PMATH 367 Topology, Solutions to Assignment 4

1: (a) Let $\mathbb{R}_{cf}$ be the set $\mathbb{R}$ using the co-finite topology. Determine whether $\mathbb{R}_{cf}$ is first countable, whether it is second countable, whether it is Lindelöff, and whether it is separable.

Solution: We claim that $\mathbb{R}_{cf}$ is not first countable, hence also not second countable. Suppose, for a contradiction, that $\mathbb{R}_{cf}$ is first countable. Let $a \in \mathbb{R}$ and choose an (at most) countable set $\mathcal{B}$ of open sets in $\mathbb{R}_{cf}$ such that for every open U in $\mathbb{R}_{cf}$ with $a \in U$ there exists $B \in \mathcal{B}$ with $a \in B \subseteq U$. Write $\mathcal{B} \setminus \{X\} = \{B_1, B_2, \cdots\}$. Since B_k^c is finite for every $k \geq 1$, the set $\bigcup_{k\geq 1} B_k^c$ is at most countable, so its complement $\left(\bigcup_{k\geq 1} B_k^c\right)^c = \bigcap_{k\geq 1} B$ is infinite. Choose $b \in \bigcap_{k\geq 1} B_k$ with $b \neq a$. Then $\{b\}^c$ is an open set in $\mathbb{R}_{cf}$ with $a \in \{b\}^c$, but there is no element $B \in \mathcal{B}$ with $a \in B \subseteq \{b\}^c$.

We claim that $\mathbb{R}_{cf}$ is compact, hence Lindelöff. Let $\mathcal{S}$ be an open cover of $\mathbb{R}_{cf}$. Choose $U_0 \in \mathcal{S}$. If $U_0 = X$ then $\{U_0\}$ is a finite subcover of $\mathcal{S}$. Suppose that $U_0 \neq X$. Then U_0^c is finite, say $U_0 = \{a_1, a_2, \cdots, a_n\}$. For each $k \in \{1, 2, \cdots, n\}$, choose $U_k \in \mathcal{S}$ with $a_k \in U_k$. Then $\{U_0, U_1, \cdots, U_k\}$ is a finite subcover of $\mathcal{S}$.

Finally, we claim that $\mathbb{R}_{cf}$ is separable. Let $A \subseteq \mathbb{R}$ be any infinite, countable set. Since $A \subseteq \overline{A}$, it follows that $\overline{A}$ is also infinite. Since $\mathbb{R}$ is the only infinite closed set in $\mathbb{R}_{cf}$, we have $\overline{A} = \mathbb{R}$.

(b) Let $\mathbb{R}_\ell$ be the set $\mathbb{R}$ using the lower limit topology. Prove that $\mathbb{R}$ is Lindelöff.

Solution: Let $\mathcal{S}$ be an open cover of $\mathbb{R}_\ell$. For each $a \in \mathbb{R}$ choose $U_a \in \mathcal{S}$ with $a \in U_a$ then choose $b_a > a$ so that $[a, b_a) \subseteq U_a$. Note that $\left\{[a, b_a) \,\middle|\, a \in \mathbb{R}\right\}$ is an open cover of $\mathbb{R}_\ell$. Let $V = \bigcup_{a\in\mathbb{R}}(a, b_a)$. We claim that $V^c = \mathbb{R}\setminus V$ is (at most) countable. For each $a \in V^c$, choose $r_a \in (a, b_a) \cap \mathbb{Q}$. Define $f : V^c \to \mathbb{Q}$ by $f(a) = r_a$. Note that f is injective because if $a_1, a_2 \in V^c$ with $a_1 < a_2$ then we must have $r_{a_1} < r_{a_2}$ since otherwise we would have $a_1 < a_2 < r_{a_2} \leq r_{a_1}$ so that $a_2 \in (a_1, r_{a_1}) \subseteq V$. Since $f : V^c \to \mathbb{Q}$ is injective, it follows that V^c is (at most) countable, as claimed.

