PMATH 367 Topology, Solutions to Assignment 1

1: (a) Let $X=\{1,2\}$. Find every topology on X.

Solution: The set of all subsets of X, that is the power set of X, is the set $\mathcal{P}(X)=\big\{\emptyset,\{1\},\{2\},X\big\}$. Each topology on X is a subset of $\mathcal{P}(X)$ containing $\emptyset$ and X. The number of such subsets is $2^2=4$ (to form a subset, there are 2 choices as to whether to include $\{1\}$, and there are 2 choices as to whether to include $\{2\}$), and the list of all such subsets is

$$\mathcal{T}_1=\big\{\emptyset,X\big\}\ ,\ \mathcal{T}_2=\big\{\emptyset,\{1\},X\big\}\ ,\ \mathcal{T}_3=\big\{\emptyset,\{2\},X\big\}\ ,\ \mathcal{T}_4=\big\{\emptyset,\{1\},\{2\},X\big\}.$$

Since each of these sets $\mathcal{T}_k$ is finite, in order to determine whether it is closed under unions and intersections, it suffices (by induction) to consider only the union and intersection of two sets $A,B\in\mathcal{T}_k$. Also, since $A\cup\phi=A$, $A\cup X=X$, $A\cap\phi=\phi$, $A\cap X=A$, $A\cup A=A$ and $A\cap A=A$, it suffices to consider the case that $A,B\in\mathcal{T}_k$ with $A,B\neq\phi$, $A,B\neq X$ and $A\neq B$. For the sets $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ there is nothing to check, and they are all closed under unions and intersections. For $\mathcal{T}_4$ the only case we need to check is the case that A and B are the sets $\{1\}$ and $\{2\}$, and we have $\{1\}\cup\{2\}=X\in\mathcal{T}_4$ and $\{1\}\cap\{2\}=\emptyset\in\mathcal{T}_4$ so that $\mathcal{T}_4$ is also closed under unions and intersections (and we did not really need to check at all because $\mathcal{T}_4=\mathcal{P}(X)$ which is clearly closed under unions and intersections). Thus X has 4 topologies, namely the sets $\mathcal{T}_k$.

(b) Let $X=\{1,2,3\}$, let $\mathcal{R}=\big\{\emptyset,\{1\},\{1,2\},X\big\}$ and let $\mathcal{S}=\big\{\emptyset,\{1\},\{2,3\},X\big\}$. Find the largest topology on X which is contained in $\mathcal{R}\cap\mathcal{S}$ and find the smallest topology on X which contains $\mathcal{R}\cup\mathcal{S}$.

Solution: We have $\mathcal{R}\cap\mathcal{S}=\big\{\emptyset,\{1\},X\big\}$. Note that this is clearly closed under unions and intersections (as explained in Part (a), we only need to consider $A,B\in\mathcal{R}\cap\mathcal{S}$ with $A,B\neq\emptyset$, $A,B\neq X$ and $A\neq B$, so there is nothing to check), and so $\mathcal{R}\cap\mathcal{S}$ is a topology on X. Since $\mathcal{R}\cap\mathcal{S}=\big\{\emptyset,\{1\},X\big\}$ is a topology on X, it is the largest topology on X which is contained in $\mathcal{R}\cap\mathcal{S}$.

We have $\mathcal{R}\cup\mathcal{S}=\big\{\emptyset,\{1\},\{1,2\},\{2,3\},X\big\}$. Note that this is not closed under intersections because $\{1,2\}\cap\{2,3\}=\{2\}\notin\mathcal{R}\cup\mathcal{S}$. Let $\mathcal{T}$ be the smallest topology on X which contains $\mathcal{R}\cup\mathcal{S}$, in other words, let $\mathcal{T}$ be the topology on X which is generated by $\mathcal{R}\cup\mathcal{S}$. Since $\mathcal{T}$ is closed under unions and intersections, we must have $\{2\}=\{1,2\}\cap\{2,3\}\in\mathcal{T}$ and hence

$$\mathcal{R}\cup\mathcal{S}\cup\{2\}=\big\{\emptyset,\{1\},\{2\},\{1,2\},\{2,3\},X\big\}\subseteq\mathcal{T}.$$

