PMATH 367 Topology, Assignment 2 Due Oct 7, 11:00 pm

1: Let $X = \mathbb{R}$ using the lower limit topology $\mathcal{T}$, generated by the sets of the form $[a, b)$ where $a, b \in \mathbb{R}$ with $a < b$, and let $Y = \mathbb{R}$ using the topology $\mathcal{S}$ generated by the sets of the form $[a, b)$ where $a, b \in \mathbb{Q}$ with $a < b$. Let $A = (0, \sqrt{2}) \subseteq \mathbb{R}$ and let $B = (\sqrt{2}, 3) \subseteq \mathbb{R}$. Find the closures $\text{Cl}_X(A)$, $\text{Cl}_X(B)$, $\text{Cl}_Y(A)$ and $\text{Cl}_Y(B)$.

2: When X is an ordered set, the **dictionary order** on $X^2 = X \times X$ is the order given by stipulating that

$$(a, b) < (c, d) \iff \big(a < c \text{ or } (a = c \text{ and } b < d)\big).$$

(a) Let $I = [0, 1] \subseteq \mathbb{R}$ and let $X = I^2$ using the order topology for the dictionary order. Find $\overline{A} = \text{Cl}_X A$ where $A = \big\{(x, 0) \,\big|\, 0 < x < 1\big\}$.

(b) Let $I = [0, 1] \subseteq \mathbb{R}$ and let $X = I^2 \subseteq \mathbb{R}^2$. Let $\mathcal{T}_1$ be the order topology on X using the dictionary order, let $\mathcal{T}_2$ be the product topology on X using the order topology on each copy of I, and let $\mathcal{T}_3$ be the subspace topology that X inherits from $Y = \mathbb{R}^2$ using the order topology for the dictionary order. For each $k, \ell \in \{1, 2, 3\}$ with $k < \ell$, determine whether $\mathcal{T}_k \subseteq \mathcal{T}_\ell$ and whether $\mathcal{T}_\ell \subseteq \mathcal{T}_k$.

3: Determine (with proof) which of the following statements are true for all topological spaces X, Y and Z.

(a) If X is connected and $\sim$ is an equivalence relation on X, then $X/\sim$ is connected.

(b) If $a \in X$ then X is not homeomorphic to $X \setminus \{a\}$.

(c) If $X \subseteq Z$ is closed in Z, $Y \subseteq Z$ and $X \cong Y$ then Y is closed in Z.

(d) If $\{\pm 1\}$ acts on $\mathbb{C}$ by multiplication, then $\mathbb{C}/\{\pm 1\} \cong \mathbb{C}$.

4: (a) The **open Möbius strip** is the quotient space $\mathbb{M}^2 = \big([0, 1] \times (0, 1)\big)/\sim$ where

$$(a, b) \sim (c, d) \iff \Big((a, b) = (c, d) \text{ or } \big(\{a, c\} = \{0, 1\} \text{ and } b + d = 1\big)\Big).$$

A line in $\mathbb{R}^2$ is determined by its equation $ax + by + c = 0$ where $(a, b) \neq (0, 0)$, and two such equations determine the same line if and only if they differ by a non-zero scalar multiple, so we can identify the set of all lines with the quotient space $X/\mathbb{R}^*$ where $X = \big\{(a, b, c) \in \mathbb{R}^3 \big| (a, b) \neq (0, 0)\big\}$ and where the group of non-zero numbers $\mathbb{R}^*$ acts by scalar multiplication. Show that $X/\mathbb{R}^*$ is homeomorphic to $\mathbb{M}^2$.

(b) Let $M_n(\mathbb{R})$ be the vector space of $n \times n$ matrices with entries in $\mathbb{R}$ using its standard inner product given by $\langle A, B \rangle = \text{trace}(B^T A)$. The **special linear group** is the group $SL_n(\mathbb{R}) = \big\{A \in M_n(\mathbb{R}) \,\big|\, \det A = 1\big\}$. Show that $SL_2(\mathbb{R})$ is homeomorphic to $\mathbb{S}^1 \times \mathbb{R}^2$.