PMATH 367 Topology, Solutions to Assignment 2

1: Let $X=\mathbb{R}$ using the lower limit topology $\mathcal{T}$, generated by the sets of the form $[a,b)$ where $a,b\in\mathbb{R}$ with $a<b$, and let $Y=\mathbb{R}$ using the topology $\mathcal{S}$ generated by the sets of the form $[a,b)$ where $a,b\in\mathbb{Q}$ with $a<b$. Let $A=(0,\sqrt{2})=\{x\in\mathbb{R}\,|\,0<x<\sqrt{2}\}$ and let $B=(\sqrt{2},3)=\{x\in\mathbb{R}\,|\,\sqrt{2}<x<3\}$. Find the closures $\mathrm{Cl}_X(A)$, $\mathrm{Cl}_X(B)$, $\mathrm{Cl}_Y(A)$ and $\mathrm{Cl}_Y(B)$.

Solution: Let $I=(r,s)$ where $r,s\in\mathbb{R}$ with $r<s$. We claim that in X, we have $\overline{I}=\mathrm{Cl}_X I=[r,s)$. For $x<r$ (so $x\notin I$) we have $x\notin I'$ because for $U=\big[x,\frac{x+r}{2}\big)$ we have $x\in U$ and $(U\setminus\{x\})\cap I=\big(x,\frac{x+r}{2}\big)\big)\cap(r,s)=\emptyset$. For $x=r$ we have $x\in I'$ because when $x=r\in U=[a,b)$ where $a,b\in\mathbb{R}$ with $a<b$, we have $a\le x=r<b$ so that $(U\setminus\{x\})\cap I\supseteq(r,b)\cap(r,s)\neq\emptyset$. For $r<x<s$ we have $x\in I$. Finally, for $x\ge s$ (so $x\neq I$) we have $x\notin I'$ because for $U=[x,x+1)$ we have $x\in U$ and $(U\setminus\{x\})\cap I=(x,x+1)\cap(r,s)=\emptyset$. Thus $\overline{I}=I\cup I'=[r,s)$, as claimed.

Again, let $I=(r,s)$ where $r,s\in\mathbb{R}$ with $r<s$. We claim that in Y, if $s\in\mathbb{Q}$ then $I'=\mathrm{Cl}_Y I=[r,s)$ and if $s\notin\mathbb{Q}$ then $\overline{I}=\mathrm{Cl}_Y I=[r,s]$. For $x<r$ (so $x\notin I$) we have $x\notin I'$ because we can choose $a,b\in\mathbb{Q}$ with $a\le x<b<r$ and then for $U=[a,b)$ we have $x\in U$ and $(U\setminus\{x\})\cap I\subseteq[a,b)\cap(r,s)=\emptyset$. For $x=r$ we have $x\in I'$ because when $x\in U=[a,b)$ where $a,b\in\mathbb{Q}$ with $a<b$, we have $a\le x=r<b$ so that $(U\setminus\{x\})\cap I\supseteq(r,b)\cap(r,s)\neq\emptyset$. For $r<x<s$ we have $x\in I$. For $x=s\in\mathbb{Q}$ (so $x\notin I$) we have $x\notin I'$ because for $U=[s,s+1)$ we have $x\in U$ and $(U\setminus\{x\})\cap I=(s,s+1)\cap(r,s)=\emptyset$. For $x=s\notin\mathbb{Q}$ we have $x\in I'$ because when $x\in U=[a,b)$ where $a,b\in\mathbb{Q}$ with $a<b$, we have $a\le x<b$ and $a\neq x$ (since $x=s\notin\mathbb{Q}$ and $a\in\mathbb{Q}$) so that $a<x<b$, and hence $(U\setminus\{x\})\cap I=\big([a,s)\cup(s,b)\big)\cap(r,s)\supseteq[a,s)\cap(r,s)\neq\emptyset$. Finally, when $x>s$ (so $x\notin I$) we have $x\notin I'$ because we can choose $a,b\in\mathbb{Q}$ with $s<a<x<b$ and then for $U=[a,b)$ we have $x\in U$ with $(U\setminus\{x\})\cap I\subseteq[a,b)\cap(r,s)=\emptyset$. Thus, if $s\in\mathbb{Q}$ we have $\overline{I}=I\cup I'=[r,s)$, and if $s\notin\mathbb{Q}$ we have $\overline{I}=I\cup I'=[r,s]$, as claimed.

