PMATH 367 Topology, Solutions to Assignment 3

1: For each of the following sets $A \subseteq \mathbb{R}^n$, determine whether A is closed, whether A is compact, and whether A is connected.

(a) $A = \left\{ \left(\frac{2t}{t^2+1} \, , \, \frac{t^2-1}{t^2+1} \right) \in \mathbb{R}^2 \,\middle|\, t \in \mathbb{R} \right\}$

Solution: Let $p = (0,1) \in \mathbb{S}^1 \subseteq \mathbb{R}^2$. Note that the stereographic projection $f : \mathbb{S}^1 \setminus \{p\} \to \mathbb{R}^1$ and its inverse $g = f^{-1} : \mathbb{R} \to \mathbb{S}^1 \setminus \{p\}$ are given by $f(x,y) = \frac{x}{1-y}$ and $g(t) = \left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1} \right)$, so the given set A is the range of g, that is $A = \mathbb{S}^1 \setminus \{p\}$. Since A is the image of the connected set $\mathbb{R}$ under the continuous map g, A is connected. We claim that A is not closed, indeed we claim that p (which does not lie in A) lies in $\overline{A}$. It suffices to show that for every $r > 0$, $B(p,r) \cap A \neq \emptyset$. Let $r > 0$. Since $\frac{2t}{t^2+1} \to 0$ and $\frac{t^2-1}{t^2+1} \to 1$ as $t \to \infty$, we can choose $t > 0$ such that $\left| \frac{2t}{t^2+1} \right| < \frac{r}{2}$ and $\left| \frac{t^2-1}{t^2+1} - 1 \right| < \frac{r}{2}$, and then we have

$$\left| g(t) - p \right| = \left| \left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1} - 1 \right) \right| \leq \left| \frac{2t}{t^2+1} \right| + \left| \frac{t^2-1}{t^2+1} - 1 \right| < \frac{r}{2} + \frac{r}{2} = r$$

so that $g(t) \in B(p,r)$, and hence $B(p,r) \cap A \neq \emptyset$, as required. Thus A is not closed, hence also not compact.

(b) $A = \left\{ (u,v,w,x,y,z) \in \mathbb{R}^6 \,\middle|\, \mathrm{rank}\left(\begin{smallmatrix} u & v & w \\ x & y & z \end{smallmatrix} \right) \neq 2 \right\}$

Solution: Note that we have $\mathrm{rank}\left(\begin{smallmatrix} u & v & w \\ x & y & z \end{smallmatrix} \right) = 2$ if and only if some pair of columns is linearly independent if and only if one of the three 2×2 submatrices $\left(\begin{smallmatrix} u & v \\ x & y \end{smallmatrix} \right)$, $\left(\begin{smallmatrix} u & w \\ x & z \end{smallmatrix} \right)$ and $\left(\begin{smallmatrix} v & w \\ y & z \end{smallmatrix} \right)$ is invertible if and only if one of the three determinants $uy - vx$, $uz - wx$ and $vz - wy$ is non-zero. Thus we have

$$\mathrm{rank}\left(\begin{smallmatrix} u & v & w \\ x & y & z \end{smallmatrix} \right) \neq 2 \iff \Big(uy - vx = 0 \text{ and } uz - wx = 0 \text{ and } vz - wy = 0 \Big)$$

and hence

$$A = f^{-1}\big(\{0\}\big) \cap g^{-1}\big(\{0\}\big) \cap h^{-1}\big(\{0\}\big)$$

where $f, g, h : \mathbb{R}^6 \to \mathbb{R}$ are given by

$$\begin{aligned} f(u,v,w,x,y,z) &= uy - vx \, , \\ g(u,v,w,x,y,z) &= uz - wx \, , \\ h(u,v,w,x,y,z) &= vz - wy \, . \end{aligned}$$