To determine whether $\mathcal{R}\cup\mathcal{S}\cup\{2\}$ is closed under unions and intersections, it suffices to consider $A,B\in\mathcal{R}\cap\mathcal{S}$ with $A,B\neq\emptyset$, $A,B\neq X$ and $A\neq B$. Since $\{1\}\cup\{2\}=\{1,2\}$, $\{1\}\cup\{1,2\}=\{1,2\}$, $\{1\}\cup\{2,3\}=\{1,2,3\}$, $\{2\}\cup\{1,2\}=\{1,2\}$, $\{2\}\cup\{2,3\}=\{2,3\}$ and $\{1,2\}\cup\{2,3\}=\{1,2,3\}$, $\mathcal{R}\cup\mathcal{S}\cup\{2\}$ is closed under unions, and since $\{1\}\cap\{2\}=\emptyset$, $\{1\}\cap\{1,2\}=\{1\}$, $\{1\}\cap\{2,3\}=\emptyset$, $\{2\}\cap\{1,2\}=\{2\}$, $\{2\}\cap\{2,3\}=\{2\}$ and $\{1,2\}\cap\{2,3\}=\{2\}$, it is closed under intersections. Thus $\mathcal{R}\cup\mathcal{S}\cup\{2\}$ is a topology which contains $\mathcal{R}\cup\mathcal{S}$. Since $\mathcal{T}$ is the smallest such topology, $\mathcal{T}\subseteq\mathcal{R}\cup\mathcal{S}\cup\{2\}$ Thus

$$\mathcal{T}=\mathcal{R}\cup\mathcal{S}\cup\{2\}=\big\{\emptyset,\{1\},\{2\},\{1,2\},\{2,3\},X\big\}.$$

(c) Let $X = \{1,2,3,4\}$ and let $\mathcal{T} = \big\{\emptyset, \{1\}, \{2\}, \{1,2\}, \{3,4\}, \{1,3,4\}, \{2,3,4\}, X\big\}$. Given that $\mathcal{T}$ is a topology on X, find the number of bases for $\mathcal{T}$.

Solution: Let $\mathcal{B} = \big\{\{1\}, \{2\}, \{3,4\}\big\}$. We claim that $\mathcal{B}$ is a basis for $\mathcal{T}$. By Theorem 1.21, it suffices to show that for every $U \in \mathcal{T}$ and every $a \in U$ there exists $B \in \mathcal{B}$ with $a \in B \subseteq \mathcal{B}$. We consider all possible cases for U and a: When $U = \emptyset$ there is no $a \in U$. When $U = \{1\}$ and $a = 1$ we can take $B = \{1\}$. When $U = \{2\}$ and $a = 2$ we can take $B = \{2\}$. When $U = \{1,2\}$, if $a = 1$ we can take $B = \{1\}$ and if $a = 2$ we can take $B = \{2\}$. When $U = \{3,4\}$ and $a \in \{3,4\}$ we can take $B = \{3,4\}$. When $U = \{1,3,4\}$, if $a = 1$ we can take $B = \{1\}$ and if $a \in \{3,4\}$ we can take $B = \{3,4\}$. When $U = \{2,3,4\}$, if $a = 2$ we can take $B = \{2\}$ and if $a \in \{3,4\}$ we can take $B = \{3,4\}$. Finally, when $U = X$, if $a = 1$ we can take $B = \{1\}$, if $a = 2$ we can take $B = \{2\}$, and if $a \in \{3,4\}$ we can take $B = \{3,4\}$. This covers all cases, and so $\mathcal{B}$ is a basis for $\mathcal{T}$.

