PMATH 367 Topology, Solutions to Assignment 2.5

1: For each of the following subsets $A \subseteq \mathbb{R}^n$, determine whether A is closed, whether A is compact, and whether A is connected.

(a) $A = \left\{ (a,b,c,d) \in \mathbb{R}^4 \,\middle|\, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \right\}$.

Solution: We claim that A is closed. For $a,b,c,d \in \mathbb{R}$ we have $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^2 = \left(\begin{smallmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{smallmatrix}\right)$ so that $(a,b,c,d) \in A$ when $\big(a^2+bc, ab+bd, ac+cd, bc+d^2\big) = (1,0,0,1)$, and so $A = f^{-1}(p)$ where $f : \mathbb{R}^4 \to \mathbb{R}^4$ is given by $f(a,b,c,d) = \big(a^2+bc, ab+bd, ac+cd, bc+d^2\big)$ and $p = (1,0,0,1) \in \mathbb{R}^4$. The map f is continuous (it is a polynomial map) and $\{p\}$ is closed in $\mathbb{R}^4$, and so $A = f^{-1}\big(\{p\}\big)$ is closed in $\mathbb{R}^4$.

Note that A is not bounded because for $r > 0$ we have $(1, r, 0, -1) \in A$ and $\big|(1,r,0,-1)\big| = \sqrt{2+r^2} \to \infty$ as $r \to \infty$. Since A is not bounded in $\mathbb{R}^4$, it is not compact (by the Heine Borel Theorem).

We claim that A is not connected. For $a,b,c,d \in \mathbb{R}$, if $(a,b,c,d) \in A$ then we have $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^2 = I$ so that $\det \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \pm 1$, that is $ad - bc = \pm 1$. It follows that A can be separated by the two open sets $U \cap A$ and $V \cap A$ where $U = \big\{(a,b,c,d) \big| ad - bc > 0\big\}$ and $V = \big\{(a,b,c,d)\big| ad - bc < 0\big\}$. Note that U is open in $\mathbb{R}^4$ because $U = g^{-1}\big((0,\infty)\big)$ where $g : \mathbb{R}^4 \to \mathbb{R}$ is given by $g(a,b,c,d) = ad - bc$ (and g is continuous and $(0,\infty)$ is open), and similarly V is open in $\mathbb{R}^4$ because $V = g^{-1}\big((-\infty, 0)\big)$. Also, note that we have $U \cap A \neq \emptyset$ because for example $(1,0,0,1) \in U \cap A$, and we have $V \cap A \neq \emptyset$ because for example $(0,1,1,0) \in V \cap A$. Thus the sets $U \cap A$ and $V \cap A$ are open sets in A which separate A.

(b) A is the set of points $(a,b,c) \in \mathbb{R}^3$ such that the polynomial $p(x) = x^3 + ax^2 + bx + c$ has three distinct real roots which all lie in the closed interval $[-1,1]$.

Solution: We claim that A is not closed in $\mathbb{R}^3$. For $n \in \mathbb{Z}^+$, let $u_n = (a_n, b_n, c_n) = \big(0, -\frac{1}{n^2}, 0\big) \in \mathbb{R}^3$. Note that $u_n \in A$ since the polynomial $p_n(x) = x^3 + a_n x^2 + b_n x + c_n = x^3 - \frac{1}{n^2}x = \big(x + \frac{1}{n}\big)\big(x - 0\big)\big(x - \frac{1}{n}\big)$ has 3 distinct real roots, namely $-\frac{1}{n}$, 0, and $\frac{1}{n}$, which all lie in $[-1,1]$. Note that $u_n \neq 0 = (0,0,0)$ and $u_n \to 0$ so that $0 \in A' \subseteq \overline{A}$. But $0 \notin A$ because the polynomial $p(x) = x^3 + 0x^2 + 0x + 0 = x^3$ does not have three distinct real roots (it has the single triple root, 0). Thus A is not closed in $\mathbb{R}^3$, as claimed. Since A is not closed in $\mathbb{R}^3$, it is not compact (by the Heine-Borel Theorem).

