PMATH 367 Topology, Assignment 4 Due Nov 18, 11:00 pm

1: (a) Let $\mathbb{R}_{cf}$ be the set $\mathbb{R}$ using the co-finite topology. Determine whether $\mathbb{R}_{cf}$ is first countable, whether it is second countable, whether it is Lindelöff, and whether it is separable.

(b) Let $\mathbb{R}_\ell$ be the set $\mathbb{R}$ using the lower limit topology. Prove that $\mathbb{R}$ is Lindelöff.

Hint: let $\mathcal{S}$ be an open cover of $\mathbb{R}_\ell$. For each $a \in \mathbb{R}$ choose $U_a \in \mathcal{S}$ with $a \in U_a$ then choose $b_a > a$ so that $[a, b_a) \subseteq U_a$. Let $V = \bigcup_{a\in\mathbb{R}}(a, b_a)$ and show that $V^c = \mathbb{R}\setminus V$ is (at most) countable.

(c) Show that the Moore plane Γ is not Lindelöff.

2: (a) Show that the image of a separable space under a continuous map is separable and that the image of a Lindelöff space under a continuous map is Lindelöff.

(b) Show that the image under a continuous map of a first or second countable space need not be first or second countable.

(c) A map $f : X \to Y$ between topological spaces is called **open** when $f(U)$ is open in Y for every open set U in X. Show that the image of a first countable space under an open continuous map is first countable, and the image of a second countable space under an open continuous map is second countable.

3: (a) A topological space X is **locally compact** when for every $a \in X$ there is a compact subspace K of X and an open set U in X with $a \in U \subseteq K$. Show that every locally compact Hausdorff space is regular.

(b) Show that every regular Lindelöff space is normal.

(c) Show that the Moore plane Γ is not normal.

4: (a) For each $k \in \mathbb{Z}^+$, let X_k be a metric space with metric d_k such that $d_k(x_k, y_k) \leq 1$ for all $x_k, y_k \in X_k$. For $x, y \in \prod_{k=1}^\infty X_k$, define $d(x, y) = \sup\left\{\frac{d_k(x_k,y_k)}{k} \,\middle|\, k \in \mathbb{Z}^+\right\}$. Show that d is a metric on $\prod_{k=1}^\infty X_k$ which induces the product topology.

(b) Recall that the limit of a sequence in a topological space was defined in Question 4 of Assignment 1. Recall, also, that when X is a metric space, and $A \subseteq X$, and $a \in X$, we have $a \in \overline{A}$ if and only if there is a sequence $(x_n)_{n\geq 1}$ in A with $\lim_{n\to\infty} x_n = a$.

Let $A = \left\{x \in \mathbb{R}^\omega \,\middle|\, \forall\, k \in \mathbb{Z}^+\ x_k > 0\right\}$. Show that when $\mathbb{R}^\omega$ uses the box topology, we have $0 \in \overline{A}$, but there is no sequence $(x_n)_{n\geq 1}$ in A with $\lim_{n\to\infty} x_n = 0$ (and hence $\mathbb{R}^\omega$ is not metrizable, using the box topology).

(c) Let K be an uncountable set. Let $A = \left\{x \in \mathbb{R}^K \,\middle|\, x_k = 1 \text{ for all but finitely many } k \in K\right\}$. Show that when $\mathbb{R}^K$ uses the product topology, we have $0 \in \overline{A}$ but there is no sequence $(x_n)_{n\geq 1}$ in A with $\lim_{n\to\infty} x_n = 0$ (and hence $\mathbb{R}^K$ is not metrizable using the product topology).