We claim that the bases for $\mathcal{T}$ are the sets $\mathcal{C}$ with $\mathcal{B} \subseteq \mathcal{C} \subseteq \mathcal{T}$. Let $\mathcal{C}$ be any basis for $\mathcal{T}$. Since $\mathcal{T}$ is the topology generated by $\mathcal{C}$ we have $\mathcal{C} \subseteq \mathcal{T}$. By Part 2 of Theorem 1.20, we know that for every $U \in \mathcal{T}$ and every $a \in U$, there exists $C \in \mathcal{C}$ with $a \in C \subseteq U$. Taking $U = \{1\}$ and $a = 1$ we see that $C = \{1\} \in \mathcal{C}$. Taking $U = \{2\}$ and $a = 2$ we see that $C = \{2\} \in \mathcal{C}$. Taking $U = \{3,4\}$ and $a = 3$ we see that $C = \{3,4\} \in \mathcal{C}$. Thus we must have $\mathcal{B} \subseteq \mathcal{C}$. This proves that if $\mathcal{C}$ is a basis for $\mathcal{T}$ then $\mathcal{B} \subseteq \mathcal{C} \subseteq \mathcal{T}$.

Suppose, on the other hand, that $\mathcal{C}$ is any set with $\mathcal{B} \subseteq \mathcal{C} \subseteq \mathcal{T}$. By Theorem 1.21, in order to show that $\mathcal{C}$ is a basis for $\mathcal{T}$ it suffices to show that for every $U \in \mathcal{T}$ and every $a \in U$ there exists $C \in \mathcal{C}$ such that $a \in C \subseteq U$. Let $U \in \mathcal{T}$ and let $a \in U$. Since $\mathcal{B}$ is a basis for $\mathcal{T}$, we can choose $B \in \mathcal{B}$ such that $a \in B \subseteq U$. Since $\mathcal{B} \subseteq \mathcal{C}$, we also have $B \in \mathcal{C}$ so we can choose $C = B \in \mathcal{C}$ to get $a \in C \subseteq U$.

This completes the proof that the bases for $\mathcal{T}$ are the sets $\mathcal{C}$ with $\mathcal{B} \subseteq \mathcal{C} \subseteq \mathcal{T}$. To form such a set $\mathcal{C}$, there are 2 choices as to whether to include each of the 5 elements in $\mathcal{T} \setminus \mathcal{B} = \big\{\emptyset, \{1,2\}, \{1,3,4\}, \{2,3,4\}, X\big\}$, so the number of such sets $\mathcal{C}$ is $2^5 = 32$.

2: Let X be a topological space, let $\mathcal{B}$ be a basis for the topology on X, let $A \subseteq X$, and let $a \in X$. Prove each of the following using only definitions and proven results from the lecture notes.

(a) $a \in A^o$ if and only if there exists $B \in \mathcal{B}$ with $a \in B$ such that $B \subseteq A$.

Solution: Suppose that $a \in A^o$. By the definition of A^o, this means that we can choose an open set U in X with $a \in U \subseteq A$. Since $\mathcal{B}$ is a basis for the topology on X, by Theorem 1.20 we can choose $B \in \mathcal{B}$ with $a \in B \subseteq U$. Since we have $a \in B$ and $B \subseteq U \subseteq A$, as required.

Suppose, conversely, that there exists $B \in \mathcal{B}$ with $a \in B \subseteq A$. Since B is an open set in X with $a \in B \subseteq A$, and A^0 is the union of all such sets, we have $a \in A^o$.

(b) $a \in \overline{A}$ if and only if for every $B \in \mathcal{B}$ with $a \in B$ we have $B \cap A \neq \emptyset$.

Solution: Suppose that $a \in \overline{A}$. By the definition of $\overline{A}$, this means that $a \in K$ for every closed set K in X with $A \subseteq K$. Let $B \in \mathcal{B}$ with $a \in B$. We need to show that $B\mathcal{A} \neq \emptyset$. Suppose, for a contradiction, that $B \cap A = \emptyset$. Since $B \cap A = \emptyset$, we have $A \subseteq B^c$ where $B^c = X \setminus B$. Since B^c is a closed set in X with $A \subseteq B^c$, we have $a \in B^c$, that is $\notin B$. This gives the desired contradiction since $a \in B$.