Since f, g and h are continuous (they are polynomials) and $\{0\}$ is closed in $\mathbb{R}$, it follows (from Theorem 1.31) that the sets $f^{-1}\big(\{0\}\big)$, $g^{-1}\big(\{0\}\big)$ and $h^{-1}\big(\{0\}\big)$ are all closed, and hence$A$ is closed (by Theorem 1.7). On the other hand, A is not bounded because for $e_1 = (1,0,0,0,0,0)$ we have $re_1 \in A$ for all $r \in \mathbb{R}$ and $\|re_1\| = |r|$. Since A is not bounded, it is not compact (by the Heine-Borel Theorem). Finally, we note that A is path-connected (hence connected), indeed given $a, b \in A$, the map $\alpha : [0,1] \subseteq \mathbb{R} \to \mathbb{R}^6$ given by

$$\alpha(t) = \left\{ \begin{array}{l} (1-2t)a \text{ for } 0 \leq t \leq \frac{1}{2} \\ (2t-1)b \text{ for } \frac{1}{2} \leq t \leq 1 \end{array} \right\}$$

is continuous (by the Glueing Lemma) with $\alpha(0) = a$, $\alpha\left(\frac{1}{2}\right) = 0$ and $\alpha(1) = b$, and we have $\alpha(t) \in A$ for all t (because when X is a matrix and $r \in \mathbb{R}$, we have $\mathrm{rank}(rX) = \mathrm{rank}(X)$ when $r \neq 0$, and we have $\mathrm{rank}(rX) = 0$ when $r = 0$).

2: (a) Let X be a topological space and let $A \subseteq X$. For this problem, let us say that A is connected in X when there do not exist open sets U and V in X such that $U \cap A \neq \emptyset$, $V \cap A \neq \emptyset$, $U \cap V = \emptyset$ and $A \subseteq U \cup V$. Prove that if X is a metric space then A is connected in X if and only if A is connected (in itself), and find an example of a topological space X and a subspace $A \subseteq X$ such that A is connected in X but A is not connected (in itself).

Solution: Let X be a metric space. Suppose that A is not connected in X. Choose open sets U and V in X such that $U \cap A \neq \emptyset$, $V \cap A \neq \emptyset$, $U \cap V = \emptyset$, and $A \subseteq U \cup V$. Then the sets $U \cap A$ and $V \cap A$ are open sets in A which separate A (in itself).

Suppose, conversely, that A is not connected (in itself). Choose (disjoint) open sets E and F in A which separate A (in itself), so we have $E \neq \emptyset$, $F \neq \emptyset$, $E \cap F = \emptyset$, and $A = E \cup F$. Choose open sets U and V in X such that $E = U \cap A$ and $F = V \cap A$. We remark that U and V might not be disjoint in X. For each $a \in E$ choose $r_a > 0$ such that $B(a, 2r_a) \subseteq U$ and then let $U_0 = \bigcup_{a \in E} B(a, r_a)$. Note that U_0 is open in X (since it is a union of open sets in X) and that we have $E \subseteq U_0 \subseteq U$. Similarly, for each $b \in F$ choose $s_b > 0$ so that $B(b, 2s_b) \subseteq V$, and then let $V_0 = \bigcup_{b \in F} B(b, s_b)$. Note that V_0 is open in X and $F \subseteq V_0 \subseteq V$. We have $\emptyset \neq E \subseteq A \cap U_0$, $\emptyset \neq F \subseteq A \cap V_0$ and $A = E \cup F \subseteq U_0 \cup V_0$. It remains to show that $U_0 \cap V_0 = \emptyset$. Suppose, for a contradiction, that $U_0 \cap V_0 \neq \emptyset$. Choose $x \in U_0 \cap V_0$. Since $x \in U_0 = \bigcup_{a \in E} B(a, r_a)$ we can choose $a \in E$ such that $x \in B(a, r_a)$. Similarly, we can choose $b \in F$ so that $x \in B(b, s_b)$. Suppose that $r_a \geq s_b$ (the case that $s_b \geq r_a$ is similar). By the Triangle Inequality, it follows that $|b - a| \leq |b - x| + |x - a| < s_b + r_a \leq 2r_a$ and so we have $b \in B(a, 2r_a) \subseteq U$. Since $b \in F \subseteq A$ and $b \in U$ we have $b \in U \cap A = E$. Thus we have $b \in E \cap F$ which contradicts the fact that $E \cap F = \emptyset$, and so $U_0 \cap V_0 = \emptyset$, as required.