In particular, $\mathrm{Cl}_X(A)=[\,0,\sqrt{2}\,)$, $\mathrm{Cl}_X(B)=[\sqrt{2},3)$, $\mathrm{Cl}_Y(A)=[0,\sqrt{2}\,]$ and $\mathrm{Cl}_Y(B)=[\sqrt{2},3)$.

2: When X is an ordered set, the **dictionary order** on $X^2 = X \times X$ is the order given by stipulating that

$$(a,b) < (c,d) \iff \big(a < c \text{ or } (a = c \text{ and } b < d)\big).$$

(a) Let $I = [0,1] \subseteq \mathbb{R}$ and let $X = I^2$ using the order topology for the dictionary order. Find $\overline{A} = \text{Cl}_X A$ where $A = \big\{(x,0)\,\big|\, 0 < x < 1\big\}$.

Solution: Let $p, q \in I$ so that $(p,q) \in I^2 = X$. When $0 < p < 1$ and $q = 0$ we have $(p,q) \in A \subseteq \overline{A}$.
Recall, from Question 2(b) on Assignment 1, that $(p,q) \in \overline{A}$ if and only if for every basic open set B in X with $(p,q) \in B$, we have $B \cap A \neq \emptyset$.
When $(p,q) = (0,0)$ we have $(p,q) \notin \overline{A}$ because $(0,0) = \min X$ so $B = \big[(0,0),(0,1)\big)_X = \big\{(0,y)\,\big|\, 0 \le y < 1\big\}$ is a basic open set in X with $(0,0) \in B$ and with $B \cap A = \emptyset$.
When $(p,q) = (1,1)$ we have $(p,q) \notin \overline{A}$ because $(1,1) = \max X$ so $B = \big((1,0),(1,1)\big]_X = \big\{(1,y)\,\big|\, 0 < y \le 1\big\}$ is a basic open set in X with $(1,1) \in B$ and with $B \cap A = \emptyset$.
When $(p,q) = (1,0)$ we have $(p,q) \in \overline{A}$ because every basic open set B in X with $(1,0) \in B$ contains an interval of the form $J = \big((a,b),(1,0)\big]_X$ with $a < 1$ and then $\big(\frac{a+1}{2}, 0\big) \in J \cap A \subseteq B \cap A$.
When $0 \le p \le 1$ and $0 < q < 1$ we have $(p,q) \notin \overline{A}$ because the interval $B = \big((p,0),(p,1)\big)_X$ is a basic open set with $(p,q) \in B$ and $B \cap A = \emptyset$.
Finally, when $0 \le p < 1$ and $q = 1$ we have $(p,q) \in \overline{A}$ because every basic open set B in X with $(p,1) \in B$ contains an interval of the form $J = \big[(p,1)\,\big|\,(c,d)\big)$ with $p < c$ and then $\big(\frac{p+c}{2}, 0\big) \in J \cap A \subseteq B \cap A$.
Thus we have $\overline{A} = \big\{(p,0)\,\big|\, 0 < p \le 1\big\} \cup \big\{(p,1)\,\big|\, 0 \le p < 1\big\} = \big((0,1] \times \{0\}\big) \cup \big([0,1) \times \{1\}\big)$.

(b) Let $I = [0,1] \subseteq \mathbb{R}$ and let $X = I^2 \subseteq \mathbb{R}^2$. Let $\mathcal{T}_1$ be the order topology on X using the dictionary order, let $\mathcal{T}_2$ be the product topology on X using the order topology on each copy of I, and let $\mathcal{T}_3$ be the subspace topology that X inherits from $Y = \mathbb{R}^2$ using the order topology for the dictionary order. For each $k, \ell \in \{1,2,3\}$ with $k < \ell$, determine whether $\mathcal{T}_k \subseteq \mathcal{T}_\ell$ and whether $\mathcal{T}_\ell \subseteq \mathcal{T}_k$.