Suppose, conversely, that for every $B \in \mathcal{B}$ with $a \in B$ we have $B \cap A \neq \emptyset$. Suppose, for a contradiction, that $a \notin \overline{A}$. By the definition of $\overline{A}$, this means that we can choose a closed set K in X with $A \subseteq K$ such that $a \notin K$. Let $U = K^c = X \setminus K$, which is open in X with $a \in U$. Since $\mathcal{B}$ is a basis for the topology on X, by Theorem 1.20 we can choose $B \in \mathcal{B}$ with $a \in B \subseteq U$. Since $B \subseteq U$ we have $B \cap U^c = \emptyset$, that is $B \cap K = \emptyset$. Since $B \cap K = \emptyset$ and $A \subseteq K$, we also have $B \cap A = \emptyset$. This contradicts the assumption made at the start of this paragraph, so we must have $a \in \overline{A}$, as required.

(c) $a \in \partial A$ if and only if for every $B \in \mathcal{B}$ with $a \in B$ we have $B \cap A \neq \emptyset$ and $B \cap (X \setminus A) \neq \emptyset$.

Solution: By Part (a) we have $a \notin A^o$ if and only if for every $B \in \mathcal{B}$ with $a \in B$ we have $B \not\subseteq A$. Also note that $B \subseteq A \iff B \cap A^c = \emptyset$ where $A^c = X \setminus A$, and hence $B \not\subseteq A \iff B \cap A^c \neq \emptyset$. Thus

$$\begin{aligned} a \in \partial A &\iff a \in \overline{A} \text{ and } a \notin A^o \\ &\iff \big(\forall\, B \in \mathcal{B} \text{ with } a \in B \;\; B \cap A \neq \emptyset\big) \text{ and } \big(\forall\, B \in \mathcal{B} \text{ with } a \in B \;\; B \not\subseteq A\big) \\ &\iff \big(\forall\, B \in \mathcal{B} \text{ with } a \in B \;\; B \cap A \neq \emptyset\big) \text{ and } \big(\forall\, B \in \mathcal{B} \text{ with } a \in B \;\; B \cap A^c \neq \emptyset\big) \\ &\iff \forall\, B \in \mathcal{B} \text{ with } a \in B \, (B \cap A \neq \emptyset \text{ and } B \cap A^c \neq \emptyset). \end{aligned}$$

3: Let X be a topological space.

(a) Show that if $A,B\subseteq X$ then $\overline{A}\cup\overline{B}=\overline{A\cup B}$.

Solution: We remark that, from the definition of the closure, that for $C,D\subseteq X$, if $C\subseteq D$ then $\overline{C}\subseteq\overline{D}$: indeed when $C\subseteq D$, $\overline{D}$ is a closed set in X which contains D, hence also C. Let $A,B\subseteq X$.

Since $A\subseteq A\cup B$ we have $\overline{A}\subseteq\overline{A\cup B}$. Since $B\subseteq A\cup B$ we have $\overline{B}\subseteq\overline{A\cup B}$. Since $\overline{A}\subseteq\overline{A\cup B}$ and $\overline{B}\subseteq\overline{A\cup B}$, it follows that $\overline{A}\cup\overline{B}\subseteq\overline{A\cup B}$.

Since $A\subseteq\overline{A}\subseteq\overline{A}\cup\overline{B}$ and $B\subseteq\overline{B}\subseteq\overline{A}\cup\overline{B}$, we have $A\cup B\subseteq\overline{A\cup B}$. Since $\overline{A}\cup\overline{B}$ is a closed set in A which contains $A\cup B$, we have $\overline{A\cup B}\subseteq\overline{A}\cup\overline{B}$.