As an example of a subspace $A \subseteq X$ with A connected (in itself) but A not connected in X, we can take $X = \{0, 1, 2\}$ with topology $\mathcal{T} = \big\{\emptyset, \{0\}, \{0, 1\}, \{0, 2\}, X\big\}$, and let $A = \{1, 2\}$

(b) It is likely that you proved in PMATH 351 that when X is a metric space, X is compact if and only if every infinite subset of X has a limit point. Show that when X is a topological space, if X is compact then every infinite subset of X has a limit point, and find an example of a non-compact topological space X in which every infinite subset has a limit point.

Solution: Let X be a compact topological space, let $A \subseteq X$, and suppose that $A' = \emptyset$. We need to show that A is finite. For each $a \in A$, since $a \notin A'$ we can choose an open set U_a in X with $a \in U_a$ such that $(U_a \setminus \{a\}) \cap A = \emptyset$. Since $a \in U_a \cap A$ and $(U \setminus \{a\}) \cap A = \emptyset$, we have $U_a \cap A = \{a\}$. Also note that since $A' = \emptyset$, we have $A' \subseteq A$ so that A is closed, hence A^c is open, so the set $\mathcal{S} = \{U_a \,|\, a \in A\} \cup \{A^c\}$ is an open cover of X. Since X is compact, we can choose a finite subcover, so we can choose $a_1, a_2, \cdots, a_n \in A$ such that $X = \big(\bigcup_{k=1}^n U_{a_k}\big) \cup A^c$. Then we have $A = X \cap A = \big(\big(\bigcup_{k=1}^n U_{a_k}\big) \cup A^c\big) \cap A = \bigcup_{k=1}^n (U_{a_k} \cap A) = \bigcup_{k=1}^n \{a_k\} = \{a_1, a_2, \cdots, a_n\}$ so that A is finite, as required.

As an example as a noncompact space X in which every infinite subset has a limit point, we can give $\mathbb{Z}^+$ its standard discrete topology, and give $\{0, 1\}$ the trivial topology, then take $X = \mathbb{Z}^+ \times \{0, 1\}$. The sets $U_n = \big\{(n, 0), (n, 1)\big\}$ with $n \in \mathbb{Z}^+$ are disjoint open sets, and $\mathcal{S} = \big\{U_n \,|\, n \in \mathbb{Z}^+\big\}$ is an open cover of X with no finite subcover, and every nonempty subset $\emptyset \neq A \subseteq X$ has a limit point: indeed if $(n, 0) \in A$ then $(n, 1) \in A'$, and if $(n, 1) \in A$ then $(n, 0) \in A'$.

3: For $x \in \mathbb{R}^\omega$, when $1 \le p < \infty$ we define the p-norm of x to be $\|x\|_p = \left(\sum_{k=1}^\infty |x_k|^p\right)^{1/p}$, and we define the ∞-norm of x to be $\|x\|_\infty = \sup\left\{|x_k| \,\middle|\, k \in \mathbb{Z}^+\right\}$. For $1 \le p \le \infty$ we define $\ell_p = \left\{x \in \mathbb{R}^\omega \,\middle|\, \|x\|_p < \infty\right\}$. You may assume, without proof, that for $1 \le p \le \infty$, ℓ_p is a normed linear space using the p-norm. Note that $\ell_p \subseteq \mathbb{R}^\omega = \prod_{k=1}^\infty \mathbb{R}$. Show that the subspace topology on ℓ_p inherited from the product topology on $\mathbb{R}^\omega$ is strictly coarser than the p-norm topology on ℓ_p which, in turn, is strictly coarser than the subspace topology on ℓ_p inherited from the box topology on $\mathbb{R}^\omega$.

Solution: First, consider $\mathbb{R}^\omega$ using the product topology. Let $U \subseteq \ell_p$ be open in the subspace topology, and let $a \in U$. Choose an open set $V \subseteq \mathbb{R}^\omega$ such that $U = V \cap \ell_p$. Choose a basic open set B in $\mathbb{R}^\omega$ with $a \in B \subseteq V$, say $B = U_1 \times U_2 \times \cdots \times U_n \times \mathbb{R} \times \mathbb{R} \times \cdots$ where each U_k is open in $\mathbb{R}$ with $a_k \in U_k$. For $1 \le k \le n$, choose $r_k > 0$ such that $B(a_k, r_k) \subseteq U_k$ and let $r = \min\{r_1, \cdots, r_n\}$. When $x \in B_p(a, r) = \left\{x \in \ell_p \,\middle|\, \|x - a\|_p < r\right\}$, we have $|x_k - a_k| \le \|x - a\|_p < r \le r_k$ for $1 \le k \le n$, and hence $B_p(a, r) \subseteq B \subseteq V$. Since $B_p(a, r) \subseteq \ell_p$ and $B_p(a, r) \subseteq V$, we have $a \in B_p(a, r) \subseteq V \cap \ell_p = U$. Thus U is also open in the p-norm topology. This shows that the subspace topology is coarser or equal to the p-norm topology.

