PMATH 367 Topology, Assignment 5 Due Dec 2, 11:00 pm

1: By counting vertices and determining orientation, determine the topological type (the homeomorphism class) of the polygons with edges identified in pairs according to the following words.

(a) $a\,b\,c\,a^{-1}d\,e\,b\,d^{-1}c\,e^{-1}$

(b) $a\,b\,c\,a^{-1}d^{-1}c^{-1}e\,b^{-1}d\,f\,e^{-1}f^{-1}$.

(c) $a_1a_2\cdots a_n{a_1}^{-1}{a_2}^{-1}\cdots{a_n}^{-1}$

2: (a) Determine the topological type of the space obtained from the disjoint union of four squares with edges identified in pairs according to the words $a\,b\,c\,d$, $e\,f\,g\,h$, $a\,c\,e\,g$ and $b\,d\,f\,h$.

(b) Two disjoint closed squares are chosen on a sphere, their interiors are removed, and then the edges of one are identified with the edges of the other according to $a\,b\,c\,d$ (listed clockwise, from the outside) on the first and $a\,d\,b\,c$ (again listed clockwise) on the second. Determine the topological type of the resulting surface.

(c) Carry out the cut-and-paste algorithm to determine the topological type of the octagon with its edges identified in pairs according to the word $a\,b\,c\,a^{-1}d\,b^{-1}d\,c$.

3: (a) Prove that $(\mathbb{R}^3\setminus\{0\})/\mathbb{R}^* \cong \mathbb{S}^2/\{\pm1\} \cong D^2/\sim$ where D^2 is the closed unit disc $D^2=\big\{z\in\mathbb{C}\,\big|\,\|z\|\leq 1\big\}$ and for $z,w\in D^2$ we have $z\sim w$ if and only if $z=w$ or ($\|z\|=1$ and $w=-z$).

(b) Prove that $D^2/\sim\,\cong I^2/\approx\,\cong T^2$ where D^2 is the closed unit disc $D^2=\big\{z\in\mathbb{C}\,\big|\,\|z\|\leq 1\big\}$ and for $z,w\in D^2$ with $z=x+iy$ and $w=u+iv$, we have $z\sim w$ if and only if $z=w$ or ($\|z\|=1$, $xy\geq 0$ and $w=-i\,\overline{z}$) or ($\|z\|=1$, $xy\leq 0$ and $w=i\,\overline{z}$) or ($z\in\{\pm1,\pm i\}$ and $w\in\{\pm1,\pm i\}$), and I^2 is the closed solid unit square $I^2=\big\{(x,y)\in\mathbb{R}^2\,\big|\,0\leq x\leq 1\,,\,0\leq y\leq 1\big\}$ and for $(x,y),(u,v)\in I^2$ we have $(x,y)\approx(u,v)$ if and only if $x-u\in\{0,\pm1\}$ and $y-v\in\{0,\pm1\}$, and where $T^2=\big\{(x,y,z)\in\mathbb{R}^3\,\big|\,16\,(x^2+y^2)=(x^2+y^2+z^2+3)^2\big\}\subseteq\mathbb{R}^3$.

4: (a) Let X be a normal topological space and let $\mathcal{S}=\{U_1,U_2,\cdots,U_\ell\}$ be a finite open cover of X. Prove that there exists a set $\{\rho_1,\rho_2,\cdots,\rho_\ell\}$ of continuous functions $\rho_k:X\to[0,1]$ with $\sum_{k=1}^{\ell}\rho_k(x)=1$ for all $x\in X$ such that $\overline{\rho_k^{-1}((0,1])}\subseteq U_k$ for all indices k (such a set of functions is called a **partition of unity**, subordinate to the cover). Hint: show that there is an open cover $\{V_1,\cdots,V_\ell\}$ with $\overline{V}_k\subseteq U_k$, use Urysohn's Lemma to obtain suitable functions $f_k:X\to[0,1]$, and let $\rho_k=f_k\big/\sum_{i=1}^{\ell}f_i$.

(b) Let X be a compact n-manifold. Prove that X is homeomorphic to a subspace of $\mathbb{R}^m$ for some $m\in\mathbb{Z}^+$. Hint: cover X by coordinate neighbourhoods $U_1,\cdots,U_\ell$ with homeomorphisms $\phi_k:U_k\subseteq X\to\phi_k(U_k)\subseteq\mathbb{R}^n$, let $\rho_k:X\to[0,1]$ be as in Part (a), then define $f:X\to\mathbb{R}^{\ell+\ell n}$ by $f=\big(\rho_1,\cdots,\rho_\ell,\rho_1\phi_1,\cdots,\rho_\ell\phi_\ell\big)$.