Solution: We remark that $\mathcal{T}_1$ is the topology used in Part (a) and $\mathcal{T}_2$ is the standard topology on I^2.
$\mathcal{T}_1 \not\subseteq \mathcal{T}_2$: for example, for $a, b, c \in I$ with $b < d$, the interval $B = \big((a,b),(a,d)\big)_{I^2} = \big\{(a,y)\,\big|\, b < y < d\big\}$ is a basic open set in $\mathcal{T}_1$, but $B \notin \mathcal{T}_2$.
$\mathcal{T}_2 \not\subseteq \mathcal{T}_1$: for example, the rectangle $R = [0,1] \times [0,1) = \big\{(x,y)\,\big|\, 0 \le x \le 1\,,\ 0 \le y < 1\big\}$ is a basic open set in $\mathcal{T}_2$, but $R \notin \mathcal{T}_1$ because, for example, $(1,0) \in R$ and every basic open set B in $\mathcal{T}_1$ with $(1,0) \in B$ contains an interval of the form $J = \big((a,b),(1,0)\big]$ with $a < 1$, and then $\big(\frac{a+1}{2}, 1\big) \in J$ so that $J \not\subseteq A$ hence $B \not\subseteq A$.
$\mathcal{T}_1 \subseteq \mathcal{T}_3$: indeed, every basic open set in $\mathcal{T}_1$ is a union of sets of the forms

$$\{a\} \times [0,d) = \big\{(a,y)\,\big|\, 0 \le y < d\big\} \text{ where } 0 \le a \le 1\,,\ 0 < d \le 1$$
$$\{a\} \times (c,1] = \big\{(a,y)\,\big|\, c < y \le 1\big\} \text{ where } 0 \le a \le 1\,,\ 0 \le c \le 1$$
$$\{a\} \times (c,d) = \big\{(a,y)\,\big|\, c < y < d\big\} \text{ where } 0 \le a \le 1\,,\ 0 \le c < d \le 1$$

and each of these is in $\mathcal{T}_3$.
$\mathcal{T}_3 \not\subseteq \mathcal{T}_1$: for example, for the set $U = \{1\} \times [0,1] = \big\{(1,y)\,\big|\, 0 \le y \le 1\big\}$, we have $U \in \mathcal{T}_3$ (because $U = V \cap X$ where V is the interval $V = \big((1,-2),(1,2)\big)_{\mathbb{R}^2} = \big\{(1,y)\,\big|\, -2 < y < 2\big\}$, which is a basic open set in $\mathbb{R}^2$ using the order topology for the dictionary order), but $U \notin \mathcal{T}_1$ because $(1,0) \in U$ and every basic open set B in $\mathcal{T}_1$ with $(1,0) \in B$ contains an interval of the form $J = \big((a,b),(1,0)\big]$ with $a < 1$, and then $\big(\frac{a+1}{2}, 0\big) \in J$ so that $J \not\subseteq U$ hence $B \not\subseteq U$.
$\mathcal{T}_2 \subseteq \mathcal{T}_3$ for the same reasn that $\mathcal{T}_1 \subseteq \mathcal{T}_3$: every basic open set in $\mathcal{T}_2$ is a union of sets of the above forms $\{a\} \times [0,d)$, $\{a\} \times (c,1]$ and $\{a\} \times (c,d)$ each of which is open in $\mathcal{T}_3$.
$\mathcal{T}_3 \not\subseteq \mathcal{T}_2$ for the same reason that $\mathcal{T}_1 \not\subseteq \mathcal{T}_2$: when $a, b, c \in I$ with $b < d$, and $B = \big\{(a,y)\,\big|\, b < y < d\big\}$ we have $B \in \mathcal{T}_3$, but $B \notin \mathcal{T}_2$.

3: Determine (with proof) which of the following statements are true for all topological spaces X, Y and Z.

(a) If X is connected and $\sim$ is an equivalence relation on X, then $X/\sim$ is connected.

Solution: This is true. Indeed, suppose that X is connected and let $\sim$ be an equivalence relation on X. Let $q : X \to X/\sim$ be the quotient map. Since X is connected and q is continuous, the image $q(X) = X/\sim$ is connected (by Theorem 3.2).

(b) If $a \in X$ then X is not homeomorphic to $X \setminus \{a\}$.

Solution: This is false. For example, if $X = \{0, 1, 2, \cdots\} \subseteq \mathbb{R}$, then $X \setminus \{0\} = \{1, 2, 3, \cdots\}$ is homeomorphic to X. Indeed the map $f(x) = x + 1$ is a homeomorphism from X to $X \setminus \{0\}$ with inverse $g(u) = u - 1$.

