PMATH 367 Topology, Assignment 3 — Due Nov 4, 11:00 pm

1: For each of the following sets $A\subseteq\mathbb{R}^n$, determine whether A is closed, whether A is compact, and whether A is connected.

(a) $A=\left\{\left(\frac{2t}{t^2+1}\,,\,\frac{t^2-1}{t^2+1}\right)\in\mathbb{R}^2\,\middle|\,t\in\mathbb{R}\right\}$

(b) $A=\left\{(u,v,w,x,y,z)\in\mathbb{R}^6\,\middle|\,\text{rank}\left(\begin{smallmatrix}u&v&w\\x&y&z\end{smallmatrix}\right)\neq 2\right\}$

2: (a) Let X be a topological space and let $A\subseteq X$. For this problem, let us say that A is connected in X when there do not exist open sets U and V in X such that $U\cap A\neq\emptyset$, $V\cap A\neq\emptyset$, $U\cap V=\emptyset$ and $A\subseteq U\cup V$. Prove that if X is a metric space then A is connected in X if and only if A is connected (in itself), and find an example of a topological space X and a subspace $A\subseteq X$ such that A is connected in X but A is not connected (in itself).

(b) It is likely that you proved in PMATH 351 that when X is a metric space, X is compact if and only if every infinite subset of X has a limit point. Show that when X is a topological space, if X is compact then every infinite subset of X has a limit point, and find an example of a non-compact topological space X in which every infinite subset has a limit point.

3: For $x\in\mathbb{R}^\omega$, when $1\leq p<\infty$ we define the p-norm of x to be $\|x\|_p=\left(\sum_{k=1}^\infty|x_k|^p\right)^{1/p}$, and we define the ∞-norm of x to be $\|x\|_\infty=\sup\left\{|x_k|\,\middle|\,k\in\mathbb{Z}^+\right\}$. For $1\leq p\leq\infty$ we define $\ell_p=\left\{x\in\mathbb{R}^\omega\,\middle|\,\|x\|_p<\infty\right\}$. You may assume, without proof, that for $1\leq p\leq\infty$, ℓ_p is a normed linear space using the p-norm. Note that $\ell_p\subseteq\mathbb{R}^\omega=\prod_{k=1}^\infty\mathbb{R}$. Show that the subspace topology on ℓ_p inherited from the product topology on $\mathbb{R}^\omega$ is strictly coarser than the p-norm topology on ℓ_p which, in turn, is strictly coarser than the subspace topology on ℓ_p inherited from the box topology on $\mathbb{R}^\omega$.

4: (a) Show that for $a,b\in\mathbb{R}^\omega$, if $b-a\in\mathbb{R}^\infty$ then there is a path from a to b in $\mathbb{R}^\omega$, using the box topology.

(b) Let $b\in\mathbb{R}^\omega$ with $b-a\notin\mathbb{R}^\infty$. For each $k\in\mathbb{Z}^+$, let $d_k=|b_k-a_k|$ and note that $d_k>0$ for infinitely many $k\in\mathbb{Z}^+$. For $k,\ell\in\mathbb{Z}^+$, let $I_{k,\ell}=\left(a_k-\frac{d_k}{2^\ell}\,,\,a_k+\frac{d_k}{2^\ell}\right)\subseteq\mathbb{R}$ when $d>0$ and let $I_{k,\ell}=\mathbb{R}$ when $d_k=0$. For $\ell\in\mathbb{Z}^+$, let $V_\ell=\left\{x\in\mathbb{R}^\omega\,\middle|\,\{k\in\mathbb{Z}^+\,|\,x_k\notin I_{k,\ell}\}\text{ is infinite}\right\}$. Let $V=\bigcup_{\ell=1}^\infty V_\ell$ and $U=V^c=\mathbb{R}^\omega\setminus V$. Note that U and V separate $\mathbb{R}^\omega$ with $a\in U$ and $b\in V$. Show that U and V are open in $\mathbb{R}^\omega$, using the box topology.

(c) Show that the connected components and the path components of $\mathbb{R}^\omega$, using the box topology, are the elements in the quotient space $\mathbb{R}^\omega/\mathbb{R}^\infty$ (that is the sets of the form $a+\mathbb{R}^\infty$ with $a\in\mathbb{R}^\omega$).