To show that it is strictly coarser, we shall show that $B_p(0, 1)$, which is open in the p-norm topology, is not open in the subspace topology. Suppose, for a contradiction, that $B_p(0, 1)$ is open in the subspace topology. Choose an open set $V \subseteq \mathbb{R}^\omega$ such that $B_p(0, 1) = V \cap \ell_p$. Choose a basic open set $B \subseteq \mathbb{R}^\omega$ such that $0 \in B \subseteq V$, say $B = U_1 \times U_2 \times \cdots \times U_n \times \mathbb{R} \times \mathbb{R} \times \cdots$ where each U_k is open in $\mathbb{R}$ with $0 \in U_k$. Then we have $e_{n+1} \in B \subseteq V$ and $e_{n+1} \in \ell_p$ so that $e_{n+1} \in V \cap \ell_p = B_p(0, 1)$, giving the desired contradiction.

Now consider $\mathbb{R}^\omega$ using the box topology. Let $U \subseteq \ell_p$ be open in the p-norm topology, and let $a \in U$. Choose $r > 0$ such that $B_p(a, r) \subseteq U$. Let $s = (s_k)_{k \ge 1}$ be given by $s_k = \frac{1}{2^k}$ and note that $s \in \ell_p$ for all $1 \le p \le \infty$ with $\|s\|_p \le 1$: indeed when $1 \le p < \infty$ we have $\|s\|_p = \left(\sum\limits_{k=1}^\infty \frac{1}{2^{pk}}\right)^{1/p} \le \left(\sum\limits_{k=1}^\infty \frac{1}{2^k}\right)^{1/p} = 1^{1/p} = 1$, and we have $\|s\|_\infty = \frac{1}{2} \le 1$. Let $B \subseteq \mathbb{R}^\omega$ be the basic open set $B = B(a_1, \frac{r}{2^1}) \times B(a_2, \frac{r}{2^2}) \times B(a_3, \frac{r}{2^3}) \times \cdots$. For all $x \in B$ we have $|x_k - a_k| < \frac{r}{2^k}$ for all $k \in \mathbb{Z}^+$ so that $\|x - a\|_p < r\|s\|_p = r$, and so $a \in B \subseteq B_p(a, r)$ (we remark that when $x \in \mathbb{R}^\omega$ with $\|x - a\|_p < r$ we have $\|x\|_p \le \|x - a\|_p + \|a\|_p < r + \|a\|_p < \infty$ so that $x \in \ell_p$, and so $B_p(a, r) = \left\{x \in \ell_p \,\middle|\, \|x - a\|_p < r\right\} = \left\{x \in \mathbb{R}^\omega \,\middle|\, \|x - a\|_p < r\right\}$). Thus U is also open in the subspace topology. This shows that the p-norm topology is coarser or equal to the subspace topology.

To show that it is strictly coarser, note that the basic open set $B = B\left(0, \frac{1}{2^1}\right) \times B\left(0, \frac{1}{2^2}\right) \times B\left(0, \frac{1}{2^3}\right) \times \cdots$ is open in $\mathbb{R}^\omega$, using the box topology, and, as remarked above, we have $B \subseteq \ell_p$. But B is not open in the p-norm topology because $0 \in B$ but there is no $r > 0$ such that $B_p(0, r) \subseteq B$: indeed given $r > 0$ we can choose $n \in \mathbb{Z}^+$ so that $\frac{1}{2^n} < \frac{r}{2}$ and then we have $\frac{r}{2}\, e_n \in B_p(0, r)$ but $\frac{r}{2}\, e_n \notin B$.

4: (a) Show that for $a, b \in \mathbb{R}^\omega$, if $b - a \in \mathbb{R}^\infty$ then there is a path from a to b in $\mathbb{R}^\omega$, using the box topology.