Let $\mathbb{R}_s$ be the set $\mathbb{R}$ using its standard topology, and let V_s be the set V using the standard subspace topology in $\mathbb{R}_s$. Since $\mathbb{R}_s$ is second countable, so is V_s, and hence V_s is Lindelöff. Since $\left\{(a, b_a) \,\middle|\, a \in \mathbb{R}\right\}$ is an open cover of V_s, we can choose an (at most) countable subcover, so we can choose an (at most) countable set $C \subseteq \mathbb{R}$ so that $V = \bigcup\left\{(a, b_a) \,\middle|\, a \in C\right\}$. It follows that $V \subseteq \bigcup\left\{[a, b_a) \,\middle|\, a \in C\right\}$. On the other hand, V^c is (at most) countable, and of course $V^c \subseteq \bigcup\left\{[a, b_a) \,\middle|\, a \in V^c\right\}$. Thus $\mathbb{R} = V \cup V^c = \bigcup\left\{[a, b_a) \,\middle|\, a \in C \cup V^c\right\}$, and hence $\left\{U_a \,\middle|\, a \in C \cup V^c\right\}$ is an (at most) countable subcover of $\mathcal{S}$.

(c) Show that the Moore plane Γ is not Lindelöff.

Solution: Let U be the open upper half plane $U = \left\{(a, b) \,\middle|\, b > 0\right\}$ and note that U is open in Γ (because given $(a, b) \in U$ we have $(a, b) \in B\big((a, b), b\big)$, which is a basic open set). For each $a \in \mathbb{R}$, let V_a be the basic open set $V_a = B\big((a, 1), 1\big) \cup \left\{(a, 0)\right\}$. Then $\mathcal{S} = \left\{V_a \,\middle|\, a \in \mathbb{R}\right\} \cup \{U\}$ is an uncountable open cover of Γ which has no proper subcover (because the point $(0, 2)$ only lies in U and, for each $a \in \mathbb{R}$, the point $(a, 0)$ only lies in V_a).

2: (a) Show that the image of a separable space under a continuous map is separable and that the image of a Lindelöff space under a continuous map is Lindelöff.

Solution: Let $f : X \to Y$ be a continuous map between topological spaces. Suppose that X is separable. Let A be an (at most) countable dense subset of X. Let $B = f(A) = \{f(a) \,|\, a \in A\}$, and note that B is also (at most) countable. Let V be any open set in $f(X)$. Then $f^{-1}(V)$ is open in X. Since A is dense in X, we have $f^{-1}(V) \cap A \neq \emptyset$. We can choose $x \in f^{-1}(V) \cap A$, and then $f(x) \in V \cap f(A) = V \cap B$ so that $V \cap B \neq \emptyset$. Since $V \cap B \neq \emptyset$ for every open set V in $f(X)$, it follows that B is dense in $f(X)$.

Now suppose that X is Lindelöff. Let $\mathcal{T}$ be an open cover of $f(X)$. Let $\mathcal{S} = \{f^{-1}(V) \,|\, V \in \mathcal{S}\}$ and note that $\mathcal{S}$ is an open cover of X. Since X is Lindelöff, $\mathcal{S}$ has an (at most) countable subcover, so we can choose $V_1, V_2, \cdots$ in $\mathcal{T}$ such that $X = \bigcup_{k \geq 1} f^{-1}(V_k)$. It follows that $f(X) = \bigcup_{k \geq 1} V_k$ so that $\{V_1, V_2, \cdots\}$ is an (at most) countable subcover of $\mathcal{T}$.

(b) Show that the image under a continuous map of a first or second countable space need not be first or second countable.

Solution: Let $\mathbb{R}_s$ be the set $\mathbb{R}$ using its standard topology and let $\mathbb{R}_{cf}$ be the set $\mathbb{R}$ using the cofinite topology. The identity map $I : \mathbb{R}_s \to \mathbb{R}_{cf}$ is continuous since, if A is a proper closed subset of $\mathbb{R}_{cf}$, then A is finite, and $I^{-1}(A) = A$, and finite sets are closed in $\mathbb{R}_s$. Athough I is continuous, and $\mathbb{R}_s$ is second countable (hence also first countable), the image $I(\mathbb{R}_s) = \mathbb{R}_{cf}$ is not first countable (hence also not second countable) by Problem 1(a).

(c) A map $f : X \to Y$ between topological spaces is called **open** when $f(U)$ is open in Y for every open set U in X. Show that the image of a first countable space under an open continuous map is first countable, and the image of a second countable space under an open continuous map is second countable.

Solution: Let $f : X \to Y$ be open and continuous. Note that when we restrict the codomain of f, the resulting map $f : X \to f(X)$ is also continuous and open: it is continuous by Theorem 1.35, and it is open since when U is open in X, $f(U)$ is open in Y and $f(U) \subseteq f(X)$ so that $f(U) = f(U) \cap f(X)$, which is open in $f(X)$ using the subspace topology.