(c) If $X \subseteq Z$ is closed in Z, $Y \subseteq Z$ and $X \cong Y$ then Y is closed in Z.

Solution: This is false. For example, the x-axis is a closed subset of $\mathbb{R}^2$, and it is homeomorphic to the open interval $(0, 1)$ along the x-axis in $\mathbb{R}^2$, but this open interval is not closed in $\mathbb{R}^2$ (its not open in $\mathbb{R}^2$ either).

(d) If $\{\pm 1\}$ acts on $\mathbb{C}$ by multiplication, then $\mathbb{C}/\{\pm 1\} \cong \mathbb{C}$.

Solution: This is true. Define $f : \mathbb{C}/\!\sim \to \mathbb{C}$ by $f([z]) = z^2$, or in cartesian coordinates by $f(x + iy) = (x^2 - y^2) + i(2xy)$. Note that f is continuous by Theorem 2.19 because, letting $q : \mathbb{C} \to \mathbb{C}/\!\sim$ be the quotient map, the map $(f \circ q) : \mathbb{C} \to \mathbb{C}$ is given by $(f \circ q)(x) = f([z]) = z^2$ so that $f \circ q$ is continuous. The inverse $g : \mathbb{C} \to \mathbb{C}/\!\sim$ is given by $g(w) = \sqrt{w}$ where $\sqrt{w}$ denotes the set of square roots of w (so that for $z, w \in \mathbb{C}$ with $z^2 = w$ we have $\sqrt{w} = \{\pm z\}$. In polar coordinates, g is given by $g\left(re^{i\theta}\right) = \left\{\pm \sqrt{r}\, e^{i\theta/2}\right\}$.

It is clear that f and g are inverses of one another, but it is not immediately clear from any of our theorems that g is continuous. To prove that g is continuous, let us express g in cartesian coordinates. For $z = x + iy$ and $w = u + iv$ we have

$$z^2 = w \iff (x^2 - y^2) + i(2xy) = u + iv \iff \big(x^2 - y^2 = u \ (1) \text{ and } 2xy = v \ (2)\big)$$

To solve this pair of equations for x, square both sides of (2) to get $4x^2y^2 = v^2$, then multiply both sides of (1) by $4x^2$ to get $4x^4 - 4x^2y^2 = 4x^2u$, so we have $4x^4 - v^2 = 4xu$, which we can write as $x^4 - ux^2 - \frac{1}{4}v^2 = 0$. This is quadratic in x^2, and the quadratic formula gives $x^2 = \frac{u \pm \sqrt{u^2+v^2}}{2}$. Since the left side is positive, we use the $+$ sign to get $x^2 = \frac{u + \sqrt{u^2+v^2}}{2}$, and hence $x = \pm\sqrt{\frac{\sqrt{u^2+v^2}+u}{2}}$. A similar calculation gives $y = \pm\sqrt{\frac{\sqrt{u^2+v^2}-u}{2}}$. Also note that in order to have $2xy = v$, x and y must be of the same sign when $v \geq 0$ and the opposite sign when $v < 0$. Thus the function g is given, in cartesian coordinates, by

$$g(u + i\,v) = \left\{ \begin{array}{ll} \left[\sqrt{\frac{\sqrt{u^2+v^2}+u}{2}} + i\sqrt{\frac{\sqrt{u^2+v^2}-u}{2}}\right] & \text{, if } v \geq 0 \\ \left[\sqrt{\frac{\sqrt{u^2+v^2}+u}{2}} - i\sqrt{\frac{\sqrt{u^2+v^2}-u}{2}}\right] & \text{, if } v \leq 0 \end{array} \right\}.$$