(b) Show that if $A_k\subseteq X$ for all $k\in K$ then $\bigcup_{k\in K}\overline{A_k}\subseteq\overline{\bigcup_{k\in K}A_k}$ and give an example of a topological space X and subsets $A_k\subseteq X$ for which the reverse inclusion does not hold.

Solution: We remark, as we did at the start of Part (a), that for $C,D\in X$, if $C\subseteq D$ then we have $\overline{C}\subseteq\overline{D}$. Let $A_k\subseteq X$ for all $k\in K$.

For each $k\in K$ we have $A_k\subseteq\bigcup_{j\in K}A_j$ and hence $\overline{A_k}\subseteq\overline{\bigcup_{j\in K}A_j}$. Since $\overline{A_k}\subseteq\overline{\bigcup_{j\in K}A_j}$ for every $k\in K$ it follows that $\bigcup_{k\in K}\overline{A_k}\subseteq\overline{\bigcup_{j\in K}A_j}$.

To see that the reverse inclusion does not always hold, Let $X=\mathbb{R}$, let $K=\mathbb{Q}$, and for each $k\in K=\mathbb{Q}$, let $A_k=\{k\}$. Then $\bigcup_{k\in K}\overline{A_k}=\bigcup_{k\in\mathbb{Q}}\overline{\{k\}}=\bigcup_{k\in\mathbb{Q}}\{k\}=\mathbb{Q}$, but $\overline{\bigcup_{k\in K}A_k}=\overline{\bigcup_{k\in\mathbb{Q}}\{k\}}=\overline{\mathbb{Q}}=\mathbb{R}$.

(c) Show that if every 1-point subset of X is closed in X then for all $A\subseteq X$ we have $\overline{A}'=A'$.

Solution: First we remark that, from the definition of a limit point, for $C,D\subseteq X$, if $C\subseteq D$ then $C'\subseteq D'$: indeed, when $C\subseteq D$ and $a\in C'$, given an open set U in X with $a\in U$ we have $(U\setminus\{a\})\cap C\neq\emptyset$, hence also $(U\setminus\{a\})\cap D\neq\emptyset$.

Suppose that every 1-point subset of X is closed in X and let $A\subseteq X$. Since $A\subseteq A'$ it follows from the above remark that $A'\subseteq\overline{A}'$. It remains to prove that $\overline{A}'\subseteq A'$. Let $a\in\overline{A}'$. We need to show that $a\in A'$. let U be an open set in X with $a\in U$. It suffices to show that $(U\setminus\{a\})\cap A\neq\emptyset$. Since $a\in\overline{A}'$, we have $(U\setminus\{a\})\cap\overline{A}\neq\emptyset$. Suppose, for a contradiction, that $(U\setminus\{a\})\cap A=\emptyset$. This means that $A\subseteq(U\setminus\{a\})^c$ where $(U\setminus\{a\})^c=X\setminus(U\setminus\{a\})$. Since $\{a\}$ is closed in X, the set $\{a\}^c$ is open in X (where $\{a\}^c=X\setminus\{a\}$), and hence $U\setminus\{a\}=U\cap\{a\}^c$ is open in X, and hence the set $(U\setminus\{a\})^c$ is closed in X. Since $A\subseteq(U\setminus\{a\})^c$ and $(U\setminus\{a\})^c$ is closed, it follows that $\overline{A}\subseteq(U\setminus\{a\})^c$. But this means that $(U\setminus\{a\})\cap\overline{A}=\emptyset$, giving the desired contradiction. Thus we must have $(U\setminus\{a\})\cap A\neq\emptyset$, as required.

4: Let X be a topological space. When $(x_n)_{n\geq 1}$ is a sequence in X, for $a\in X$ we say that $(x_n)_{n\geq 1}$ **converges to** a in X, or that a is **a limit** of $(x_n)_{n\geq 1}$ in X, and we write $x_n\to a$ in X, when for every open set U in X with $a\in U$ there exists $m\in\mathbb{Z}^+$ such that $x_n\in U$ for all $n\geq m$, and we say that $(x_n)_{n\geq 1}$ is **convergent** in X when $x_n\to a$ for some $a\in X$.