Suppose that X is first countable. Let $b \in f(X)$. Choose $a \in X$ such that $f(a) = b$. Since X is first countable, we can choose an (at most) countable local basis $\mathcal{B}$ for X at a (meaning that for every open set U in X with $a \in U$ there exists $B \in \mathcal{B}$ with $a \in B \subseteq U$). Let $\mathcal{C} = \{f(B) \,|\, B \in \mathcal{B}\}$ and note that $\mathcal{C}$ is at most countable (because the map $F : \mathcal{B} \to \mathcal{C}$ given by $F(B) = f(B)$ is surjective), and that each element $C = f(B)$ is open in $f(X)$ because the map $f : X \to f(X)$ is open. We claim that $\mathcal{C}$ is a local basis for $f(X)$ at b. Let V be an open set in $f(X)$ with $b \in V$. Let $U = f^{-1}(V)$, and note that U is open in X since $f : X \to f(X)$ is continuous, and that $a \in U = f^{-1}(V)$ because $f(a) = b \in V$. Since $\mathcal{B}$ is a local basis for X at a, we can choose $B \in \mathcal{B}$ such that $a \in B \subseteq U$. Let $C = f(B) \in \mathcal{C}$. Then we have $b = f(a) \in f(B) = C$ and $C = f(B) \subseteq f(U) = V$. Thus $\mathcal{C}$ is a local basis for Y at b, as claimed.

The proof for second countability is similar. Suppose that X is second countable and let $\mathcal{B}$ be an (at most) countable basis for the topology on X. Let $\mathcal{C} = \{f(B) \,|\, B \in \mathcal{B}\}$. Note that $\mathcal{C}$ is at most countable (because $F : \mathcal{B} \to \mathcal{C}$ given by $F(B) = f(B)$ is surjective) and that each element $C = f(B) \in \mathcal{C}$ is open in $f(X)$ (because the map $f : X \to f(X)$ is open). We claim that $\mathcal{C}$ is a basis for the topology on $f(X)$ (using the subspace topology in Y). Let V be an open set in $f(X)$ and let $b \in V$. Choose $a \in X$ such that $f(a) = b$. Let $U = f^{-1}(V)$ and note that $a \in U$ (since $f(a) = b \in V$) and U is open in X (since the map $f : X \to f(X)$ is continuous). Since $\mathcal{B}$ is a basis for the topology on X, we can choose $B \in \mathcal{B}$ such that $a \in B \subseteq U$. Let $C = f(B) \in \mathcal{C}$. Since $a \in B$ we have $b = f(a) \in f(B) = C$ and since $B \subseteq U$ we have $C = f(B) \subseteq f(U) = V$. Thus $\mathcal{C}$ is a basis for the topology on $f(X)$, as claimed.

3: (a) A topological space X is **locally compact** when for every $a \in X$ there is a compact subspace K of X and an open set U in X with $a \in U \subseteq K$. Show that every locally compact Hausdorff space is regular.

Solution: Let X be a locally compact Hausdorff space. Let $a \in X$ and let B be a closed set in X with $a \notin B$. Since X is locally compact, we can choose a compact set K in X and an open set W in X with $a \in W \subseteq K$. Since B is closed in X, $B \cap K$ is closed in K. Since $B \cap K$ is closed in K and K is compact, it follows that $B \cap K$ is compact. For each $b \in B \cap K$, since X is Hausdorff we can choose disjoint open sets U_b and V_b in X with $a \in U_b$ and $B \subseteq V_b$. The set $\left\{V_b \,\middle|\, b \in B \cap K\right\}$ is an open cover of $B \cap K$, which is compact, so we can choose $b_1, b_2, \cdots, b_n \in B \cap K$ so that $B \cap K \subseteq \bigcup_{k=1}^n V_{b_k}$. Let $U = \bigcap_{k=1}^n U_{b_k} \cap W$ and let $V = \bigcup_{k=1}^n V_{b_k} \cup K^c$, where $K^c = X \setminus K$. Note that since K is a compact subset of the Haudorff space X, it follows that K is closed so that K^c is open, and so the sets U and V are open in X. Note that $a \in U$ because $a \in W$ and $a \in U_{b_k}$ for every k. Note that $B \subseteq V$ because $B \subseteq (B \cap K) \cup K^c \subseteq \bigcup_{k=1}^n V_{b_k} \cup K^c = V$. And note that $U \cap V = \emptyset$ because if $x \in V$ the either $x \in K^c$ in which case $x \notin W$ hence $x \notin U$, or else $x \in V_{b_k}$ for some k and then $x \notin U_{b_k}$ so that $x \notin U$.