Let $A = \{u + iv \,|\, v \geq 0\}$ and $B = \{u + iv \,|\, v \leq 0\}$. Let $g_1 : A \to \mathbb{C}$ and $g_2 : B \to \mathbb{C}$ be the continuous maps given by $g_1(u + iv) = \sqrt{\frac{\sqrt{u^2+v^2}+u}{2}} + i\sqrt{\frac{\sqrt{u^2+v^2}-u}{2}}$ and $g_2(u + iv) = \sqrt{\frac{\sqrt{u^2+v^2}+u}{2}} - i\sqrt{\frac{\sqrt{u^2+v^2}-u}{2}}$. Since the quotient map q is continuous, the maps $q \circ g_1 : A \to \mathbb{C}/\!\sim$ and $q \circ g_2 : B \to \mathbb{C}/\!\sim$ are continuous. Since $\mathbb{C} = A \cup B$, and A and B are closed in $\mathbb{C}$, and g is given by $g(w) = (q \circ g_1)(w)$ when $w \in A$ and by $g(w) = (q \circ g_2)(w)$ when $w \in B$, it follows that g is continuous by Theorem 1.36 (The Glueing Lemma). We also remark that g is well-defined because when $v = 0$ so that $u + iv = u$, for $u \geq 0$ we have $g_1(u) = \sqrt{u} = g_2(u)$ and for $u \leq u$ we have $g_1(u) = -\sqrt{-u} = -g_2(u)$ so, in either case, $[g_1(u)] = [g_2(u)]$.

4: (a) The **open Möbius strip** is the quotient space $\mathbb{M}^2 = \big([0,1]\times(0,1)\big)/\sim$ where

$$(a,b)\sim(c,d) \iff \Big((a,b)=(c,d) \text{ or } \big(\{a,c\}=\{0,1\} \text{ and } b+d=1\big)\Big).$$

A line in $\mathbb{R}^2$ is determined by its equation $ax+by+c=0$ where $(a,b)\neq(0,0)$, and two such equations determine the same line if and only if they differ by a non-zero scalar multiple, so we can identify the set of all lines with the quotient space $X/\mathbb{R}^*$ where $X=\big\{(a,b,c)\in\mathbb{R}^3\big|(a,b)\neq(0,0)\big\}$ and where the group of non-zero numbers $\mathbb{R}^*$ acts by scalar multiplication. Show that $X/\mathbb{R}^*$ is homeomorphic to $\mathbb{M}^2$.

Solution: The equivalence classes in X under the action of $\mathbb{R}^*$ are the lines through the origin, excluding the origin (and excluding the z-axis). Each of these lines passes through the unit sphere at two antipodal points. Let H be the closed, punctured hemisphere

$$H=\big\{(x,y,z)\in\mathbb{S}^2\,\big|\,y\geq 0, z\neq\pm1\big\}\subseteq X.$$

We have a homeomorphism $f_1:[0,1]\times(0,1)\to H$ given by

$$(x,y,z)=f_1(u,v)=\Big(2\sqrt{v-v^2}\cos(\pi u)\,,\,2\sqrt{v-v^2}\sin(\pi u)\,,\,2v-1\Big).$$

with inverse $g_1={f_1}^{-1}:H\to[0,1]\times(0,1)$ given by

$$(u,v)=g_1(x,y,z)=\Big(\tfrac{1}{\pi}\cos^{-1}\tfrac{x}{\sqrt{x^2+y^2}}\,,\,\tfrac{z+1}{2}\Big).$$

Let us use these maps to obtain a continuous map $f:\mathbb{M}^2\to X/\mathbb{R}^*$ with continuous inverse $g:X/\mathbb{R}^*\to\mathbb{M}^2$. Let $p:X\to X/\mathbb{R}^*$ and $q:[0,1]\times(0,1)\to\mathbb{M}^2$ be the quotient maps. Let $f_2=p\circ f_1:[0,1]\times(0,1)\to X/\mathbb{R}^*$, which is continuous. Note that f_2 is constant on equivalence classes in $\mathbb{M}^2$ because

$$f_1(1,1-v)=\big(-2\sqrt{(1-v)-(1-v)^2}\,,\,0\,,\,2(1-v)-1\big)=\big(-2(v-v^2),1-2v\big)=-f_1(0,v).$$

Since f_2 is continuous and constant on equivalence classes, it induces the continuous map $f:\mathbb{M}^2\to X/\mathbb{R}^*$ given by $f([x,y,z])=f_2(x,y,z)$, that is by

$$f\big([u,v]\big)=\Big[\Big(2\sqrt{v-v^2}\cos(\pi u)\,,\,2\sqrt{v-v^2}\sin(\pi u)\,,\,2v-1\Big)\Big].$$