Solution: Let $a, b \in \mathbb{R}^\omega$ with $a - b \in \mathbb{R}^\infty$. Choose $m \in \mathbb{Z}^+$ such that for all $k \in \mathbb{Z}^+$ with $k \geq m$ we have $a_k - b_k = 0$ so that $a_k = b_k$. For each $k \in \mathbb{Z}^+$, define $\alpha_k : [0,1] \subseteq \mathbb{R} \to \mathbb{R}$ by $\alpha_k(t) = a_k + t(b_k - a_k)$, then define $\alpha : [0,1] \subseteq \mathbb{R} \to \mathbb{R}^\omega$ by $\alpha(t)_k = \alpha_k(t) = a_k + t(b_k - a_k)$. Note that when $k \geq m$ we have $\alpha(t)_k = a_k = b_k$ for all $t \in [0,1]$. We claim that α is a (continuous) path from a to b in $\mathbb{R}^\omega$. It is clear that $\alpha(0) = a$ and $\alpha(1) = b$, so it suffices to show that α is continuous. Let U be any basic open set in $\mathbb{R}^\omega$ (using the box topology), say $U = \prod_{k=1}^\infty U_k$ where each U_k is open in $\mathbb{R}$. Then

$$\alpha^{-1}(U) = \left\{t \in [0,1] \,\middle|\, \alpha(t)_k \in U_k \text{ for all } k \in \mathbb{Z}^+\right\} = \bigcap_{k=1}^\infty \alpha_k^{-1}(U_k).$$

If there exists $k \in \mathbb{Z}^+$ with $k \geq m$ such that $a_k \notin U_k$, then we have $\alpha_k^{-1}(U_k) = \emptyset$ and hence $\alpha^{-1}(U) = \emptyset$, which is open. If $a_k \in U_k$ for all $k \geq m$ then we have $\alpha_k^{-1}(U_k) = [0,1]$ for all $k \geq m$ so that $\alpha^{-1}(U) = \bigcap_{k=1}^{m-1} \alpha_k^{-1}(U_k)$, which is open (since it is the intersection of finitely many open sets). Since $\alpha^{-1}(U)$ is open in $[0,1]$ for every basic open set U in $\mathbb{R}^\omega$, it follows that α is continuous, as required.

(b) Let $b \in \mathbb{R}^\omega$ with $b - a \notin \mathbb{R}^\infty$. For each $k \in \mathbb{Z}^+$, let $d_k = |b_k - a_k|$ and note that $d_k > 0$ for infinitely many $k \in \mathbb{Z}^+$. For $k, \ell \in \mathbb{Z}^+$, let $I_{k,\ell} = \left(a_k - \frac{d_k}{2^\ell}, a_k + \frac{d_k}{2^\ell}\right) \subseteq \mathbb{R}$ when $d > 0$ and let $I_{k,\ell} = \mathbb{R}$ when $d_k = 0$. For $\ell \in \mathbb{Z}^+$, let $V_\ell = \left\{x \in \mathbb{R}^\omega \,\middle|\, \{k \in \mathbb{Z}^+ \,|\, x_k \notin I_{k,\ell}\} \text{ is infinite}\right\}$. Let $V = \bigcup_{\ell=1}^\infty V_\ell$ and $U = V^c = \mathbb{R}^\omega \setminus V$. Note that U and V separate $\mathbb{R}^\omega$ with $a \in U$ and $b \in V$. Show that U and V are open in $\mathbb{R}^\omega$, using the box topology.

Solution: We claim that V is open. Let $c \in V = \bigcup_{\ell=1}^\infty V_\ell$. Choose $\ell \in \mathbb{Z}^+$ such that $c \in V_\ell$. Let $K = \left\{k \in \mathbb{Z}^+ \,\middle|\, c_k \notin I_{k,\ell}\right\}$ and note that K is infinite (since $c \in V_\ell$). Let B be the basic open set $B = \prod_{k=1}^\infty J_{k,\ell}$ where $J_{k,\ell} = \left(c_k - \frac{d_k}{2^{\ell+1}}, c_k + \frac{d_k}{2^{\ell+1}}\right)$. Let $x \in B$. For all $k \in \mathbb{Z}^+$ we have $x_k \in J_{k,\ell}$ so that $|x_k - c_k| < \frac{d_k}{2^{\ell+1}}$, and for all $k \in K$ we have $c_k \notin I_{k,\ell}$ so that $|c_k - a_k| \geq \frac{d_k}{2^\ell}$. Thus for all $k \in K$ we have $|a_k - x_k| \geq |a_k - c_k| - |c_k - x_k| > \frac{d_k}{2^\ell} - \frac{d_k}{2^{\ell+1}} = \frac{d_k}{2^{\ell+1}}$ so that $x \notin I_{k,\ell}$. Since $x \notin I_{k,\ell}$ for all $k \in K$, and K is infinite, we have $x \in V_\ell$. Since $x \in B$ was arbitrary, we have $B \subseteq V_\ell \subseteq V$. Thus, for each $c \in V$ we have a basic open set B with $c \in B \subseteq V$, and hence V is open.