(b) Show that every regular Lindelöff space is normal.

Solution: We immitate the proof of Theorem 4.26. Let X be a regular Lindelöff space. Let A and B be disjoint closed sets in X. For each $a \in A$, since X is regular, and $a \in B^c$ which is open, we can choose an open set C_a in X with $a \in C_a \subseteq \overline{C}_a \subseteq B^c$. Since $\overline{C_a} \subseteq B^c$, we have $\overline{C_a} \cap B = \emptyset$. The set $\left\{C_a \,\middle|\, a \in A\right\} \cup \{A^c\}$ is an open cover of X so, since X is Lindelöff, we can choose $a_1, a_2, \cdots$ in X such that $A \subseteq \bigcup_{k \geq 1} C_{a_k}$. Similarly, for each $b \in B$ we can choose an open set D_b in X with $b \in D_b$ and with $\overline{D_b} \cap A = \emptyset$, and we can choose $b_1, b_2, \cdots$ in B so that $B \subseteq \bigcup_{k \geq 1} D_{b_k}$. As in the proof of Theorem 4.26, we let $U = \bigcup_{n \geq 1} U_n$ where $U_n = C_{a_n} \setminus \bigcup_{k=1}^n \overline{D_{b_n}}$ and let $V = \bigcup V_n$ where $V_n = D_{b_n} \setminus \bigcup_{k=1}^n \overline{c_{a_k}}$. Then U and V are disjoint open sets in X with $A \subseteq U$ and $B \subseteq V$.

(c) Show that the Moore plane Γ is not normal.

Solution: In the Moore plane Γ, the subspace topology on the line $L = \left\{(a, 0) \,\middle|\, a \in \mathbb{R}\right\}$ is the discrete topology since for every $a \in \mathbb{R}$ we have $\big(B\big((a, r), r\big) \cup \{(a, 0)\}\big) \cap L = \{(a, 0)\}$ so that the 1-point set $\{(a, 0)\}$ is open in L using the subspace topology. Also, the line L is closed in Γ since $L^c = \left\{(a, b) \middle| b > 0\right\}$, which is open in Γ, because it is the union of the basic open sets $B\big((a, b), r\big)$ with $0 < r < b$. Since the subspace topology on L is the discrete topology, and L is closed in Γ, it follows that every subset $A \subseteq L$ is closed in Γ.

Suppose, for a contradiction, that Γ is normal. For each $A \subseteq L$, note that A and $L \setminus A$ are disjoint closed subsets of Γ, so we can choose disjoint open sets U_A and V_A in Γ with $A \subseteq U_A$ and $L \setminus A \subseteq V_A$. Define $F : \mathcal{P}(L) \to \mathcal{P}(\mathbb{Q}^2)$ (where $\mathcal{P}(X)$is the set of subsets of X) by $F(A) = U_A \cap \mathbb{Q}^2$. We claim that F is injective. Let $A, B \subseteq L$ with $A \neq B$. One of the sets A and B contains a point which is not contained in the other, say $p \in A \setminus B$ (the case $p \in B \setminus A$ is similar). Since $p \in A$ we have $p \in U_A$ and since $p \in L \setminus B$ we have $p \in V_B$, so that $p \in U_A \cap V_B$. Since $\mathbb{Q}^2$ is dense in Γ (because every basic open set contains points in $\mathbb{Q}^2$) and since $U_A \cap V_B$. is a nonempty open set in Γ, we can choose $q \in U_A \cap V_B \cap \mathbb{Q}^2$. Note that since $q \in V_B$ and $U_B \cap V_B = \emptyset$, we have $q \notin U_B$. Thus $q \in U_A \cap \mathbb{Q}^2 = F(A)$ but $q \notin U_B \cap \mathbb{Q}^2 = F(B)$, and hence $F(A) \neq F(B)$. This proves that F is injective, as claimed. Since $F : \cap P(L) \to \mathcal{P}(\mathbb{Q}^2)$ is injective, it follows that $|\mathcal{P}(L)| \leq |\mathcal{P}(\mathbb{Q}^2)|$. This is not possible since $|\mathcal{P}(L)| = 2^{|L|} = 2^{2^{\aleph_0}}$ but $|\mathcal{P}(\mathbb{Q}^2)| = 2^{|\mathbb{Q}^2|} = 2^{\aleph_0}$.