For the inverse map, let $A=\big\{(a,b,c)\in X\,\big|\,b\geq0\big\}$ and $B=\big\{(a,b,c)\in X\,\big|\,b\leq0\big\}$, and note that A and B are closed in X with $X=A\cup B$. Define continuous maps $h:A\to H$ by $h(w)=\frac{w}{\|w\|}$ and $k:B\to H$ by $k(w)=-\frac{w}{\|w\|}$. These give continuous maps $(q\circ g_1\circ h):A\to\mathbb{M}^2$ and $(q\circ g_1\circ k):B\to\mathbb{M}^2$. These maps agree on $A\cap B$, so they combine to give a continuous map $g_2:X\to\mathbb{M}^2$ given by $g_2(w)=q(g_1(h(w)))=\big[g_1\big(\frac{w}{\|w\|}\big)\big]$ when $w\in A$, and by $g_2(w)=q(g_1(k(w)))=\big[g_1\big(-\frac{w}{\|w\|}\big)\big]$ when $w\in B$. This map is constant on equivalence classes in $X/\mathbb{R}^*$, so it induces the continuous map $g:X/\mathbb{R}^*\to\mathbb{M}^2$ given by $g(w)=g_2([w])$, that is by

$$g\big([a,b,c]\big)=\left\{\begin{array}{ll}\Big[\tfrac{1}{\pi}\cos^{-1}\tfrac{b}{\sqrt{a^2+b^2+c^2}}\,,\,\tfrac12\big(1+\tfrac{c}{\sqrt{a^2+b^2+c^2}}\big)\Big] & \text{if } b\geq0\\ \Big[\tfrac{1}{\pi}\cos^{-1}\tfrac{-b}{\sqrt{a^2+b^2+c^2}}\,,\,\tfrac12\big(1-\tfrac{c}{\sqrt{a^2+b^2+c^2}}\big)\Big] & \text{if } b\leq0\end{array}\right\}$$

(b) Let $M_n(\mathbb{R})$ be the vector space of $n\times n$ matrices with entries in $\mathbb{R}$ using its standard inner product given by $\langle A,B\rangle=\text{trace}(B^TA)$. The **special linear group** is the group $SL_n(\mathbb{R})=\big\{A\in M_n(\mathbb{R})\,\big|\,\det A=1\big\}$. Show that $SL_2(\mathbb{R})$ is homeomorphic to $\mathbb{S}^1\times\mathbb{R}^2$.

Solution: The first row of a matrix in $SL(2,\mathbb{R})$ lies in $\mathbb{R}^2\setminus\{(0,0)\}$, and we have $\mathbb{R}^2\setminus\{(0,0)\}\cong\mathbb{S}^1\times\mathbb{R}^1$ with a homeomorphism $f:\mathbb{R}^2\setminus\{(0,0)\}\to\mathbb{S}^1\times\mathbb{R}^1$ given by $f(a,b)=\big(\big(\frac{a}{\sqrt{a^2+b^2}},\frac{b}{\sqrt{a^2+b^2}}\big),\ln\sqrt{a^2+b^2}\big)$. Given the first row (a,b) for a matrix in $SL(2,\mathbb{R})$, the second row (c,d) satisfies $ad-bc=1$, so it lies on the line $ay-bx=1$ which is given parametrically by $(x,y)=\big(\frac{-b}{a^2+b^2},\frac{a}{a^2+b^2}\big)+t(a,b)$. Note that when $(c,d)=\big(\frac{-b}{a^2+b^2},\frac{a}{a^2+b^2}\big)+t(a,b)$, we have $(c,d)\cdot(a,b)=(a^2+b^2)\,t$ and so $t=\frac{ac+bd}{a^2+b^2}$. We use this to obtain a homeomorphism $g:SL(2,\mathbb{R})\to\mathbb{S}^1\times\mathbb{R}^2$ given by

$$g\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)=\Big(\big(\tfrac{a}{\sqrt{a^2+b^2}},\tfrac{b}{\sqrt{a^2+b^2}}\big),\ln\sqrt{a^2+b^2},\tfrac{ac+bd}{a^2+b^2}\Big).$$

You can verify that the inverse $h=g^{-1}:\mathbb{S}^1\times\mathbb{R}^2\to SL(2,\mathbb{R})$ is given by

$$h\left((x,y),r,t\right)=\begin{pmatrix}xe^r & ye^r\\ txe^r-ye^{-r} & tye^r+xe^{-r}\end{pmatrix}.$$