We claim that U is open. Let $c \in U = V^c$, so $c \notin V_\ell$ for every $\ell \in \mathbb{Z}^+$. Then for every $\ell \in \mathbb{Z}^+$ there exist finitely many $k \in \mathbb{Z}^+$ such that $c_k \notin I_{k,\ell}$. Choose $1 \leq m_1 < m_2 < \cdots$ such that for all $k, \ell \in \mathbb{Z}^+$, $k \geq m_\ell \Longrightarrow c_k \in I_{k,\ell}$. Let B be the basic open set $B = \prod_{k \in \mathbb{Z}^+} J_k$ where J_k is as follows: when $k < m_1$ we take $J_k = \mathbb{R}$ and when $m_\ell \leq k < m_{\ell+1}$ we take $J_k = I_{k,\ell}$ and note that $c \in B$. When $x \in B$, for every $\ell \in \mathbb{Z}^+$ we have $x_k \in I_{k,\ell}$ for all $k \geq m_\ell$ (indeed when $n \geq \ell$ and $m_n \leq k < m_{n+1}$ we have $x_k \in I_{k,n} \subseteq I_{k,\ell} = J_k$) so that $x_k \notin I_{k,\ell}$ for finitely many k, and hence $x \notin V_\ell$ for every ℓ, and hence $x \in U$. Thus B is a basic open set with $c \in B \subseteq U$, and hence U is open.

(c) Show that the connected components and the path components of $\mathbb{R}^\omega$, using the box topology, are the elements in the quotient space $\mathbb{R}^\omega / \mathbb{R}^\infty$ (that is the sets of the form $a + \mathbb{R}^\infty$ with $a \in \mathbb{R}^\omega$).

Solution: Let $a \in \mathbb{R}^\omega$, let C be the connected component of a and let P be the path component of a. By Part (a), the set $a + \mathbb{R}^\infty$ is path-connected, hence connected, so we have $a + \mathbb{R}^\infty \subseteq P \subseteq C$. For each $b \in (a + \mathbb{R}^\infty)^c = \mathbb{R}^\omega \setminus (a + \mathbb{R}^\infty)$, by Part (b) we can choose disjoint open sets U_b and V_b which separate $\prod_{k \in K} X_k$ with $a \in U_b$ and $b \in V_b$. By Lemma 3.4, since C is connected, either $C \subseteq U_b$ or $C \subseteq V_b$. Since $a \in C \cap U_b$ and $a \notin V_b$, we must have $C \subseteq U_b$. Since $C \subseteq U_b$ and $U_b \cap V_b = \emptyset$, we have $C \cap V_b = \emptyset$. Since $C \cap V_b = \emptyset$ for every $b \in (a + \mathbb{R}^\infty)^c$, we have $C \cap V = \emptyset$ where $V = \bigcup_{b \in (a + \mathbb{R}^\infty)^c} V_b$. Since $b \in V_b$ for every $b \in (a + \mathbb{R}^\infty)^c$, we have $(a + \mathbb{R}^\infty)^c \subseteq V$. Since $C \cap V = \emptyset$ and $(a + \mathbb{R}^\infty)^c \subseteq V$, we have $C \cap (a + \mathbb{R}^\infty)^c = \emptyset$ so that $C \subseteq a + \mathbb{R}^\infty$. Since $a + \mathbb{R}^\infty \subseteq P \subseteq C \subseteq a + \mathbb{R}^\infty$, it follows that $P = C = a + \mathbb{R}^\infty$.