4: (a) For each $k \in \mathbb{Z}^+$, let X_k be a metric space with metric d_k such that $d_k(x_k, y_k) \leq 1$ for all $x_k, y_k \in X_k$. For $x, y \in \prod_{k=1}^{\infty} X_k$, define $d(x,y) = \sup\left\{\frac{d_k(x_k,y_k)}{k} \,\middle|\, k \in \mathbb{Z}^+\right\}$. Show that d is a metric on $\prod_{k=1}^{\infty} X_k$ which induces the product topology.

Solution: To verify that d is a metric on $\prod_{k=1}^{\infty} X_k$, let $x, y, z \in \prod_{k=1}^{\infty} X_k$. It is clear that $d(x,y) = d(y,x)$ and that $d(x,y) \geq 0$ with $d(x,y) = 0 \iff x = y$. For each $k \in \mathbb{Z}^+$, since $d_k(x_k, z_k) \leq d_k(x_k, y_k) + d_k(y_k, z_k)$, we have $\frac{d(x_k,z_k)}{k} \leq \frac{d_k(x_k,y_k)}{k} + \frac{d_k(y_k,z_k)}{k} \leq d(x,y) + d(y,z)$, and hence $d(x,z) = \sup\left\{\frac{d_k(x_k,z_k)}{k}\right\} \leq d(x,y) + d(y,z)$. Thus d is a metric on $\prod_{k=1}^{\infty} X_k$.

Let $W \subseteq \prod_{k=1}^{\infty} X_k$. Suppose that W is open using the metric topology. Let $a \in W$. Choose $r > 0$ so that $B(a,r) \subseteq W$. Choose $m \in \mathbb{Z}^+$ with $\frac{1}{m} < r$. Let U be the basic open set $U = \prod_{k=1}^{\infty} U_k$ where $U_k = B_k(a_k, kr) \subseteq X_k$ for $1 \leq k \leq m$ and $U_k = X_k$ for $k > m$. Let $x \in U$. For $1 \leq k \leq m$, since $x_k \in B_k(a_k, kr)$, we have $d_k(x_k, a_k) < kr$ so that $\frac{d_k(x_k,a_k)}{k} < r$. For $k > m$, since $d_k(x_k, a_k) \leq 1$ we have $\frac{d_k(x_k,a_k)}{k} \leq \frac{1}{k} < \frac{1}{m}$. It follows that $d(x,a) = \sup\left\{\frac{d_k(x_k,a_k)}{k} \,\middle|\, k \in \mathbb{Z}^+\right\} < r$ so that $x \in B(a,r)$. Thus we have $a \in U \subseteq B(a,r) \subseteq W$. This shows that if W is open using the metric topology, then W is open using the product topology.

Now suppose that W is open using the product topology. Let $a \in W$. Choose a basic open set U in $\prod_{k=1}^{\infty} X_k$ with $a \in U \subseteq W$. Say $U = \prod_{k=1}^{\infty} U_k$ where each U_k is open in X_k (using the metric d_k) with $a_k \in U_k$, and we have $U_k = X_k$ for all but finitely many $\mathbb{Z}^+$. Choose $m \in \mathbb{Z}^+$ so that $U_k = X_k$ for all $k > m$. For $1 \leq k \leq m$, choose $r_k > 0$ so that $B_k(a_k, r_k) \subseteq U_k$. Let $r = \min\left\{\frac{r_1}{1}, \frac{r_2}{2}, \cdots, \frac{r_m}{m}\right\}$. Let $x \in B(a,r)$. Note that since $d(x,a) = \sup\left\{\frac{d_k(x_k,a_k)}{k}\right\} < r$ we have $\frac{d_k(x_k,a_k)}{k} < r$ for all $k \in \mathbb{Z}^+$. For $1 \leq k \leq m$, since $\frac{d_k(x_k,a_k)}{k} < r \leq \frac{r_k}{k}$ we have $d_k(x_k, a_k) < r$ so that $x_k \in B_k(x_k, a_k) \subseteq U_k$. For $k > m$, since $U_k = X_k$ we have $x_k \in U_k$. Since $x_k \in U_k$ for all $k \in \mathbb{Z}^+$, we have $x \in U$. Thus $B(a,r) \subseteq U \subseteq W$. This shows that if W is open using the product topology, then W is open using the metric topology.