We claim that A is connected. Let $C = \big\{(r,s,t) \in \mathbb{R}^3 \big| -1 \leq r < s < t \leq 1\big\}$ and define $f : C \to A$ by $f(r,s,t) = \big(-(r+s+t), st+tr+rs, -rst\big)$. Note that f is continuous (all polynomial functions are continuous), and f takes values in A and is surjective because $x^3 - (r+s+t)x^2 + (st+tr+rs)x - rst = (x-r)(x-s)(x-t)$. Note that $C = C_1 \cap C_2 \cap C_3 \cap C_4$ where $C_1 = \big\{(r,s,t)\big| -1 \leq r\big\}$, $C_2 = \big\{(r,s,t)\big| r < s\big\}$, $C_3 = \big\{(r,s,t)\big| s < t\big\}$ and $C_4 = \big\{(r,s,t)\big| t \leq 1\big\}$. Each of these sets C_k is easily seen to be convex: for example, C_2 is convex because if $u_1 = (r_1, s_1, t_1) \in C_2$ (so $r_1 < s_2$) and $u_2 = (r_2, s_2, t_2) \in C_2$ (so $r_2 < s_2$) then for all $\lambda \in [0,1]$ we have $(1-\lambda)r_1 + \lambda r_2 < (1-\lambda)s_1 + \lambda s_2$ so that

$$(1-\lambda)u_1 + \lambda u_2 = \big((1-\lambda)r_1 + \lambda r_2, (1-\lambda)s_1 + \lambda s_2, (1-\lambda)t_1 + \lambda t_2\big) \in C_2.$$

Since C is the intersection of four convex sets, it follows that C is convex: indeed given $a, b \in C$, we have $a, b \in C_k$ so that $[a,b] \subseteq C_k$ for every index k, and hence $[a,b] \subseteq C = \bigcap_{k=1}^4 C_k$. Since C is convex, it is path connected, and hence connected. Since f is continuous and C is connected and $A = f(C)$, it follows that A is connected.

2: (a) Let $A \subseteq \mathbb{R}^2$. Show that if A is countable then $A^c = \mathbb{R}^2 \setminus A$ is path-connected.

Solution: Suppose that A is countable. Let $b, c \in \mathbb{R}^2 \setminus A$. There are uncountably many lines through b, and only countably many of these lines pass through the points in A, so we can choose a line L through b which does not intersect with any of the points in A. If $c \in L$ then the linear path given by $\alpha(t) = b + t(c-b)$ is a path from b to c in $\mathbb{R}^2 \setminus A$. Suppose that $c \notin L$. There exist uncountably many lines through c, and only one of these is parallel to L and only countably many pass through the points in A, so we can choose a line M through c such that M is not parallel to L and M does not pass through any of the points in A. Let p be the point of intersection of L and M. Then the path $\alpha\beta$ where $\alpha(t) = b + t(p-b)$ and $\beta(t) = p + t(c-p)$ is a path from b to c in $\mathbb{R}^2 \setminus A$.

(b) Let $I = [0,1] \subseteq \mathbb{R}$. Find the path-components of $X = I^2$ using the dictionary order. topology

Solution: We claim that the path components of X are the vertical line segments $\{a\} \times [0,1]$ with $a \in [0,1]$. Note first that the basic open sets in X intersected with the vertical line segment $\{a\} \times [0,1]$ are the sets of one of the forms $\{a\} \times [0,b)$, $\{a\} \times (b,1]$ or $\{a\} \times (b,c)$ (or the empty set). These are precisely the basic open sets for $\{a\} \times [0,1]$ using its standard topology (as a subspace of $\mathbb{R}^2$). Thus $\{a\} \times [0,1]$ is homeomorphic to $[0,1]$, which is path-connected.

To complete the proof of our claim, it suffices to show that when $0 \le a < b \le 1$, there is no path in X from $\left(a, \frac{1}{2}\right)$ to $\left(b, \frac{1}{2}\right)$. Suppose, for a contradiction, that there is such a path $\alpha : [0,1] \to X$. Note that for every $c \in (a,b)$ there exists $s \in (0,1)$ such that $\alpha(s) = \left(c, \frac{1}{2}\right)$, since otherwise, for $U = \left\{(x,y) \middle| (x,y) < \left(c, \frac{1}{2}\right)\right\}$ and $V = \left\{(x,y) \middle| (x,y) > \left(c, \frac{1}{2}\right)\right\}$, the two nonempty disjoint open sets $\alpha^{-1}(U)$ and $\alpha^{-1}(V)$ would separate the connected set $[0,1]$. But this implies that we have uncountably many disjoint open sets $\alpha^{-1}\big(\{c\} \times (0,1)\big)$ in $(0,1)$, which is not possible (because we can choose a rational number in each of the disjoint sets, but there are only countably many rational numbers).