(a) Show that if X is Hausdorff then every convergent sequence in X has a unique limit in X.

Solution: Suppose that X is Hausdorff. Let $(x_n)_{n\geq 1}$ be a convergent sequence in X. Let $a,b\in X$ and suppose that $x_n\to a$ in X and $x_n\to b$ in X. We need to show that $a=b$. Suppose, for a contradiction, that $a\neq b$. Since X is Hausdorff, we can choose disjoint open sets u and V in X with $a\in U$ and $b\in V$, Since $x_n\to a$ in X we can choose $m_1\in\mathbb{Z}^+$ so that $n\geq m_1\implies x_n\in U$. Since $x_n\to b$ in X we can choose $m_2\in\mathbb{Z}^+$ so that $n\geq m_2\implies x_n\in V$. But then for $n\geq\max\{m_1,m_2\}$ we have $x_n\in U$ and $x_n\in V$, which contradicts the fact that U and V are disjoint.

(b) Show that if every convergent sequence in X has a unique limit in X then every 1-point subset of X is closed in X.

Solution: Suppose that every convergent sequence in X has a unique limit in X. Let $a\in X$. Let $(x_n)_{n\geq 1}$ be the constant sequence given by $x_n=a$ for all $n\in\mathbb{Z}^+$. Note that $x_n\to a$ in X since given any open set U in X with $a\in U$, we can choose $m=1$ and then for all $n\in\mathbb{Z}^+$ with $n\geq m$ we have $x_n=a\in U$. Let $b\in\{a\}^c=X\setminus\{a\}$. Since $x_n\to a$ in X and $a\neq b$, the fact that convergent sequences have unique limits implies that $x_n\not\to b$ in X. This means that we can choose an open set U_b in X with $b\in U_b$ such that for all $m\in\mathbb{Z}^+$ there exists $n\in\mathbb{Z}^+$ with $n\geq m$ such that $x_n\notin U_b$. Take $m=1$ and choose any $n\in\mathbb{Z}^+$ with $n\geq 1$, and we have $a=x_n\notin U_b$. Thus for each $b\in X$ with $b\neq a$ we have an open set U_b in X with $b\in U_b$ and $a\notin U_b$. Note that $\bigcup_{b\in\{a\}^c}U_b=\{a\}^c$, and so $\{a\}^c$ is a union of open sets in X. Thus $\{a\}^c$ is open in X, and hence $\{a\}$ is closed in X.

(c) In the case that $X=\mathbb{Z}$ with the co-finite topology $\mathcal{T}=\big\{A\subseteq\mathbb{Z}\,|\,A=\emptyset\text{ or }\mathbb{Z}\setminus A\text{ is finite}\big\}$, and $x_n=n^2$ for all $n\in\mathbb{Z}^+$, find all points $a\in X$ for which $x_n\to a$ in X.

Solution: Let $X=\mathbb{Z}$ with the co-finite topology, and let $x_n=n^2$ for all $n\in\mathbb{Z}^+$. We claim that $x_n\to a$ for every $a\in X$. Let $a\in X$. Let U be any open set in X with $a\in U$. Since U is open in X, and $a\in U$ so that $U\neq\emptyset$, the complement $U^c=X\setminus U$ is finite. Since U^c is finite, there are only finitely many $n\in\mathbb{Z}^+$ for which $n^2\in U^c$. Choose $m\in\mathbb{Z}^+$ which is larger than all of the $n\in\mathbb{Z}^+$ for which $n^2\in U^c$. Then for all $n\in\mathbb{Z}^+$ with $n\geq m$ we have $n^2\notin U^c$ so that $x_n=n^2\in U$. This proves that $x_n\to a$. Since $a\in X$ was arbitrary, we have $x_n\to a$ for every $a\in X$, as claimed.