(b) Recall that the limit of a sequence in a topological space was defined in Question 4 of Assignment 1. Recall, also, that when X is a metric space, and $A \subseteq X$, and $a \in X$, we have $a \in \overline{A}$ if and only if there is a sequence $(x_n)_{n\geq 1}$ in A with $\lim_{n\to\infty} x_n = a$.

Let $A = \left\{x \in \mathbb{R}^\omega \,\middle|\, \forall\, k \in \mathbb{Z}^+\ x_k > 0\right\}$. Show that when $\mathbb{R}^\omega$ uses the box topology, we have $0 \in \overline{A}$, but there is no sequence $(x_n)_{n\geq 1}$ in A with $\lim_{n\to\infty} x_n = 0$ (and hence $\mathbb{R}^\omega$ is not metrizable, using the box topology).

Solution: To show that $0 \in \overline{A}$, let U be a basic open set in $\mathbb{R}^\omega$ (using the box topology) with $0 \in U$, say $U = \prod_{k=1}^{\infty} U_k$ where each U_k is open in $\mathbb{R}$ with $0 \in U_k$. For each k we can choose $r_k > 0$ such that $(-2r_k, 2r_k) \subseteq U_k$, then for $r = (r_1, r_2, \cdots)$ we have $r \in U \cap A$ so that $U \cap A \neq \emptyset$. This proves that $0 \in \overline{A}$.

Let $(x_n)_{n\geq 1}$ be any sequence in A. For each $k \in \mathbb{Z}^+$, note that $x_{k,k} > 0$ (since $x_k \in A$) and let $U_k = (-x_{k,k}, x_{k,k}) \subseteq \mathbb{R}$. Then $U = \prod_{k=1}^{\infty} U_k$ is a basic open set in $\mathbb{R}^\infty$ and $x_n \notin U$ for any $n \in \mathbb{Z}^+$ (since $x_{n,n} \notin U_n$). Since U is open in $\mathbb{R}^\omega$ with $0 \in U$ and $x_n \notin U$ for any $n \in \mathbb{Z}^+$, it follows (from the definition of a limit) that $x_n \not\to 0$ in $\mathbb{R}^\omega$.

(c) Let K be an uncountable set. Let $A = \left\{x \in \mathbb{R}^K \,\middle|\, x_k = 1 \text{ for all but finitely many } k \in K\right\}$. Show that when $\mathbb{R}^K$ uses the product topology, we have $0 \in \overline{A}$ but there is no sequence $(x_n)_{n\geq 1}$ in A with $\lim_{n\to\infty} x_n = 0$ (and hence $\mathbb{R}^K$ is not metrizable using the product topology).

Solution: To show that $0 \in \overline{A}$, let U be a basic open set in $\mathbb{R}^K$ (using the product topology) with $0 \in U$, say $U = \prod_{k\in K} U_k$ where each U_k is open in $\mathbb{R}$ with $0 \in U_k$ and $U_k = \mathbb{R}$ for all $k \notin J$ where $J \subseteq K$ is finite. Let $a \in \mathbb{R}^K$ be the element with $a_k = 0$ for all $k \in J$ and $a_k = 1$ for all $k \notin J$. Then we have $a \in U \cap A$ so that $U \cap A \neq \emptyset$. Since $U \cap A \neq \emptyset$ for every basic open set U in $\mathbb{R}^K$ with $0 \in U$, it follows that $0 \in \overline{A}$.

Let $(x_n)_{n\geq 1}$ be any sequence in A. For each $n \in \mathbb{Z}^+$, let $J_n = \left\{k \in K \,\middle|\, x_{n,k} \neq 1\right\}$ and note that J_n is finite since $x_n \in A$. Let $J = \bigcup_{n=1}^{\infty} J_n$ and note that J is (at most) countable. Since K is uncountable, we can choose $\ell \in K$ with $\ell \notin J$. For every $n \in \mathbb{Z}^+$, since $\ell \notin J$, we have $\ell \notin J_n$, and hence $x_{n,\ell} = 1$. Let U be the basic open set $U = \prod_{k\in K} U_k$ where $U_\ell = (-1,1) \subseteq \mathbb{R}$ and $U_k = \mathbb{R}$ for $k \neq \ell$. For every $n \in \mathbb{Z}^+$, since $x_{n,\ell} = 1 \notin U_\ell$, we have $x_n \notin U$. Since U is open in $\mathbb{R}^K$ with $0 \in U$ and $x_n \notin U$ for any $n \in \mathbb{Z}^+$, it follows (from the definition of a limit) that $x_n \not\to 0$ in $\mathbb{R}^K$.