x,y,z)\big)=f_1\big(1-\tfrac{1}{2\pi}\cos^{-1}\tfrac{x}{\sqrt{x^2+y^2}}\,,\ \tfrac{1}{2\pi}\cos^{-1}z\big)\\ &=\Big(\big(2+\sin(\cos^{-1}z)\big)\cos\big(2\pi-\cos^{-1}\tfrac{x}{\sqrt{x^2+y^2}}\big),\big(2+\sin(\cos^{-1}z)\big)\sin\big(2\pi-\cos^{-1}\tfrac{x}{\sqrt{x^2+y^2}}\big),\cos(\cos^{-1}z)\Big)\\ &=\Big((2+\sqrt{1-z^2}\,)\tfrac{x}{\sqrt{x^2+y^2}}\,,\ (2+\sqrt{1-z^2}\,)\tfrac{-|y|}{\sqrt{x^2+y^2}}\,,\ z\Big)=(x,y,z).\end{aligned}$$

4: (a) Let X be a normal topological space and let $\mathcal{S} = \{U_1, U_2, \cdots, U_\ell\}$ be a finite open cover of X. Prove that there exists a set $\{\rho_1, \rho_2, \cdots, \rho_\ell\}$ of continuous functions $\rho_k : X \to [0,1]$ with $\sum_{k=1}^{\ell} \rho_k(x) = 1$ for all $x \in X$ such that $\overline{\rho_k^{-1}((0,1])} \subseteq U_k$ for all indices k (such a set of functions is called a **partition of unity**, subordinate to the cover).

Solution: First we claim that given any finite open cover $\{U_1, \cdots, U_\ell\}$ of a normal topological space X, there exists another open cover $\{V_1, V_2, \cdots, V_\ell\}$ of X with $\overline{V}_k \subseteq U_k$ for all indices k. Let $n \in \mathbb{Z}$ with $1 \leq n \leq \ell$ and suppose, inductively, that we have constructed open sets $V_1, V_2, \cdots, V_{n-1}$ such that $\{V_1, V_2, \cdots, V_{n-1}, U_n, U_{n+1}, \cdots, U_\ell\}$ is an open cover of X with $\overline{V}_k \subseteq U_k$ for $1 \leq k < n$. Let $S = \left(\bigcup_{k=1}^{n-1} V_k\right) \cup \left(\bigcup_{k=n+1}^{\ell} U_k\right)$ and let $A = S^c = X \setminus S$. Note that A is closed (since S is open), and note that, since $\{V_1, \cdots, V_{n-1}, U_n, U_{n+1}, \cdots, U_\ell\}$ covers X, we have $S \cup U_n = X$ so that $A \cap U_n^c = (S \cup U_n)^c = \emptyset$. Since X is normal we can choose disjoint open sets V_n and W_n with $A \subseteq V_n$ and $U_n^c \subseteq W_n$. Since $X = S \cup A$ and $A \subseteq V_n$, we also have $X = S \cup V_n$ so that $\{V_1, \cdots, V_n, U_{n+1}, \cdots, U_\ell\}$ is an open cover of X. Since $V_n \cap W_n = \emptyset$ we have $V_n \subseteq W_n^c$, which is closed, so that $\overline{V}_n \subseteq W_n^c$. Since $U_n^c \subseteq W_n$ we have $W_n^c \subseteq U_n$, and hence $\overline{V}_n \subseteq W_n^c \subseteq U_n$. By induction, we obtain an open cover $\{V_1, \cdots, V_\ell\}$ of X with $\overline{V}_k \subseteq U_k$ for all k.

