PMATH 347 Groups and Rings, Assignment 6 Due Tues July 30, 11:00 pm

1: Let $A=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$, $B=\begin{pmatrix}0&i\\i&0\end{pmatrix}$ and $C=\begin{pmatrix}i&0\\0&-i\end{pmatrix}$, and let $Q_8=\langle A,B\rangle\leq GL_2(\mathbb{C})$.

(a) Show that $Q_8=\{I,A,B,C,-I,-A,-B,-C\}$ and make the multiplication table for Q_8.

(b) Find the number of elements of each order in Q_8.

(c) Find an abelian group which has the same number of elements of each order as $\mathbb{Z}_2\times Q_8$.

2: (a) Find a group of the form $\mathbb{Z}_{n_1}\times\cdots\times\mathbb{Z}_{n_l}$, with $n_i\big|n_{i+1}$ for all i, which is isomorphic to $\mathbb{Z}_{18}\times\mathbb{Z}_{60}\times\mathbb{Z}_{70}\times\mathbb{Z}_{100}$.

(b) Find a group of the form $\mathbb{Z}_{n_1}\times\cdots\times\mathbb{Z}_{n_l}$, with $n_i\big|n_{i+1}$ for all i, which is isomorphic to $U_{100}\big/\langle 21\rangle$.

(c) Find the number of distinct abelian groups of order 2,000,000 (up to isomorphism).

(d) Determine which abelian group of order 72 has the most elements of order 6.

3: (a) How many ways (up to D_9 symmetry) can the elements of C_9 be coloured using 3 colours?

(b) How many ways (up to rotational symmetry) can the 12 vertices of a regular icosahedron be coloured using 2 colours?

4: (a) Show that if G is a finite group with $|G|$ odd, and $a\in G$ with $\big|\mathrm{Cl}(a)\big|=3$, then G is not simple.

(b) Show that if a group G has a proper subgroup of finite index, then G has a proper normal subgroup of finite index.

(c) Show that if G is a group with $|G|=p^k$ where p is prime and $k\in\mathbb{Z}^+$, then $Z(G)\neq\{e\}$.