PMATH 367 Topology, Assignment 1 Due Sept 23, 11:00 pm

1: (a) Let $X=\{1,2\}$. Find every topology on X.

(b) Let $X=\{1,2,3\}$, let $\mathcal{R}=\big\{\emptyset,\{1\},\{1,2\},X\big\}$ and let $\mathcal{S}=\big\{\emptyset,\{1\},\{2,3\},X\big\}$. Find the largest topology on X which is contained in $\mathcal{R}\cap\mathcal{S}$ and find the smallest topology on X which contains $\mathcal{R}\cup\mathcal{S}$.

(c) Let $X=\{1,2,3,4\}$ and let $\mathcal{T}=\big\{\emptyset,\{1\},\{2\},\{1,2\},\{3,4\},\{1,3,4\},\{2,3,4\},X\big\}$. Given that $\mathcal{T}$ is a topology on X, find the number of bases for $\mathcal{T}$.

2: Let X be a topological space, let $\mathcal{B}$ be a basis for the topology on X, let $A\subseteq X$, and let $a\in X$. Prove each of the following using only definitions and proven results from the lecture notes.

(a) $a\in A^o$ if and only if there exists $B\in\mathcal{B}$ with $a\in B$ such that $B\subseteq A$.

(b) $a\in\overline{A}$ if and only if for every $B\in\mathcal{B}$ with $a\in B$ we have $B\cap A\neq\emptyset$.

(c) $a\in\partial A$ if and only if for every $B\in\mathcal{B}$ with $a\in B$ we have $B\cap A\neq\emptyset$ and $B\cap(X\setminus A)\neq\emptyset$.

3: Let X be a topological space.

(a) Show that if $A,B\subseteq X$ then $\overline{A}\cup\overline{B}=\overline{A\cup B}$.

(b) Show that if $A_k\subseteq X$ for all $k\in K$ then $\bigcup_{k\in K}\overline{A_k}\subseteq\overline{\bigcup_{k\in K}A_k}$ and give an example of a topological space X and subsets $A_k\subseteq X$ for which the reverse inclusion does not hold.

(c) Show that if every 1-point subset of X is closed in X then for all $A\subseteq X$ we have $\overline{A}'=A'$.

4: Let X be a topological space. When $(x_n)_{n\geq1}$ is a sequence in X, for $a\in X$ we say that $(x_n)_{n\geq1}$ **converges to** a in X, or that a is **a limit** of $(x_n)_{n\geq1}$ in X, and we write $x_n\to a$ in X, when for every open set U in X with $a\in U$ there exists $m\in\mathbb{Z}^+$ such that $x_n\in U$ for all $n\geq m$, and we say that $(x_n)_{n\geq1}$ is **convergent** in X when $x_n\to a$ for some $a\in X$.

(a) Show that if X is Hausdorff then every convergent sequence in X has a unique limit in X.

(b) Show that if every convergent sequence in X has a unique limit in X then every 1-point subset of X is closed in X.

(c) In the case that $X=\mathbb{Z}$ with the co-finite topology $\mathcal{T}=\big\{A\subseteq\mathbb{Z}\,|\,A=\emptyset\text{ or }\mathbb{Z}\setminus A\text{ is finite}\big\}$, and $x_n=n^2$ for all $n\in\mathbb{Z}^+$, find all points $a\in X$ for which $x_n\to a$ in X.