Let X be the given normal topological space and let $\mathcal{S} = \{U_1, U_2, \cdots, U_\ell\}$ be the given finite open cover. By the claim, we can choose a second cover $\{V_1, V_2, \cdots, V_\ell\}$ with $\overline{V}_k \subseteq U_k$ for all indices k, and then we can choose a third open cover $\{W_1, W_2, \cdots, W_\ell\}$ of X with $\overline{W}_k \subseteq U_k$ for all indices k. For each $k \in \{1, 2, \cdots, \ell\}$, since $\overline{W}_k \subseteq V_k$ so that $\overline{W}_k \cap V_k^c = \emptyset$, by Urysohn's Lemma we can choose a continuous map $f_k : X \to [0,1]$ with $f_k(x) = 1$ for all $x \in \overline{W}_k$ and $f_k = 0$ for all $x \in V_k^c$. Define $g : X \to \mathbb{R}$ by $g(x) = \sum_{k=1}^{\ell} f_k(x)$. Note that, since the sets W_k cover X, we have $g(x) \neq 0$ for all $x \in X$ (indeed, given $x \in X$ we can choose an index k so that $x \in W_k$, and then we have $f_k(x) = 1$ hence $g(x) = \sum_{j=1}^{\ell} f_j(x) \geq f_k(x) = 1$). For each index k, define $\rho_k : X \to \mathbb{R}$ by $\rho(x) = \frac{f_k(x)}{g(x)}$, and note that ρ_k is well-defined and continuous since f_k and g are continuous with $g(x) \neq 0$ for all $x \in X$. Finally, note that $\sum_{k=1}^{\ell} \rho(x) = 1$ for all $x \in X$ and that for all indices k, since $\rho_k(x) = 0$ for all $x \notin V_k$, we have $\rho_k^{-1}(0,1] \subseteq V_k$ so that $\overline{\rho_k^{-1}(0,1]} \subseteq \overline{V}_k \subseteq U_k$.

(b) Let X be a compact n-manifold. Prove that X is homeomorphic to a subspace of $\mathbb{R}^m$ for some $m \in \mathbb{Z}^+$.

Solution: Since X is an n-manifold, we can choose an open cover $\mathcal{S}$ of X such that each $U \in \mathcal{S}$ is homeomorphic to an open set in $\mathbb{R}^n$. Since X is compact, we can choose a finite cover, say $\{U_1, U_2, \cdots, U_\ell\}$. For each index k, choose a homeomorphism $\phi_k : U_k \subseteq X \to \phi_k(U_k) \subseteq \mathbb{R}^n$ with each set $\phi_k(U_k)$ open in $\mathbb{R}^n$. Since every n-manifold is metrizable, hence normal, using the result of Part (a) we can choose continuous maps $\rho_k : X \to \mathbb{R}$ with $\sum_{k=1}^{\ell} \rho_k(x) = 1$ for all x such that $\overline{\rho^{-1}(0,1]} \subseteq U_k$. Let $f : X \to \mathbb{R}^{\ell + \ell n}$ be the continuous map given by $f = \left(\rho_1, \cdots, \rho_\ell, \rho_1\phi_1, \cdots, \rho_\ell\phi_\ell\right)$.

We claim that f is injective. Let $a, b \in X$ with $f(a) = f(b)$. Since $\rho_k(a) \geq 0$ for all k and $\sum_{k=1}^{n} \rho_k(a) = 1$, we can choose k so that $\rho_k(a) > 0$. Since $f(a) = f(b)$, we have $\rho_k(b) = \rho_k(a)$. Since $\rho_k(a) \neq 0$ and $\rho_k(b) \neq 0$ and $\rho_k(x) = 0$ for all $x \notin U_k$, we have $a, b \in U_k$. Since $f(a) = f(b)$ we have $\rho_k(a)\phi_k(a) = \rho_k(b)\phi_k(b)$, and hence $\phi_k(a) = \phi_k(b)$ (since $\rho_k(a) = \rho_k(b) > 0$). Since ϕ_k is injective, we have $a = b$. Thus f is injective, as claimed.

Since f is continuous and injective, the map $f : X \to f(X) \subseteq \mathbb{R}^{\ell + \ell n}$ is continuous and bijective. Since X is compact and $\mathbb{R}^{\ell + n\ell}$ is Hausdorff, this map $f : X \to f(X)$ is a homeomorphism (by Theorem 3.30).